# THE HOCHSCHILD COHOMOLOGY RING OF A GROUP ALGEBRA

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Abstract. There is a standard additive decomposition of the Hochschild cohomology ring of the group algebra of a finite group $G$ as the direct sum of the cohomology rings of the centralizers of representatives of the conjugacy classes of $G$. A special case of our main result describes the cup product in terms of this decomposition. As applications, we determine presentations for the Hochschild cohomology rings of (1) the mod-3 group algebra of the symmetric group $S_3$, (2) the mod-2 group algebra of the alternating group $A_4$, and (3) the mod-2 group algebras of the dihedral 2-groups.

## 1. Introduction

**Historical background.** The theory of the cohomology of associative algebras was initiated by G. Hochschild in 1945 [16] and evolved in tandem with group cohomology. To any associative algebra $\Lambda$, and any $\Lambda$-bimodule $M$, Hochschild assigned Abelian groups $H^n(\Lambda, M)$ $(n \geq 0)$. In modern language these may be defined by

$$H^n(\Lambda, M) = \mathrm{Ext}^n_{\Lambda\otimes_R\Lambda^{\mathrm{op}}}(\Lambda, M). \tag{1.1}$$

Hochschild assumed the ground ring $R$ was a field, but there is no problem with taking $R$ to be an arbitrary commutative ring as long as $\Lambda$ is $R$-projective.

Around this time, Eilenberg and Mac Lane gave a purely algebraic definition of the cohomology groups $H^n(G, M)$ for a group $G$ and left $RG$-module $M$ [9]. They also showed that the Hochschild and ordinary theories agree, in the following sense. If $M$ is an $RG$-bimodule then it may be considered a left $RG$-module *via* conjugation. Eilenberg and Mac Lane observed that the Hochschild cohomology of $RG$ with coefficients in $M$ is isomorphic to the ordinary cohomology of $G$ with coefficients in that left module. (They used $R = \mathbb{Z}$ but the same arguments go through for any commutative ring.) This shows that for group algebras, nothing new is obtained by considering bimodules instead of just left modules. Moreover, since any left module arises in this way (just let $RG$ act trivially on the right), this also shows that Hochschild cohomology is a true generalization of group cohomology.

For an arbitrary associative algebra $\Lambda$, there is a natural choice for the bimodule $M$, namely the bimodule $\Lambda$ itself. In fact, the groups $H^*(\Lambda, \Lambda)$ have been investigated for many different classes of algebras. In low degrees, these groups are related to "algebraic deformations," and have found applications in the study of quantum groups (see [13] and references cited).

Eilenberg and Mac Lane defined cup products for group cohomology, and in essence for Hochschild cohomology as well ([9, §§4,5]). The cup product gives

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$H^*(G, R)$ or $H^*(\Lambda, \Lambda)$ the structure of an associative graded algebra. Much progress has been made in describing the ring structure of the former. In particular, the landmark results of Quillen [23, 24] give a complete description, up to homeomorphism, of the spectrum of the ordinary cohomology ring of $G$ with coefficients in a field of prime characteristic. Much less is known about Hochschild cohomology rings. One of the few general results, due to Gerstenhaber [12], is that $H^*(\Lambda, \Lambda)$ is graded-commutative, generalizing another fact from group cohomology.

It follows from Eilenberg and Mac Lane's observation that $H^*(RG, RG)$ is isomorphic to the cohomology of $G$ with coefficients in $RG$, where $RG$ is considered a module by conjugation. From this and the Eckmann-Shapiro Lemma, one sees that $H^*(RG, RG)$ is isomorphic to the direct sum of the cohomology of the centralizers of representatives of the conjugacy classes of $G$. The isomorphism, however, is *not* in general multiplicative. In fact, very little is known at present about the multiplicative structure of the Hochschild cohomology of group algebras. The case where $G$ is Abelian was completely explained by Holm [18] and Cibils and Solotar [8]—in this case the Hochschild ring is just the tensor product of $RG$ and the ordinary cohomology ring—but almost nothing else is known.

One of the intriguing aspects of Hochschild cohomology for representation theorists is that it may be applied to a block of a group algebra. With suitable restrictions on $R$, one may write $RG$ as a direct sum of ideals, each of which is indecomposable as an algebra. These ideals turn out to be unique, and are called the *blocks* of $RG$ ([3, §1.8]). One of the central themes in the representation theory of finite groups involves how each aspect of the theory "breaks up" into blocks. Since these algebras are not, in general, augmented, they do not come with trivial modules, and therefore the ordinary definition of cohomology does not apply.

But Hochschild cohomology does. Moreover, it can be shown that the Hochschild cohomology of the group algebra is isomorphic, as an algebra, to the direct sum of the Hochschild cohomology rings of the blocks. J. Rickard showed that under appropriate hypotheses, two blocks of $RG$ which are *derived equivalent* have isomorphic Hochschild cohomology algebras ([26, Prop. 2.5]). Using this, Holm was able to calculate the even part of the Hochschild cohomology ring of a block with cyclic defect, and show that this ring distinguished between derived equivalence classes of those blocks. But almost no other cases appear in the literature. One of the reasons for this is undoubtedly the complexity of the calculations required to compute products.

**Overview.** Our main result, Theorem 5.1, gives a formula for the product in $H^*(RG, RG)$ in terms of the additive decomposition, proving a conjecture of Cibils [7] and Cibils and Solotar [8]. It reduces the computation of Hochschild products to standard operations within the ordinary cohomology rings of certain subgroups of $G$. As applications, we use this formula to determine presentations for the Hochschild cohomology rings of (1) the mod-3 group algebra of the symmetric group $S_3$, (2) the mod-2 group algebra of the alternating group $A_4$, and (3) the mod-2 group algebras of the dihedral 2-groups $D_{4m}$.

The material is organized as follows. In §2 we review the basic definitions of Hochschild cohomology and cup product, using arbitrary projective resolutions in place of the bar resolutions used by Hochschild and Eilenberg-Mac Lane. In §3 we specialize to the group algebra case, making explicit the isomorphism observed by Eilenberg-Mac Lane, and also showing that that isomorphism preserves the cup product. In this section we also introduce a generalization of the Hochschild cohomology ring of a group algebra: if a second group $H$ acts on $G$ then we may consider the rings $H^*(H, RG)$; the Hochschild cohomology ring is the special case where $H = G$ acts on $G$ by conjugation. Working on this level of generality, in §4 we prove an explicit version of the additive decomposition described above, and in §5 we prove our main result, the Product Formula.

Some authors use the term *Hochschild cohomology of* $\Lambda$ to refer to $H^*(\Lambda, \Lambda^*)$, where $\Lambda^*$ is the dual bimodule $\mathrm{Hom}_R(\Lambda, R)$ (e.g., Loday [20, §1.5.5] and Benson [4, §2.11]). This definition has certain advantages (e.g., it is a functor in $\Lambda$) though in general it does not seem to have an obvious multiplicative structure. In the group algebra case, however, a natural product can be defined, and in this case we show that the additive isomorphism not only holds, but is in fact multiplicative. This is covered in §6.

The next three sections contain the specific applications. The calculations are relatively straightforward and, we hope, demonstrate the usefulness of the Product Formula. In §10, we try to obtain general structure theorems for the Hochschild cohomology ring of a group algebra, and meet with some limited successes. If $G$ is a $p$-group, we show that the mod-$p$ Hochschild and ordinary cohomology rings are isomorphic modulo radicals. We then show that our rings $H^*(H, RG)$ have the structure of a *Green functor*, and exploit this theory to obtain a description of $H^*(H, RG)$ similar to the description of Hochschild cohomology for $H = G$ Abelian, but modulo a certain ideal. These results were inspired by analogy with the Grothendieck ring of the category of modules for the *quantum double* of $G$ (see [30]), an analogy that was essentially noted by Cibils [7]. We conclude in §11 with some intriguing questions which are suggested by the examples and these results.

**Notation.** Throughout this paper, $R$ denotes an arbitrary commutative ring. $G$ will always denote a finite group, and in §§7–9 a specific finite group. All rings and algebras are assumed to possess a unit, and all modules are assumed to be left modules unless stated otherwise. We will abbreviate $H^*(G, R)$ by $H^*(G)$ when the coefficient ring is clear. If $H$ is a subgroup of $G$ then restriction and corestriction (transfer) are denoted $\mathrm{res}^G_H$ and $\mathrm{cor}^G_H$, respectively. If $M$ is an $RG$-module then $M{\downarrow}^G_H$ denotes $M$ considered as an $RH$-module; this is *the restriction of* $M$ *from* $G$ *to* $H$. If $U$ is an $RH$-module then $U{\uparrow}^G_H$ denotes the $RG$-module $RG \otimes_{RH} U$; this is *the module induced from* $H$ *to* $G$. If $g, x \in G$ then ${}^g x = gxg^{-1}$. More generally, if $G$ is acting on a set $X$ then we will often write ${}^g x$ to denote the action of $g$ on an element $x$ of $X$. We let $\delta_g$ denote the $R$-linear function from $RG$ to $R$ which takes the value 1 on $g$ and 0 on any other element of $G$.

## 2. Definitions of Hochschild cohomology and products

Let $R$ be a commutative ring and $\Lambda$ an $R$-algebra which is finitely generated and projective as an $R$-module. Let $\Lambda^e = \Lambda \otimes_R \Lambda^{\mathrm{op}}$. If $M$ is a $\Lambda$-bimodule (i.e., a $\Lambda^e$-module) then the *Hochschild cohomology of* $\Lambda$ *with coefficients in* $M$ is defined to be

$$H^*(\Lambda, M) = \mathrm{Ext}^*_{\Lambda^e}(\Lambda, M), \tag{2.1}$$

where $\Lambda$ is considered to be a $\Lambda$-bimodule in the usual way. These were originally defined by Hochschild [16] when $R$ is a field; see [6, IX§4] for the more general definition and basic properties. Clearly $H^*(\Lambda, -)$ is a covariant functor from the category of $\Lambda$-bimodules to the category of graded $R$-modules.

If $\Lambda$ happens to be an augmented $R$-algebra, then any left $\Lambda$-module $M$ may be considered a $\Lambda$-bimodule with trivial right action. Then (2.1) agrees with the usual definition of $H^*(\Lambda, M)$ ([4, §2.11]). The advantage of Hochschild cohomology is that it may be applied to algebras which are not necessarily augmented, such as blocks of group algebras.

If $N$ is another $\Lambda$-bimodule then there is a Hochschild cup product

$$H^*(\Lambda, N) \otimes_R H^*(\Lambda, M) \xrightarrow{\smile} H^*(\Lambda, N \otimes_\Lambda M),$$

which can be defined as follows. Let $\mathbf{X} \xrightarrow{\epsilon} \Lambda$ be a $\Lambda^e$-projective resolution of $\Lambda$, and let $D \colon \mathbf{X} \to \mathbf{X} \otimes_\Lambda \mathbf{X}$ be a *Hochschild diagonal approximation map*, i.e., a $\Lambda^e$-chain map such that $(\epsilon \otimes \epsilon) \circ D = \epsilon$. In the literature it is common to take $\mathbf{X}$ to be the *standard complex* [6, IX§6], in which case there is an explicit formula for $D$ [28, 1.2]. However, we will show below through standard homological arguments that a map $D$ not only exists but is unique up to chain homotopy. Assuming this, let $\alpha$ and $\beta$ be cohomology classes represented (resp.) by cocycles $f \in \operatorname{Hom}_{\Lambda^e}(\mathbf{X}, N)$ and $g \in \operatorname{Hom}_{\Lambda^e}(\mathbf{X}, M)$. Then $\alpha \smile \beta$ is represented by the cocycle $(f \otimes g) \circ D$. It is easily verified that the cup product is independent of the choices of $\mathbf{X}$ and $D$, and of the cocycles $f$ and $g$ representing $\alpha$ and $\beta$.

Our claims about the existence and uniqueness of $D$ will follow from the Comparison Theorem [3, Thm. 2.4.2] if we can show that

$$\mathbf{X} \otimes_\Lambda \mathbf{X} \xrightarrow{\epsilon \otimes \epsilon} \Lambda$$

is also a $\Lambda^e$-projective resolution. To see that $\mathbf{X} \otimes_\Lambda \mathbf{X}$ is projective in each degree, it suffices to show that $\Lambda^e \otimes_\Lambda \Lambda^e$ is a projective $\Lambda^e$-module. But as $\Lambda^e$-modules, $\Lambda^e \otimes_\Lambda \Lambda^e \cong \Lambda^e \otimes_R \Lambda$. By hypothesis, $\Lambda$ is a summand of a direct sum of copies of $R$; hence $\Lambda^e \otimes_\Lambda \Lambda^e$ is isomorphic to a summand of a direct sum of copies of $\Lambda^e$, i.e., it is projective. To see that the complex is acyclic, we argue as follows. Since $\Lambda \otimes_R \Lambda$ is projective as a right $\Lambda$-module (again because the left copy of $\Lambda$ is $R$-projective), we have that $\mathbf{X} \to \Lambda$ is a projective resolution of the free right $\Lambda$-module $\Lambda$. Hence the differentials in $\mathbf{X}$ are all split as right $\Lambda$-homomorphisms, and the cycles, boundaries, and homology in $\mathbf{X}$ are all projective right $\Lambda$-modules. In particular they are $\Lambda$-flat, so we may apply the Künneth Theorem [3, Thm. 2.7.1] to the right $\Lambda$-complex $\mathbf{X}$ and the left $\Lambda$-complex $\mathbf{X}$. The Tor-term vanishes since the homology of $\mathbf{X}$ is $\Lambda$-projective, and so

$$H_*(\mathbf{X} \otimes_\Lambda \mathbf{X}) \cong H_*(\mathbf{X}) \otimes_\Lambda H_*(\mathbf{X}) \cong \Lambda \otimes_\Lambda \Lambda \cong \Lambda,$$

as required.

One important case is where $N = M = \Lambda$. In this case, the cup product gives $H^*(\Lambda, \Lambda)$ the structure of an associative graded algebra with unit over $R$, called the *Hochschild cohomology ring of* $\Lambda$. Gerstenhaber [12, §7, Cor. 1] showed that in fact $H^*(\Lambda, \Lambda)$ is graded-commutative (see also [28, Prop. 1.2] for a simplified proof). Another important case is where only $N$ is replaced by $\Lambda$; this gives $H^*(\Lambda, M)$ the structure of a graded $H^*(\Lambda, \Lambda)$-module.

## 3. The group algebra case

**Reduction to ordinary cohomology.** We are interested in the case where $G$ is a finite group and $\Lambda = RG$, where as above $R$ is any commutative ring. In this case the definitions above can all be reduced to ordinary group cohomology with no mention of bimodules. For in this case, $\Lambda^e \cong R[G \times G]$, and if $M$ is any $RG$-bimodule then by the Eckmann-Shapiro Lemma [3, Cor. 2.8.4],

$$\begin{aligned} H^*(RG, M) &\cong \operatorname{Ext}^*_{R[G\times G]}(RG, M) \cong \operatorname{Ext}^*_{R[G\times G]}(R{\uparrow}^{G\times G}_{\Delta G}, M) \\ &\cong \operatorname{Ext}^*_{RG}(R, M{\downarrow}^{G\times G}_{\Delta G}) \cong H^*(G, M), \end{aligned} \tag{3.1}$$

where $\Delta G = \{(g,g) \mid g \in G\}$, and $M$ is considered a left $RG$-module through the diagonal action $g.x = gxg^{-1}$ ($g \in G$, $x \in M$). Hence the Hochschild cohomology of $G$ with coefficients in the bimodule $M$ is just the ordinary cohomology of $G$ with coefficients in $M$ under the diagonal action (cf. [21, Ch. X, Thm. 5.5], [9, §5]). In particular,

$$H^*(RG, RG) \cong H^*(G, RG), \tag{3.2}$$

where $RG$ is considered a left $RG$-module by conjugation: $g.a = gag^{-1}$ ($g \in G$, $a \in RG$).

Conversely, given any left $RG$-module $M$, we may consider it a bimodule by having $RG$ act trivially on the right; and when we restrict this to the diagonal action we end up with the original left $RG$-module $M$. So we may as well forget all about bimodules and just consider left modules $M$, keeping in mind that $M$ may be considered a bimodule in this way if the context demands.

**The cup products are the same.** Let $M$ and $N$ be $RG$-modules. Recall that there is already the (ordinary) cup product

$$H^*(G, N) \otimes_R H^*(G, M) \to H^*(G, N \otimes_R M).$$

The definition is entirely analogous to that of the Hochschild cup product; see [3, §3.2] or [10, Ch. 3] for details (but note that the restrictions placed on the coefficient ring in [10] are not needed for the proofs of the results which we will use). Furthermore, given an $RG$-homomorphism $\mu\colon N \otimes_R M \to L$, we may follow the cup product with the map $\mu^*$ from $H^*(G, N \otimes_R M)$ to $H^*(G, L)$. We call the resulting product the *cup product with respect to the pairing* $\mu$ (cf. [10, §3.1]). One important case is where $L = M = N$ is an $R$-algebra on which $G$ acts as automorphisms and $\mu$ is the structure map. In this case the product gives $H^*(G, L)$ the structure of an associative graded algebra.

We will be interested here in the case $N = RG$, where $RG$ is again considered a module *via* conjugation. In this case the structure map $\mu\colon RG \otimes_R M \to M$ is an $RG$-homomorphism. We may therefore form the cup product with respect to $\mu$:

$$H^*(G, RG) \otimes_R H^*(G, M) \to H^*(G, RG \otimes_R M) \xrightarrow{\mu^*} H^*(G, M). \tag{3.3}$$

Our claim is that under the isomorphism described above, this cup product corresponds to the Hochschild cup product. Specifically,

**Proposition 3.1.** *The following diagram commutes:*

$$\begin{tikzcd} H^*(RG, RG) \otimes_R H^*(RG, M) \arrow[r, "\smile"] \arrow[d, "\cong"] & H^*(RG, M) \arrow[d, "\cong"] \\ H^*(G, RG) \otimes_R H^*(G, M) \arrow[r, "\smile"] & H^*(G, M). \end{tikzcd}$$

*Proof.* Let $\mathbf{P} \to R$ be a projective resolution of the trivial $RG$-module. We will use this to construct an $R[G \times G]$-projective resolution $\mathbf{X}$ of $RG$ as follows. First, we may consider any $RG$-module as an $R\Delta G$-module *via* the diagonal map; we will abuse notation slightly by using the same symbol to denote either module, letting the context determine which one is meant. The same applies to complexes of $RG$-modules. With this in mind, set $\mathbf{X} = \mathbf{P}{\uparrow}_{\Delta G}^{G \times G}$. Since induction is exact and takes projectives to projectives, and $R{\uparrow}_{\Delta G}^{G \times G} \cong RG$, this yields a projective resolution of $RG$.

We can make the isomorphism (3.1) explicit as follows. There is an $R\Delta G$-chain map $\iota \colon \mathbf{P} \hookrightarrow \mathbf{X}$ defined by $\iota(x) = (1,1) \otimes x$ ($x \in \mathbf{P}$). Now given a cohomology class in $H^*(RG, M)$ represented by the cocycle $f \colon \mathbf{X} \to M$, the corresponding class in $H^*(G, M)$ is represented by the cocycle $f \circ \iota$.

Now let $D \colon \mathbf{P} \to \mathbf{P} \otimes_R \mathbf{P}$ be an $RG$-diagonal map. We may use this to construct a Hochschild diagonal map $D' \colon \mathbf{X} \to \mathbf{X} \otimes_{RG} \mathbf{X}$ as follows. There is an isomorphism of $R[G \times G]$-complexes

$$\begin{aligned} &(\mathbf{P} \otimes_R \mathbf{P}){\uparrow}_{\Delta G}^{G \times G} \xrightarrow{\theta} \mathbf{X} \otimes_{RG} \mathbf{X} \\ &\quad (g,h) \otimes x \otimes y \longmapsto \big((g,1) \otimes x\big) \otimes \big((1,h) \otimes y\big) \qquad g,h \in G,\ x,y \in \mathbf{P}. \end{aligned}$$

Set $D' = \theta \circ D{\uparrow}_{\Delta G}^{G \times G}$; this is the desired diagonal map.

Given cocycles $f \colon \mathbf{X} \to RG$ and $g \colon \mathbf{X} \to M$, it follows from these definitions that we have a commutative diagram

$$\begin{tikzcd} \mathbf{X} \arrow[r, "D'"] & \mathbf{X} \otimes_{RG} \mathbf{X} \arrow[r, "f \otimes g"] & RG \otimes_{RG} M \arrow[r, "\cong"] & M \\ \mathbf{P} \arrow[u, hook, "\iota"] \arrow[r, "D"] & \mathbf{P} \otimes_R \mathbf{P} \arrow[r, "(f\iota)\otimes(g\iota)"] & RG \otimes_R M \arrow[r, "\mu"] & M \arrow[u, equal] \end{tikzcd}$$

The bottom row represents the cup product in ordinary cohomology, the top row the product in Hochschild cohomology. ◻

An alternative proof is to apply the results of Sanada [28, Thm. 1.1], which show that there is only one product in Hochschild cohomology satisfying a handful of basic axioms, and then verify those axioms for the ordinary cup product.

The upshot is that we may just take (3.2) and (3.3) to be our definitions of Hochschild cohomology and product for a group algebra, and interpret the rest of this section as a proof that our definitions correspond to the ones usually found in the literature.

**A generalization.** We now describe a generalization of the Hochschild cohomology ring of a group algebra. If $H$ and $G$ are finite groups, with $H$ acting as automorphisms on $G$, then $RG$ becomes an $RH$-module. The multiplication map $RG \otimes_R RG \to RG$ is an $RH$-homomorphism, and so we obtain a ring $H^*(H, RG)$ *via* cup product. This is also an associative graded $R$-algebra with unit, and in the case $H = G$, with action given by conjugation, it is the Hochschild cohomology ring. In general however, it is not necessarily graded-commutative. (For example, $H^*(1, RG) = RG$.) The calculations in this paper are all concerned with the case $H = G$ and action given by conjugation, but in §10 we will indicate that $H \mapsto H^*(H, RG)$, for subgroups $H$ of $G$, is a Green functor and exploit this to obtain some structure theorems.

*Note.* If $R$ is Noetherian, then $H^*(H, RG)$ is a finitely generated $R$-algebra. For there is an algebra homomorphism (which is in fact injective) from $H^*(H)$ to $H^*(H, RG)$ induced by the map $R \to RG$. By Evens' Theorem, $H^*(H, RG)$ is finitely generated as an $H^*(H)$-module, and $H^*(H)$ is a finitely generated $R$-algebra [10, Thm. 7.4.1, Cor. 7.4.6]. So the union of the images of a set of algebra generators for $H^*(H)$ with a set of module generators for $H^*(H, RG)$ will generate $H^*(H, RG)$ as an algebra.

While in general $H^*(H, RG)$ can be a very complicated ring, there is a special case in which we can say exactly what it is:

**Proposition 3.2.** *If $H$ acts trivially on $G$, then $H^*(H, RG) \cong RG \otimes_R H^*(H)$ as graded $R$-algebras. In particular, if $G$ is Abelian then*

$$H^*(RG, RG) \cong RG \otimes_R H^*(G).$$

*Proof.* Let $\mathbf{P} \to R$ be an $RH$-projective resolution which is finitely generated over $RH$ in each degree. As an $RH$-module, $RG$ is trivial, and moreover it is free as an $R$-module. So we may apply the Universal Coefficient Theorem [6, VI,Thm. 3.3] to the complex $\mathrm{Hom}_{RH}(\mathbf{P}, R)$ and the $RH$-module $RG$ to obtain an isomorphism of graded $R$-modules

$$RG \otimes_R H^*(H) \xrightarrow{\theta} H^*(H, RG)$$

(see [10, §3.4]). This map may be defined as follows: let $a \in RG$, $\zeta \in H^*(H)$. If $\zeta$ is represented by the cocycle $f\colon \mathbf{P} \to R$ then $\theta(a \otimes \zeta)$ is represented by the cocycle $x \mapsto af(x)$ ($x \in \mathbf{P}$). It is immediate from the definition of cup product that $\theta$ is a ring homomorphism. □

The case where $G$ is Abelian in the Proposition was proved using very different methods by Cibils and Solotar [8, Thm. 2.1] (see also Holm [18] for the case where $R$ is a field). In §10 we obtain a similar but weaker result for a group $H$ acting not necessarily trivially on $G$.

## 4. Additive decomposition

We continue with the notation of the previous section: $R$ is a commutative ring, and $H$ and $G$ are finite groups with $H$ acting as automorphisms on $G$. The additive decomposition which we are about to describe is well-known (at least when $H = G$; cf. Burghelea [5] or [4, Thm. 2.11.2]), but for us it will be important to know the isomorphism explicitly. To do this we will need the following explicit version of the Eckmann-Shapiro Lemma (cf. [3, Cor. 2.8.4]):

**Lemma 4.1.** *Let $K \leq H$ and $U$ an $RK$-module. Let $\iota\colon H^*(K, U) \to H^*(K, U{\uparrow}_K^H)$ be the map induced by the $RK$-homomorphism $U \to U{\uparrow}_K^H$ given by $x \mapsto 1 \otimes x$. Let $\pi\colon H^*(K, U{\uparrow}_K^H) \to H^*(K, U)$ be the map induced by the $RK$-homomorphism $U{\uparrow}_K^H \to U$ which sends $h \otimes x$ to $hx$ (if $h \in K$) and $0$ (otherwise). Then*

$$\mathrm{cor}_K^H \circ \iota\colon H^*(K, U) \to H^*(H, U{\uparrow}_K^H)$$

*is an isomorphism and its inverse is $\pi \circ \mathrm{res}_K^H$.*

*Proof.* Compose in either order and obtain the identity. □

Let $g_1, \dots, g_r$ be representatives of the orbits of the action of $H$ on $G$, and let $H_i = \mathrm{Stab}_H(g_i)$ be the stabilizer of $g_i$. Next, fix $g \in G$. Then there are two $R(\mathrm{Stab}_H(g))$-homomorphisms

$$\begin{array}{rcl} \theta_g\colon R & \longrightarrow & RG \\ \lambda & \longmapsto & \lambda g \end{array} \qquad\qquad \begin{array}{rcl} \pi_g\colon RG & \longrightarrow & R \\ \sum_{a\in G}\lambda_a a & \longmapsto & \lambda_g. \end{array}$$

If $W$ is any subgroup of $\mathrm{Stab}_H(g)$ then these maps induce maps on cohomology

$$\theta_g^*\colon H^*(W) \to H^*(W, RG), \qquad \pi_g^*\colon H^*(W, RG) \to H^*(W).$$

Now define maps $\gamma_i\colon H^*(H_i) \to H^*(H, RG)$ by

$$\gamma_i(\alpha) = \mathrm{cor}_{H_i}^H \theta_{g_i}^*(\alpha), \qquad \alpha \in H^*(H_i)$$

**Lemma 4.2.** *The map*

$$\begin{aligned} H^*(H, RG) &\longrightarrow \bigoplus_i H^*(H_i) \\ \zeta &\longmapsto \big(\pi_{g_i}^* \operatorname{res}_{H_i}^H(\zeta)\big)_i \end{aligned}$$

*is an isomorphism of graded $R$-modules, and its inverse sends $\alpha \in H^*(H_i)$ to $\gamma_i(\alpha)$.*

*Proof.* (cf. [27, Exer. 6.1.45].) We have $RG = \bigoplus_i M_i$, where $M_i$ is the free $R$-module generated by the elements of the orbit containing $g_i$. Hence $H^*(H, RG) \cong \bigoplus_i H^*(H, M_i)$. Now there is an isomorphism $M_i \cong R{\uparrow}_{H_i}^H = RH \otimes_{RH_i} R$ given by $\lambda^h g_i \leftrightarrow h \otimes \lambda$, and this induces an isomorphism in cohomology. Composing that with the isomorphism of Lemma 4.1 (with $K = H_i$), one obtains the desired isomorphism. □

## 5. The Product Formula

We continue with the notation above; in particular $H$ is a finite group acting on a finite group $G$. We seek a way to describe the cup product in $H^*(H, RG)$ in terms of the direct sum decomposition described in Lemma 4.2. This is accomplished as follows. Fix $i, j \in \{1, \dots, r\}$. Let $D$ be a set of double coset representatives for $H_i \backslash H / H_j$. For each $x \in D$, there is a unique $k = k(x)$ such that

$$g_k = {}^y g_i \, {}^{yx} g_j \tag{5.1}$$

for some $y \in H$. Moreover, $k$ is independent of the choice of representative $x$ of the double coset $H_i x H_j$. The set of all $y$ satisfying (5.1) is also a double coset. To see this, let us fix one such $y = y(x)$ for which (5.1) holds. Then

$$\{y' \in H \mid g_k = {}^{y'}(g_i {}^x g_j)\} = H_k y = H_k y({}^x H_j \cap H_i) \in H_k \backslash H / {}^x H_j \cap H_i,$$

since by (5.1), ${}^{yx}H_j \cap {}^y H_i \subseteq H_k$. We may now state our main result:

**Theorem 5.1** (Product Formula)**.** *Let $\alpha \in H^*(H_i)$, $\beta \in H^*(H_j)$. Then*

$$\gamma_i(\alpha) \smile \gamma_j(\beta) = \sum_{x \in D} \gamma_k\big(\mathrm{cor}_W^{H_k}(\mathrm{res}_W^{{}^yH_i} y^*\alpha \smile \mathrm{res}_W^{{}^{yx}H_j}(yx)^*\beta)\big)$$

*where $D$ is a set of double coset representatives for $H_i \backslash H / H_j$, $k = k(x)$ and $y = y(x)$ are chosen to satisfy* (5.1)*, and $W = W(x) = {}^{yx}H_j \cap {}^y H_i$.*

The statement requires a bit of explanation. First, recall that given an $RH$-module $U$, a subgroup $K \leq H$, and $h \in H$, there is a map

$$h^*\colon H^*(K,U) \to H^*({}^hK, U),$$

which can be defined on the cochain level by fixing an $RH$-projective resolution $\mathbf{P} \to R$ and setting

$$(h^*f)(v) = hf(h^{-1}v), \quad f \in \mathrm{Hom}_{RK}(\mathbf{P}, U),\ v \in \mathbf{P}$$

(see [10, §4.1]). Now it follows from the theorem, and the fact that the cup product is well defined, that the sum in the statement of the theorem is independent of the choices for $x$ and $y$. However, this can also be seen directly. For $y$ is unique up to left multiplication by an element $z$ of $H_k$. Since $z^*$ respects the cup product and commutes with restriction and corestriction, and since $H_k$ acts trivially on its own cohomology, any term of the sum is unchanged by replacing $y$ with $zy$. If $x$ is multiplied on the right by an element of $H_j$ the term is unchanged for similar reasons. If $x$ is replaced by $wx$, where $w \in H_i$, then one must replace $y$ with $yw^{-1}$ so that (5.1) holds, and again the term is unchanged.

We now turn to the proof.

**Lemma 5.2.** *Let $h \in H$ and $a, b \in G$.*

(i) *Suppose $W \leq \mathrm{Stab}_H(a)$. Then $h^*\theta_a^* = \theta_{{}^ha}^* h^*$ as maps from $H^*(W)$ to $H^*({}^hW, RG)$.*
(ii) *Suppose $W \leq \mathrm{Stab}_H(a) \cap \mathrm{Stab}_H(b)$, and $\alpha, \beta \in H^*(W)$. Then*

$$\theta_a^*(\alpha) \smile \theta_b^*(\beta) = \theta_{ab}^*(\alpha \smile \beta).$$

(iii) *Suppose $W' \leq W \leq \mathrm{Stab}_H(a)$. Then $\theta_a^*$ and $\pi_a^*$ commute with $\mathrm{res}^W_{W'}$ and $\mathrm{cor}^W_{W'}$.*
(iv) *Suppose $W \leq \mathrm{Stab}_H(a) \cap \mathrm{Stab}_H(b)$. Then $\pi_a^*\theta_b^* = \delta_{a,b}\,\mathrm{id}$.*

*Proof.* (i) Let $f \in \mathrm{Hom}_W(\mathbf{P}, R)$. Then $h^*(\theta_a f)(x) = f(h^{-1}x)^h a = \theta_{{}^ha}(h^*(f))(x)$. (ii) Let $\mu\colon RG \otimes_R RG \to RG$ be the multiplication map. Let $D\colon \mathbf{P} \to \mathbf{P} \otimes_R \mathbf{P}$ be a diagonal approximation map. For $f, g \in \mathrm{Hom}_W(\mathbf{P}, R)$,

$$\mu \circ ((\theta_a \circ f) \otimes (\theta_b \circ g)) \circ D = \mu \circ (\theta_a \otimes \theta_b) \circ (f \otimes g) \circ D = \theta_{ab} \circ (f \otimes g) \circ D.$$

(iii) We will show $\mathrm{cor}^W_{W'}$ commutes with $\pi_a^*$; the other three cases are similar. For $f \in \mathrm{Hom}_{W'}(\mathbf{P}, RG)$,

$$\pi_a\left(\sum_{w \in W/W'} {}^w f(w^{-1}x)\right) = \sum_w \pi_{{}^{w^{-1}}a}\big(f(w^{-1}x)\big) = \sum_w (\pi_a \circ f)(w^{-1}x),$$

as ${}^{w^{-1}}a = a$. If $f$ is a cocycle representing $\alpha$ then the left side above represents $\pi_a^*(\mathrm{cor}^W_{W'}(\alpha))$ and the right side represents $\mathrm{cor}^W_{W'}(\pi_a^*(\alpha))$. (iv) Clearly $\pi_a\theta_b = \delta_{a,b}\,\mathrm{id}$; apply the cohomology functor. ∎

*Proof of Theorem 5.1.* By Lemma 4.2,

$$\begin{aligned}
\gamma_i(\alpha) \smile \gamma_j(\beta) &= \operatorname{cor}_{H_i}^{H}(\theta_{g_i}^{*}\alpha) \smile \operatorname{cor}_{H_j}^{H}(\theta_{g_j}^{*}\beta)\\
&= \operatorname{cor}_{H_i}^{H}(\theta_{g_i}^{*}\alpha \smile \operatorname{res}_{H_i}^{H} \operatorname{cor}_{H_j}^{H} \theta_{g_j}^{*}\beta)\\
&= \sum_{x\in D} \operatorname{cor}_{H_i}^{H}(\theta_{g_i}^{*}\alpha \smile \operatorname{cor}_{{}^xH_j\cap H_i}^{H_i} \operatorname{res}_{{}^xH_j\cap H_i}^{{}^xH_j} x^*\theta_{g_j}^{*}\beta)\\
&= \sum_{x} \operatorname{cor}_{{}^xH_j\cap H_i}^{H}(\operatorname{res}_{{}^xH_j\cap H_i}^{H_i} \theta_{g_i}^{*}\alpha \smile \operatorname{res}_{{}^xH_j\cap H_i}^{{}^xH_j} x^*\theta_{g_j}^{*}\beta)\\
&= \sum_{x} \operatorname{cor}_{{}^xH_j\cap H_i}^{H} \theta_{g_i{}^xg_j}^{*}(\operatorname{res}_{{}^xH_j\cap H_i}^{H_i} \alpha \smile \operatorname{res}_{{}^xH_j\cap H_i}^{{}^xH_j} x^*\beta),
\end{aligned}$$

by Lemma 5.2 (i)–(iii), as well as the double coset formula [10, Thm. 4.2.6], [10, Prop. 4.2.4], and [10, Prop. 4.2.1]. So again by Lemma 4.2,

$$\begin{aligned}
&\gamma_i(\alpha) \smile \gamma_j(\beta)\\
&\quad= \sum_k \sum_x \gamma_k\big(\pi_{g_k}^{*} \operatorname{res}_{H_k}^{H} \operatorname{cor}_{{}^xH_j\cap H_i}^{H} \theta_{g_i{}^xg_j}^{*}(\operatorname{res}_{{}^xH_j\cap H_i}^{H_i} \alpha \smile \operatorname{res}_{{}^xH_j\cap H_i}^{{}^xH_j} x^*\beta)\big)\\
&\quad= \sum_k \sum_{x,y} \gamma_k\big(\pi_{g_k}^{*} \operatorname{cor}_{W'}^{H_k} \operatorname{res}_{W'}^{{}^{yx}H_j\cap {}^yH_i} y^*\theta_{g_i{}^xg_j}^{*}(\operatorname{res}_{{}^xH_j\cap H_i}^{H_i} \alpha \smile \operatorname{res}_{{}^xH_j\cap H_i}^{{}^xH_j} x^*\beta)\big)\\
&\quad= \sum_k \sum_{x,y} \gamma_k\Big(\operatorname{cor}_{W'}^{H_k} \pi_{g_k}^{*} \theta_{{}^yg_i{}^{yx}g_j}^{*}\big(\operatorname{res}_{W'}^{{}^yH_i} y^*\alpha \smile \operatorname{res}_{W'}^{{}^{yx}H_j} (yx)^*\beta\big)\Big),
\end{aligned}$$

where $y$ runs over a set of representatives for $H_k\backslash H/{}^xH_j \cap H_i$, and

$$W' = W'(k,x,y) = {}^{yx}H_j \cap {}^yH_i \cap H_k.$$

However, Lemma 5.2 (iv) implies the only terms that can be non-zero are those for which $g_k = {}^yg_i{}^{yx}g_j$. But we have seen that each $x$ determines a unique $k$ and double coset $H_ky({}^xH_j \cap H_i)$ for which this holds. So we may take $y = y(x)$, and then ${}^{yx}H_j \cap {}^yH_i \subseteq H_k$, so $W' = W$. □

Notice that $\gamma_1$ is an algebra monomorphism. This is because it is the map in cohomology induced by the algebra homomorphism $R \to RG$ sending $\lambda$ to $\lambda 1$. (Alternatively, one may see this as the special case of Theorem 5.1 where $i = j = 1$.) So we may view $H^*(H, RG)$ as an $H^*(H)$-module. Now, each $H^*(H_i)$ may also be considered as an $H^*(H)$-module *via* restriction. As an immediate corollary of Theorem 5.1, we obtain

**Corollary 5.3.** *The additive isomorphism of Lemma 4.2*

$$H^*(H, RG) \xrightarrow{\approx} \bigoplus_i H^*(H_i)$$

*is an isomorphism of graded $H^*(H)$-modules.*

*Proof.* For $i = 1$, the Theorem reduces to $\gamma_1(\alpha) \smile \gamma_j(\beta) = \gamma_j\big(\operatorname{res}_{H_j}^{H}(\alpha) \smile \beta\big)$. □

*Note.* In the case where $H = G$ and the action is conjugation, Theorem 5.1 verifies a formula conjectured by Cibils [7] and Cibils and Solotar [8]. Our sum over $k$ corresponds to their sum over $C \in \mathcal{C}$. Our sum over $x$ corresponds to their sum over $E_{A,B}^{C}/Z_c$. Their pair $(K, L)$ is our $(y^{-1}, x^{-1}y^{-1})$.

## 6. Hochschild cohomology with dual coefficients

There is an isomorphism of $RH$-bimodules $RG \to RG^*$, sending $g$ to $\delta_{g^{-1}}$ ($g \in G$). It follows from functorality that $H^*(RH, RG) \cong H^*(RH, RG^*)$ as graded $R$-modules. Arguing as above, we have $H^*(RH, RG^*) \cong H^*(H, RG^*)$, where $RG^*$ is now considered a left $RH$-module as follows: $(h \cdot \theta)(x) = \theta(h^{-1} \cdot x)$ ($h \in H$, $\theta \in RG^*$, $x \in RG$).

There is a natural product on $H^*(H, RG^*)$ which we now describe. Since $RG$ is a bialgebra, $RG^*$ is an $R$-algebra with multiplication

$$\begin{aligned} RG^* \otimes_R RG^* &\xrightarrow{\nu} RG^* \\ \theta \otimes \phi &\mapsto \theta\phi \end{aligned}$$

where $(\theta\phi)(g) = \theta(g)\phi(g)$ ($g \in G$). One can check that $\nu$ is an $RH$-homomorphism, and therefore there is a cup product on $H^*(H, RG^*)$ with respect to the pairing $\nu$.

Using the familiar identification of $H^0(H, M)$ with the invariants $M^H$, we see that $H^0(H, RG^*)$ may be identified with the algebra of functions from $G$ to $R$ constant on each $H$-orbit of $G$, under pointwise multiplication; this is isomorphic as an algebra to the direct product of $r$ copies of $R$, where $r$ is the number of $H$-orbits on $G$. However, $H^0(H, RG) \cong (RG)^H$. As an $R$-module this is also isomorphic to $R^r$, but in general it has a completely different ring structure. For example, if $R = \mathbb{F}$ is a field of characteristic dividing $|G|$ then $(RG)^G = Z(\mathbb{F}G)$ has nilpotent elements (such as $\sum_{g \in G} g$).

We will show next that the additive decomposition of Lemma 4.2 (using $RG^* \cong RG$) is also multiplicative, by contrast to the case for $H^*(G, RG)$.

**Proposition 6.1.** *There is an isomorphism of graded $R$-algebras $H^*(H, RG^*) \cong \bigoplus_i H^*(H_i)$. In particular, $H^*(G, RG^*)$ and $H^*(LBG; R)$ are isomorphic as algebras, where $LBG$ is the space of free loops on the classifying space $BG$ of $G$.*

*Proof.* There is an ideal direct sum decomposition $RG^* = \bigoplus_i M_i$ where $M_i$ is the free $R$-module generated by the dual functions $\delta_g$ for all $g$ in the $H$-orbit of $g_i$. Therefore $H^*(H, RG^*) \cong \bigoplus_i H^*(H, M_i)$ as algebras, where $H^*(H, M_i)$ is endowed with the cup product with respect to the multiplication map $M_i \otimes M_i \to M_i$. We claim that $H^*(H, M_i) \cong H^*(H_i)$ as algebras. Indeed, we may identify $M_i$ with $R{\uparrow}_{H_i}^H$ *via* the $RH$-isomorphism $\delta_{^h g_i} \leftrightarrow h \otimes 1$. Lemma 4.1 says that there is an additive isomorphism $\pi \circ \mathrm{res}^H_{H_i} \colon H^*(H, M_i) \to H^*(H_i)$. But restriction is multiplicative, and $\pi$ is induced by the projection $M_i \to R\delta_{g_i} \cong R$, which is a ring homomorphism (and an $RH_i$-homomorphism). Hence $\pi$ is multiplicative as well, and our claim is proved. Finally, $H^*(LBG; R) \cong \bigoplus_i H^*(H_i)$ as algebras because $LBG$ is homotopy equivalent to the disjoint union of the $BH_i$ ([4, §2.12]). $\square$

In contrast, $H^*(G, \mathbb{F}G)$ cannot in general be the cohomology ring of any topological space $X$. For $H^0(X; \mathbb{F})$ is isomorphic, as an algebra, to a product of copies of $\mathbb{F}$, one for each connected component of $X$. But as we have seen, $H^0(G, \mathbb{F}G)$ may have nilpotent elements.

## 7. The symmetric group of degree 3

**Generators and relations.** Before beginning the calculations it is helpful to make some general remarks about generators and relations. Let $\mathbb{F}$ be a field and $A$ a f.g. graded-commutative $\mathbb{F}$-algebra which is finite dimensional in each degree. Suppose

$A = B \oplus U$, where $B$ is a graded subalgebra of $A$ and $U$ is a graded $B$-submodule. Such a situation arises, for example, if $A = H^*(G, \mathbb{F}G)$, $B = \gamma_1\big(H^*(G,\mathbb{F})\big)$, and $U = \sum_{i\geq 2} \gamma_i\big(H^*(H_i,\mathbb{F})\big)$.

Suppose $B$ is generated, as an algebra, by $x_1, \dots, x_N$, subject to the relations $r_i = 0$ $(i \in I)$, in addition to the graded-commutative relations. Suppose in addition that $U$ is generated, as a $B$-module, by $y_1, \dots, y_M$, subject to the relations $s_j = 0$ $(j \in J)$. (Hence $U$ is isomorphic to the graded-free $B$-module on the $y_i$ modulo the $B$-submodule generated by the $s_j$). It is assumed that all of these generators and relations are homogeneous. By hypothesis, for each $1 \leq l \leq m \leq M$, we may choose $f_{lm}^n \in B$ $(0 \leq n \leq M)$ such that

$$y_l y_m = f_{lm}^0 + \sum_{n=1}^{M} f_{lm}^n y_n. \tag{7.1}$$

Now it is clear that $A$ is generated as an $\mathbb{F}$-algebra by $x_1, \dots, x_N, y_1, \dots, y_M$. We claim one obtains a presentation for $A$ by taking those generators, together with the *relations of type 1* $r_i = 0$, the *relations of type 2* $s_j = 0$, and the *relations of type 3* given by equation (7.1), in addition to the graded-commutative relations. Indeed, let $A'$ be the algebra presented abstractly with these generators and relations, and let $B'$ be the subalgebra of $A'$ generated by the $x_i$. Clearly there is a homomorphism $\theta$ from $A'$ onto $A$, and the relations of type 1 guarantee $\theta$ maps $B'$ isomorphically onto $B$. The relations of type 3 show that every element of $A'$ can be expressed as the sum of an element of $B'$ and a $B'$-linear combination of the $y_i$. Together with the relations of type 2, this implies that the dimension, in each degree, of $A'$ is less than or equal to that of $A$, so $\theta$ is an isomorphism, and we have established the claim.

**Hochschild cohomology of $\mathbb{F}_3 S_3$.** Let $G = S_3 = \langle a, b \mid a^3 = 1 = b^2, bab = a^{-1}\rangle$. In this section we will show how to find the Hochschild cohomology ring of $\mathbb{F}_3 G$ using the Product Formula. To do this, choose conjugacy class representatives $g_1 = 1$, $g_2 = a$, and $g_3 = b$. The centralizers of these elements are, resp., $H_1 = G$, $H_2 = \langle a\rangle$, and $H_3 = \langle b\rangle$. From now on we will assume all coefficients are in $\mathbb{F}_3$ unless stated otherwise. Now $H^*(G)$ is generated by elements $u$ and $v$, of degrees (resp.) 3 and 4, subject only to the graded-commutative relations. Similarly, $H^*(\langle a\rangle)$ is generated by elements $w_1$, $w_2$, of degrees (resp.) 1, 2, subject only to the graded-commutative relations. Of course $H^*(\langle b\rangle)$ is just $\mathbb{F}_3$, concentrated in degree 0.

Now define the following elements in the Hochschild cohomology ring. Since $\gamma_1$ is an algebra monomorphism, by a slight abuse of notation we may identify any element of $H^*(G)$ with its image under $\gamma_1$. Let $E_i = \gamma_i(1)$ $(1 \leq i \leq 3)$, let $C_1 = 1 + E_2$, $C_2 = E_3$, and $X_j = \gamma_2(w_j)$ $(j = 1, 2)$.

**Theorem 7.1.** *$H^*(S_3, \mathbb{F}_3 S_3)$ is generated as an algebra by elements $u$, $v$, $C_1$, $C_2$, $X_1$, and $X_2$, of degrees (resp.) 3, 4, 0, 0, 1, and 2, subject to the relations*

$$uX_1 = 0, \quad vX_1 = uX_2, \quad uC_2 = 0 = vC_2$$
$$C_iX_j = 0 = C_iC_j \ (i,j \in \{1,2\}), \quad X_1X_2 = uC_1, \quad X_2^2 = vC_1$$

*in addition to the graded-commutative relations. In particular, the algebra monomorphism $\gamma_1 \colon H^*(S_3) \to H^*(S_3, \mathbb{F}_3 S_3)$ induces an isomorphism modulo radicals.*

*Proof.* Relations in degree 0 are most easily handled by the identification of degree-0 Hochschild cohomology with the center of the group algebra. Under this identification, $E_i$ corresponds to the sum of the group elements conjugate to $g_i$. So one obtains, for example,

$$E_2^2 = (a + a^{-1})^2 = a + a^{-1} - 1 = E_2 - 1,$$

which implies $C_1^2 = 0$. The other degree-0 relations are handled in a similar way.

Restriction from $G$ to $\langle a \rangle$ is injective and takes $u$ to $w_1 w_2$ and $v$ to $w_2^2$. It follows that $H^*(\langle a \rangle)$ is generated as an $H^*(G)$-module by 1, $w_1$, and $w_2$, subject to the relations $u.w_1 = 0$ and $v.w_1 = u.w_2$. Of course $H^*(\langle b \rangle)$ is generated as an $H^*(G)$-module by 1, subject to the relations $u.1 = 0 = v.1$. This implies $H^*(G, \mathbb{F}_3 G)$ is generated by $u$, $v$, $E_2$, $E_3$, $X_1$, $X_2$; we have replaced $E_2$ with $1 + E_2$ because it simplifies the relations somewhat. It also implies the first line of relations hold, which are the "relations of type 2."

We now turn to the "relations of type 3." According to the Product Formula,

$$\gamma_2(\alpha)\gamma_2(\beta) = \gamma_2\big(b^*(\alpha\beta)\big) + \gamma_1\big(\mathrm{cor}^G_{\langle a \rangle}\big(\alpha b^*(\beta)\big)\big).$$

for $\alpha, \beta \in H^*(\langle a \rangle)$. It is easily checked that $b^*(w_i) = -w_i$, and

$$\mathrm{cor}^G_{\langle a \rangle}(w_1 w_2^n) = \begin{cases} -uv^{(n-1)/2} & n \text{ odd} \\ 0 & n \text{ even} \end{cases} \qquad \mathrm{cor}^G_{\langle a \rangle}(w_2^n) = \begin{cases} 0 & n \text{ odd} \\ -v^{n/2} & n \text{ even.} \end{cases}$$

Letting $\alpha = 1$ and $\beta = w_1$, this yields $E_2 X_1 = -X_1$, i.e., $C_1 X_1 = 0$. Letting $\alpha = \beta = w_2$, one obtains

$$X_2^2 = v + \gamma_2(w_2^2) = v + vE_2 = vC_1.$$

The relations involving $X_1 X_2$ and $C_1 X_2$ are handled similarly. Finally, the Product Formula implies that the product of an element in the image of $\gamma_2$ with one in the image of $\gamma_3$ lies in the image of $\gamma_3$. Since this is 0 in positive degrees, we obtain the remaining relations.

It follows from our comments above that we have found a complete set of generators and relations. The remark on isomorphism modulo radicals follows from the observation that $X_2^3 = vC_1 X_2 = 0$, so $C_1$, $C_2$, $X_1$, and $X_2$ all lie in the radical of the Hochschild cohomology ring. □

**Hochschild cohomology of $\mathbb{F}_2 S_3$.** The mod-2 Hochschild cohomology of $S_3$ can be found by more elementary measures. For $\mathbb{F}_2 S_3$ has two blocks: the principal block, which is isomorphic to $\mathbb{F}_2(\mathbb{Z}/2)$, and a block which is isomorphic to the algebra of 2 by 2 matrices over $\mathbb{F}_2$. The Hochschild cohomology of the first is determined by Prop. 3.2, the second is a simple algebra and so has Hochschild cohomology isomorphic to $\mathbb{F}_2$ concentrated in degree 0. Hence

$$\begin{aligned} H^*(S_3, \mathbb{F}_2 S_3) &\cong \mathbb{F}_2(\mathbb{Z}/2) \otimes_{\mathbb{F}_2} H^*(\mathbb{Z}/2, \mathbb{F}_2) \oplus \mathbb{F}_2 \\ &\cong \mathbb{F}_2[u, v, w \mid v^2 = uw = vw = 0,\ w^2 = w], \end{aligned}$$

where $u$ and $v$ have degree 1 and $w$ has degree 0.

## 8. The Alternating group $A_4$

Let $G = A_4 = \langle a, b, c \mid a^2 = b^2 = c^3 = 1, ab = ba, c^{-1}ac = b, c^{-1}bc = ab \rangle$.

**Proposition 8.1.** *$H^*(A_4, \mathbb{F}_2 A_4)$ is generated as a commutative algebra by elements $u$, $v$, $w$, $C_1$, $C_2$, $C_3$, $X_1$, $Y_1$, $X_2$, and $Y_2$, of degrees (resp.) 2, 3, 3, 0, 0, 0, 1, 1, 2, and 2, subject to the relation of type 1*

$$u^3 + v^2 + vw + w^2 = 0,$$

*the 10 relations of type 2*

$$uX_2 + vX_1 + vY_1 + wY_1 = 0 \qquad uY_2 + vX_1 + wX_1 + wY_1 = 0$$
$$u^2Y_1 + vX_2 + wX_2 + vY_2 = 0 \qquad u^2X_1 + u^2Y_1 + vX_2 + wY_2 = 0$$
$$uC_2 = vC_2 = wC_2 = uC_3 = vC_3 = wC_3 = 0$$

*and the 28 relations of type 3*

$$X_1^2 = 0,\ X_1Y_1 = uC_1,\ X_1X_2 = (v+w)C_1,\ X_1Y_2 = wC_1,\ Y_1^2 = 0,$$
$$Y_1X_2 = vC_1,\ Y_1Y_2 = (v+w)C_1,\ X_2^2 = 0,\ X_2Y_2 = u^2C_1,\ Y_2^2 = 0,$$
$$X_1C_i = Y_1C_i = X_2C_i = Y_2C_i = 0 \quad (1 \le i \le 3),$$
$$C_iC_j = 0 \quad (1 \le i \le j \le 3).$$

*In particular, the monomorphism $H^*(A_4, \mathbb{F}_2) \overset{\gamma_1}{\hookrightarrow} H^*(A_4, \mathbb{F}_2 A_4)$ induces an isomorphism modulo radicals.*

The remainder of this section is devoted to the proof of the proposition. To begin, the conjugacy classes of $G$ with chosen representatives are

$$\{g_1 = 1\}, \{g_2 = a, b, ab\}, \{g_3 = c, ac, bc, abc\}, \{g_4 = c^2, ac^2, bc^2, abc^2\}.$$

The centralizers are $H_1 = G$, $H_2 = \langle a, b\rangle \cong \mathbb{Z}/2 \times \mathbb{Z}/2$, $H_3 = H_4 = \langle c\rangle \cong \mathbb{Z}/3$.

The cohomology rings are as follows (coefficients are assumed to be in $\mathbb{F}_2$ unless stated otherwise):

$$H^*(G) = \mathbb{F}_2[u, v, w \mid \deg(u) = 2, \deg(v) = \deg(w) = 3, u^3 + v^2 + vw + w^2 = 0],$$

(see [2, Appendix]), $H^*(H_2) = \mathbb{F}_2[x, y \mid \deg(x) = \deg(y) = 1]$, and $H^*(H_3) = H^*(H_4) = \mathbb{F}_2$. We may take $x, y$ to be the dual basis to $a, b$; restriction from $G$ to $H_2$ is injective, and is defined by

$$u \mapsto x^2 + xy + y^2,\ v \mapsto x^3 + x^2y + y^3,\ w \mapsto x^3 + xy^2 + y^3.$$

$G$ acts on $H^*(H_2)$ as follows: $a$ and $b$ act trivially, while $c$ sends $x$ to $y$ and $y$ to $x + y$, and the image of restriction from $G$ is the invariant subring. If we identify $H^*(G)$ with this subring, then corestriction from $H_2$ to $G$ is just the trace map $\sum_{i=0}^{2} (c^i)^*$.

As above, we will identify $\alpha$ with $\gamma_1(\alpha)$ ($\alpha \in H^*(G)$), and let $E_i = \gamma_i(1)$ ($1 \le i \le 4$). Let $C_1 = E_2 + 1$, $C_2 = E_3$, $C_3 = E_4$, $X_1 = \gamma_2(x)$, $Y_1 = \gamma_2(y)$, $X_2 = \gamma_2(x^2)$, and $Y_2 = \gamma_2(y^2)$.

Let $A = H^*(H_2)$, $B = H^*(G)$; we may consider $B$ as a subalgebra of $A$ *via* restriction. We first show that $A$ is generated as a $B$-module by $1, x, y, x^2, y^2$. This is clearly the case for degrees 0 and 1. In degree 2, we have $xy = u1 + x^2 + y^2$. In degree 3,

$$x^3 = ux + (v+w)1, \qquad x^2y = v1 + ux + uy,$$
$$xy^2 = w1 + ux + uy, \qquad y^3 = (v+w)1 + uy. \tag{8.1}$$

Now suppose $\alpha$ is a monomial in $x$ and $y$ of degree $n$ ($n \ge 4$). Let $\alpha = \beta\gamma$, with $\beta$ of degree 3. By induction on $n$, $\gamma$ may be written as a $B$-linear combination of

$1, x, y, x^2, y^2$, which have degree at most 2. By (8.1), $\beta$ is a $B$-linear combination of $1, x, y$, which have degree at most 1. Hence $\alpha$ is a $B$-linear combination of monomials of degree at most 3. Again using (8.1), this means $\alpha$ is a $B$-linear combination of $1, x, y$. Hence $A$ is generated by $1, x, y, x^2, y^2$, as claimed.

We next show that the first four relations of type 2 define $A$ as a $B$-module. It is straightforward to check that these relations hold. Hence there is a complex of graded $B$-modules

$$P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\delta_0} A \to 0$$

in which $P_0 = B \oplus B[1]^{\oplus 2} \oplus B[2]^{\oplus 2}$, $P_1 = B[4]^{\oplus 2} \oplus B[5]^{\oplus 2}$ ($B[n]$ is $B$ as an $\mathbb{F}_2$-module but with the grading shifted so that $B[n]_i = B_{i-n}$), and the differentials are defined by

$$\begin{aligned} \delta_0(\lambda_1, \dots, \lambda_5) &= \lambda_1 1 + \lambda_2 x + \lambda_3 y + \lambda_4 x^2 + \lambda_5 y^2 & \lambda_1, \dots, \lambda_5 \in B \\ \delta_1(\alpha_1, \dots, \alpha_4) &= \alpha_1 r_1 + \dots + \alpha_4 r_4 & \alpha_1, \dots, \alpha_4 \in B, \end{aligned}$$

where

$$\begin{aligned} r_1 &= (0, v, v+w, u, 0) & r_2 &= (0, v+w, w, 0, u) \\ r_3 &= (0, 0, u^2, v+w, v) & r_4 &= (0, u^2, u^2, v, w). \end{aligned}$$

To show that the relations suffice is equivalent to showing that this complex is exact in degree 0.

We show the complex is exact by extending it slightly to a complex

$$0 \to P_2 \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\delta_0} A \to 0, \tag{8.2}$$

and showing that this bigger complex is exact. Here, $P_2 = S[7]^{\oplus 2} \oplus S[8]^{\oplus 2} \oplus S[9]^{\oplus 2}$, where $S = \mathbb{F}_2[v, w]$. Note that $B = S \oplus Su \oplus Su^2$. In particular, $B$ is a graded-free $S$-module, so what we are showing is that (8.2) is actually a graded-free resolution of $A$ as an $S$-module. We may define $\delta_2$ by

$$\begin{aligned} \delta_2(s_1, \dots, s_6) = \big((v+w)s_1 + vs_2 &+ ((v+w)s_5 + vs_6)u + s_3 u^2, \\ vs_1 + ws_2 &+ (vs_5 + ws_6)u + s_4 u^2, \\ ws_3 + vs_4 &+ s_1 u + s_5 u^2, \\ vs_3 + (v+w)s_4 &+ s_2 u + s_6 u^2\big), \end{aligned} \tag{8.3}$$

where $s_1, \dots, s_6 \in S$.

It is tedious but straightforward to check that $\delta_1 \delta_2 = 0$, so this is actually a complex. Also, it follows easily from (8.3) that $\delta_2$ is injective. We will now show that $\mathrm{Ker}(\delta_1) = \mathrm{Im}(\delta_2)$. Suppose $\sum_{i=1}^4 \alpha_i r_i = 0$. Let $\alpha_i = \mu_i + \nu_i u + \xi_i u^2$ ($\mu_i, \nu_i, \xi_i \in S$). Setting each coordinate of $\sum_i \alpha_i r_i$ equal to 0 and using the relation of type 1, we have

$$\begin{aligned} 0 &= \mu_1 + (v+w)\nu_3 + v\nu_4 & 0 &= \nu_1 + (v+w)\xi_3 + v\xi_4 \\ 0 &= \mu_2 + v\nu_3 + w\nu_4 & 0 &= v\xi_1 + (v+w)\xi_2 + \mu_4 \\ 0 &= \nu_2 + v\xi_3 + w\xi_4 & 0 &= w\xi_1 + v\xi_2 + \mu_3. \end{aligned}$$

Hence $(\alpha_1, \dots, \alpha_4) = \delta_2(\nu_3, \nu_4, \xi_1, \dots, \xi_4)$.

We now know that the complex (8.2) is exact everywhere except possibly in degree 0. However, if $p_i(t)$ is the Poincaré series of $P_i$, then $p_2(t) - p_1(t) + p_0(t)$ is

$$\frac{2(t^7+t^8+t^9)}{(1-t^3)^2} - \frac{2(t^4+t^5)(1-t+t^2)}{(1-t)(1-t^3)} + \frac{(1+2t+2t^2)(1-t+t^2)}{(1-t)(1-t^3)} = \frac{1}{(1-t)^2},$$

which is the Poincaré series for $A$. Hence the complex is exact everywhere.

The remaining relations of type 2 follow trivially from degree considerations. To get the relations of type 3 we use the Product Formula, which yields

$$\gamma_2(\alpha)\gamma_2(\beta) = \gamma_1\big(\mathrm{cor}_{H_2}^G(\alpha\beta)\big) + \gamma_2\big(c^*(\alpha)(c^2)^*(\beta) + (c^2)^*(\alpha)c^*(\beta)\big). \tag{8.4}$$

This yields the products of any pair of generators from $\gamma_2(H^*(H_2))$; for example

$$\begin{aligned} X_2Y_2 &= \gamma_1\big(\mathrm{cor}_{H_2}^G(x^2y^2)\big) + \gamma_2\big(c^*(x^2)(c^*)^2(y^2) + (c^*)^2(x^2)c^*(y^2)\big) \\ &= \gamma_1(u^2) + \gamma_2(x^4 + x^2y^2 + y^4) = u^2C_1. \end{aligned}$$

Now for $i \in \{1,2\}$, $j \in \{3,4\}$, the Product Formula tells us that the product of any element of $\gamma_i\big(H^*(H_i)\big)$ with one of $\gamma_j\big(H^*(H_j)\big)$ will lie in $\gamma_j\big(H^*(H_j)\big)$, and will hence be 0 unless possibly both elements have degree 0. Finally, the relations $C_iC_j = 0$ are easily verified using the identification of 0-degree Hochschild cohomology with the center of $\mathbb{F}_2G$.

It follows from our comments on generators and relations that we have given a presentation of $H^*(A_4, \mathbb{F}_2A_4)$ as an $\mathbb{F}_2$-algebra. It remains only to prove the remark about the isomorphism modulo radicals. This is immediate since the $X_i$, $Y_i$ $(i = 1,2)$, and $C_j$ $(j = 1,2,3)$ all square to 0.

## 9. The dihedral 2-groups

Let $m \geq 2$ be a power of 2 and $G = D_{4m} = \langle a, b \mid a^{2m} = b^2 = 1, bab = a^{-1}\rangle$.

**Proposition 9.1.** *$H^*(D_{4m}, \mathbb{F}_2D_{4m})$ is generated as a commutative algebra by elements $u$, $v$, $w$, $C_1$, $C_2$, $C_3$, $C_4$, $T_1$, $T_2$, of degrees (resp.) 1, 1, 2, 0, 0, 0, 0, 1, 1, subject to the 10 degree-0 relations*

$$C_1^2 = XC_2 = C_1C_3 = C_1C_4 = C_2^m = C_2C_3 = C_2C_4 = C_3^2 = C_3C_4 = C_4^2 = 0,$$

*where*

$$X = C_1 + \sum_{i=0}^{m/2-2} \binom{m/2+i}{2i+2} C_2^{2i+2},$$

*the 11 degree-1 relations*

$$\begin{gathered} vC_3 = C_1T_1, \quad vC_4 = C_1T_2, \quad vC_2 = uC_3 = (u+v)C_4 = C_2T_1 = C_2T_2 = 0, \\ C_3T_1 = uX + vC_1, \quad C_3T_2 = uC_2^{m-1} = C_4T_1, \quad C_4T_2 = uX, \end{gathered}$$

*and the 6 degree-2 relations*

$$u^2 = uv, \quad uT_1 = (u+v)T_2 = T_1T_2 = 0, \quad T_1^2 = v(u+v), \quad T_2^2 = u^2.$$

*Note.* It is also true in this case that $\gamma_1$ induces an isomorphism modulo radicals, but this is a special case of the more general Theorem 10.1.

In the above and in what follows, we use the convention that the binomial coefficient $\binom{n}{k}$ is 0 if $k > n$. The rest of this section is devoted to a proof of the proposition.

**Generators.** The $m+3$ conjugacy classes of $G$ are

$$\{1\},\{a^m\},\{a^r,a^{-r}\}(1\le r\le m-1),\{a^sb\mid s\text{ even}\},\{a^sb\mid s\text{ odd}\}.$$

and from these we choose representatives $g_1=1$, $g_2=a^m$, $g_{r+2}=a^r$, $g_{m+2}=b$, $g_{m+3}=ab$. The centralizers are $H_1=H_2=G$, $H_{r+2}=\langle a\rangle$, and the Klein 4-groups $H_{m+2}=\langle a^m,b\rangle$ and $H_{m+3}=\langle a^m,ab\rangle$. We have

$$\begin{aligned} H^*(G)&=\mathbb{F}_2[u,v,w\mid \deg(u)=\deg(v)=1,\,\deg(w)=2,\,u^2=uv]\\ H^*(\langle a\rangle)&=\mathbb{F}_2[u_1,w_1\mid \deg(u_1)=1,\,\deg(w_1)=2,\,u_1^2=0]\\ H^*(H_{m+j+1})&=\mathbb{F}_2[x_j,y_j\mid \deg(x_j)=\deg(y_j)=1]\qquad (j=1,2)\end{aligned}$$

(all coefficients are assumed to be in $\mathbb{F}_2$; see [22] for $H^*(G)$). We may take $u,v$ to be the basis dual to the basis $\bar a,\bar b$ of $G/\langle a^2\rangle$, and $w$ to be the degree-2 Stiefel-Whitney class of the natural representation of $G$ on the plane. There is only one choice for $u_1$ (resp. $w_1$), namely, the unique non-zero element of $H^*(\langle a\rangle)$ in degree 1 (resp. 2). Take $x_1,y_1$ (resp. $x_2,y_2$) to be dual to $a^m,b$ (resp. $a^m,ab$). Also let $H^*(\langle a^m\rangle)=\mathbb{F}_2[z]$. As before, we will identify $\alpha$ with $\gamma_1(\alpha)$ ($\alpha\in H^*(G)$). Let $E_i=\gamma_i(1)$ $(1\le i\le m+3)$, $C_1=1+E_2$, $C_2=E_3$, $C_3=E_{m+2}$, $C_4=E_{m+3}$, and $T_j=\gamma_{m+j+1}(x_j)$ $(j=1,2)$.

**Lemma 9.2.** *The restrictions of $u$ to the subgroups $\langle a\rangle$, $H_{m+2}$, $H_{m+3}$, $\langle a^m\rangle$ are (resp.) $u_1$, $0$, $y_2$, $0$. The restrictions of $v$ are $0$, $y_1$, $y_2$, $0$. The restrictions of $w$ are $w_1$, $x_1(x_1+y_1)$, $x_2(x_2+y_2)$, $z^2$. Moreover, restriction from $\langle a\rangle$ to $\langle a^m\rangle$ maps $u_1$ to $0$ and $w_1$ to $z^2$.*

*Proof.* Restrictions from $\langle a\rangle$ to $\langle a^m\rangle$ follow from [1, Cor. II.5.7]. Of the others, only $w$ presents some difficulties. Its restrictions to $H_{m+2}$ and $H_{m+3}$ follow from [22, Prop. 2.1]. It follows that

$$\operatorname{res}^G_{\langle a^m\rangle}(w)=\operatorname{res}^{H_{m+2}}_{\langle a^m\rangle}\operatorname{res}^G_{H_{m+2}}(w)=\operatorname{res}^{H_{m+2}}_{\langle a^m\rangle}(x_1^2+x_1y_1)=z^2,$$

whence $\operatorname{res}^G_{\langle a\rangle}(w)=w_1$. □

We claim that $H^*(G,\mathbb{F}_2G)$ is generated as an $H^*(G)$-module by the $E_i$ $(1\le i\le m+3)$, $T_1$, and $T_2$. For $1\le i\le m+1$, the restriction from $G$ to $H_i$ is surjective (by Lemma 9.2, if $i\ge 3$), so $E_i$ generates the image of $\gamma_i$. By Lemma 9.2, the image of restriction to $H_{m+2+j}$ is the subring generated by $y_j$ and $x_j(x_j+y_j)$ $(j=1,2)$. Hence the proof of our claim is completed by

**Lemma 9.3.** *The polynomial ring $\mathbb{F}_2[x,y]$ is generated as a module over the subring $\mathbb{F}_2[y,x(x+y)]$ by $1$ and $x$.*

*Proof.* We show that $x^iy^j$ is in the submodule generated by 1 and $x$ by induction on $i+j$. This is clear if $i+j<2$. If $j>0$ then $x^iy^j=y(x^iy^{j-1})$. If $i\ge 2$ then $x^i=(x^2+xy)x^{i-2}+y(x^{i-1})$. □

Using the identification of degree-0 Hochschild cohomology with the center of $\mathbb{F}_2G$, one can see that $E_4=C_2^2$ and $E_{r+2}=E_{r+1}C_2+E_r$ $(3\le r\le m-1)$. By

FIGURE 1. Product Formula data for $G = D_{4m}$

| $i$ | $j$ | $x$ | $g_i{}^x g_j$ | $k$ | $y$ | ${}^yH_i$ | ${}^{yx}H_j$ | $W$ |
|---|---|---|---|---|---|---|---|---|
| 1 | $j$ | 1 | $g_j$ | $j$ | 1 | $G$ | $H_j$ | $H_j$ |
| 2 | 2 | 1 | 1 | 1 | 1 | $G$ | $G$ | $G$ |
| | $r+2$ | 1 | $a^{r-m}$ | $m-r+2$ | $b$ | $G$ | $\langle a\rangle$ | $\langle a\rangle$ |
| | $m+2$ | 1 | $a^m b$ | $m+2$ | $a^{m/2}$ | $G$ | $H_{m+2}$ | $H_{m+2}$ |
| | $m+3$ | 1 | $a^{m+1}b$ | $m+3$ | $a^{m/2}$ | $G$ | $H_{m+3}$ | $H_{m+3}$ |
| $r+2$ | $s+2$ | 1 | $a^{r+s}$ | $r+s+2$ $^{(0)}$ | 1 | $\langle a\rangle$ | $\langle a\rangle$ | $\langle a\rangle$ |
| | | | | 2 $^{(1)}$ | 1 | $\langle a\rangle$ | $\langle a\rangle$ | $\langle a\rangle$ |
| | | | | $2m-r-s+2$ $^{(2)}$ | $b$ | $\langle a\rangle$ | $\langle a\rangle$ | $\langle a\rangle$ |
| | | $b$ | $a^{r-s}$ | $r-s+2$ $^{(3)}$ | 1 | $\langle a\rangle$ | $\langle a\rangle$ | $\langle a\rangle$ |
| | | | | 1 $^{(4)}$ | 1 | $\langle a\rangle$ | $\langle a\rangle$ | $\langle a\rangle$ |
| | | | | $s-r+2$ $^{(5)}$ | $b$ | $\langle a\rangle$ | $\langle a\rangle$ | $\langle a\rangle$ |
| | $m+2$ | 1 | $a^r b$ | $m+2$ $^{(6)}$ | $a^{-r/2}$ | $\langle a\rangle$ | $\langle a^m, a^{-r}b\rangle$ | $\langle a^m\rangle$ |
| | | | | $m+3$ $^{(7)}$ | $a^{\frac{1-r}{2}}$ | $\langle a\rangle$ | $\langle a^m, a^{1-r}b\rangle$ | $\langle a^m\rangle$ |
| | $m+3$ | 1 | $a^{r+1}b$ | $m+3$ $^{(6)}$ | $a^{-r/2}$ | $\langle a\rangle$ | $\langle a^m, a^{1-r}b\rangle$ | $\langle a^m\rangle$ |
| | | | | $m+2$ $^{(7)}$ | $a^{\frac{-1-r}{2}}$ | $\langle a\rangle$ | $\langle a^m, a^{-r}b\rangle$ | $\langle a^m\rangle$ |
| $m+2$ | $m+2$ | 1 | 1 | 1 | 1 | $H_{m+2}$ | $H_{m+2}$ | $H_{m+2}$ |
| | | $a^l$ $^{(8)}$ | $a^{-2l}$ | $2+2l$ | $b$ | $H_{m+2}$ | $\langle a^m, a^{-2l}b\rangle$ | $\langle a^m\rangle$ |
| | | $a^{m/2}$ | $a^m$ | 2 | 1 | $H_{m+2}$ | $H_{m+2}$ | $H_{m+2}$ |
| | $m+3$ | $a^l$ $^{(9)}$ | $a^{-1-2l}$ | $3+2l$ | $b$ | $H_{m+2}$ | $\langle a^m, a^{-1-2l}b\rangle$ | $\langle a^m\rangle$ |
| $m+3$ | $m+3$ | 1 | 1 | 1 | 1 | $H_{m+3}$ | $H_{m+3}$ | $H_{m+3}$ |
| | | $a^l$ $^{(8)}$ | $a^{-2l}$ | $2+2l$ | $b$ | $\langle a^m, a^{-1}b\rangle$ | $\langle a^m, a^{-1-2l}b\rangle$ | $\langle a^m\rangle$ |
| | | $a^{m/2}$ | $a^m$ | 2 | 1 | $H_{m+3}$ | $H_{m+3}$ | $H_{m+3}$ |

$^{(0)}$ $1 \le r+s < m$ $^{(1)}$ $r+s=m$ $^{(2)}$ $r+s>m$ $^{(3)}$ $r>s$ $^{(4)}$ $r=s$
$^{(5)}$ $r<s$ $^{(6)}$ $r$ even $^{(7)}$ $r$ odd $^{(8)}$ $1 \le l < m/2$ $^{(9)}$ $0 \le l < m/2$

iterating this formula one obtains

$$E_k = \begin{cases} \displaystyle\sum_{i=0}^{k/2-2} \binom{k/2-1+i}{1+2i} C_2^{2+2i} & k \text{ even and } 4 \le k \le m \\ \displaystyle\sum_{i=0}^{(k-3)/2} \binom{(k-3)/2+i}{2i} C_2^{1+2i} & k \text{ odd and } 3 \le k \le m+1. \end{cases} \tag{9.1}$$

So in fact the powers of $C_2$ generate the images of $\gamma_i$ $(3 \le i \le m+1)$. From this we obtain the algebra generators of $H^*(G, \mathbb{F}_2 G)$ listed in the proposition.

**Relations.** Straightforward calculations in $Z(\mathbb{F}_2 G)$, together with formula (9.1) with $k = m+1$ in the case of the product $C_1 C_2$, yield the degree-0 relations. We now gather the data necessary to verify the remaining relations. In order to apply the Product Formula, we will use the table in Figure 1. The notation for the table is exactly as in Section 4. All values of $r$ and $s$ in the table range between 1 and $m-1$. Next, we will need the following.

**Lemma 9.4.** (i) *If $F$ is a proper subgroup of an elementary abelian 2-group $E$ then* $\operatorname{cor}_F^E = 0$.
(ii) $\operatorname{cor}_{H_{m+2}}^G$ *maps $y_1$ to $0$, $x_1$ to $u+v$, and $x_1^2$ to $v(u+v)$.*
(iii) $\operatorname{cor}_{H_{m+3}}^G$ *maps $x_2$ to $u$, $y_2$ to $0$, and $x_2^2$ to $u^2$.*

(iv) $\mathrm{cor}_{\langle a^m\rangle}^{\langle a\rangle}$ *maps* $z$ *to* $u_1$ *and* $z^2$ *to* $0$.

*Proof.* Part (i) follows from [10, Lem. 6.3.4]. To prove (ii), note that $y_1$ is in the image of restriction from $G$, so $\mathrm{cor}_{H_{m+2}}^G(y_1) = 0$. Now $\mathrm{cor}_{H_{m+2}}^G(x_1) = \lambda_1 u + \lambda_2 v$ for some $\lambda_1, \lambda_2 \in \mathbb{F}_2$. By the double coset formula,

$$\begin{aligned}\mathrm{res}_{H_{m+2}}^G \mathrm{cor}_{H_{m+2}}^G(x_1) &= x_1 + \sum_{i=1}^{m/2-1} \mathrm{cor}_{\langle a^m\rangle}^{H_{m+2}} \mathrm{res}_{\langle a^m\rangle}^{\langle a^m, a^{2i}b\rangle}\big((a^i)^* x_1\big) + (a^{m/2})^* x_1 \\ &= x_1 + 0 + (x_1 + y_1) = y_1.\end{aligned}$$

The double coset formula also yields $\mathrm{res}_{H_{m+3}}^G \mathrm{cor}_{H_{m+2}}^G(x_1) = 0$, since each term involves corestrictions from a proper subgroup of $H_{m+3}$. On the other hand,

$$\begin{aligned}\mathrm{res}_{H_{m+2}}^G \mathrm{cor}_{H_{m+2}}^G(x_1) &= \mathrm{res}_{H_{m+2}}^G(\lambda_1 u + \lambda_2 v) = \lambda_2 y_1 \\ \mathrm{res}_{H_{m+3}}^G \mathrm{cor}_{H_{m+2}}^G(x_1) &= \mathrm{res}_{H_{m+3}}^G(\lambda_1 u + \lambda_2 v) = (\lambda_1 + \lambda_2) y_2.\end{aligned}$$

This forces $\lambda_1 = \lambda_2 = 1$. Now, $x_1^2 + x_1 y_1 = \mathrm{res}_{H_{m+2}}^G(w)$ and $y_1 = \mathrm{res}_{H_{m+2}}^G(v)$, whence

$$\mathrm{cor}_{H_{m+2}}^G(x_1^2) = \mathrm{cor}_{H_{m+2}}^G(x_1 y_1) = v\,\mathrm{cor}_{H_{m+2}}^G(x_1) = v(u+v),$$

completing the proof of (ii). Part (iii) is handled in a similar way. Part (iv) follows from [1, Cor. II.5.7]. $\square$

Using this information and the Product Formula, one may obtain the remaining relations. The only complications arise in the following cases. When computing $C_3T_1$ one obtains

$$C_3T_1 = (u+v)C_1 + \sum_{j=1}^{m/2-1} \gamma_{2+2j}(u_1),$$

where the sum is empty in case $m = 2$. Using $\gamma_{2+2j}(u_1) = uE_{2+2j}$ and formula (9.1) with $k = 2 + 2j$, we obtain

$$\begin{aligned}C_3T_1 &= (u+v)C_1 + \sum_{j=1}^{m/2-1} \sum_{i=0}^{j-1} \binom{j+i}{1+2i} u C_2^{2+2i} \\ &= (u+v)C_1 + \sum_{i=0}^{m/2-2} \binom{m/2+i}{2+2i} u C_2^{2+2i}.\end{aligned}$$

Similarly, we calculate

$$C_3T_2 = \sum_{i=1}^{m/2} uE_{1+2i} = \sum_{i=0}^{m/2-1} \binom{m/2+i}{1+2i} u C_2^{1+2i}. \tag{9.2}$$

However,

**Lemma 9.5.** *If* $0 \le i \le m/2 - 2$ *then* $\binom{m/2+i}{1+2i} \equiv 0 \mod 2$.

*Proof.* Let $m/2 = 2^k$, and let $i = a_0 + a_1 2 + \cdots + a_{k-1}2^{k-1}$ be the base 2 expansion of $i$. We then have base 2 expansions

$$m/2 + i = a_0 + a_1 2 + a_2 2^2 + \cdots + a_k 2^k$$
$$1 + 2i = b_0 + b_1 2 + b_2 2^2 + \cdots + b_k 2^k,$$

where $a_k = 1$, $b_0 = 1$ and $b_j = a_{j-1}$ $(1 \le j \le k)$. By [19, Lem. 22.4], it suffices to show $a_j < b_j$ for some $j$. As $i \le 2^k - 2$, at least one of the $a_j$ must be equal to 0. Choose the smallest $j$ with $a_j = 0$. If $j = 0$, then $0 = a_0 < b_0 = 1$. If $j > 0$, then $b_j = a_{j-1} = 1$ as $j$ is smallest, so that $0 = a_j < b_j = 1$ in this case as well. $\square$

Hence the sum (9.2) reduces to $C_3T_2 = uC_2^{m-1}$. Similar arguments yield $C_4T_1 = uC_2^{m-1}$ and $C_4T_2 = uX$.

**Sufficiency.** We will now show that we have found a full set of relations by considering the algebra $A$ defined abstractly by our generators and relations. Clearly there is a homomorphism from $A$ onto $H^*(G, \mathbb{F}_2G)$. Note also that $w$ does not occur in any of our relations, so $A \cong \mathbb{F}_2[w] \otimes_{\mathbb{F}_2} B$, where $B$ is the algebra given by the remaining generators and all the relations.

Direct examination shows that

$$B_0 = \langle 1, C_1, C_2, C_2^2, \ldots, C_2^{m-1}, C_3, C_4 \rangle$$
$$B_1 = \langle u, v, T_1, T_2, uC_1, vC_1, C_1T_1, C_1T_2, uC_2, uC_2^2, \ldots, uC_2^{m-1} \rangle$$
$$B_2 = \langle uv, v^2, vT_1, vT_2, uvC_1, v^2C_1, vC_1T_1, vC_1T_2 \rangle$$

In particular, the dimensions of the degree-0, 1, and 2 components of $B$ are less than or equal to (resp.) $m + 3$, $m + 7$, and 8. Note also that $B_2$ is the degree-2 component of $Bv$, the ideal generated by $v$. This implies that $B_k$ is the degree-$k$ component of $Bv$ for all $k \ge 2$: as the generators of $B$ are all of degree 0 or 1, a monomial $f$ of degree $k$ may be factored into a degree-2 factor and a degree-$(k-2)$ factor. The degree-2 factor is in $Bv$, so the same is true of $f$. This implies that multiplication by $v$ is a surjective map from $B_k$ to $B_{k+1}$ for all $k \ge 2$. Therefore $\dim(B_k) \le 8$ for all $k \ge 2$.

The Poincaré series of a graded vector space having dimension $m + 3$ in degree 0, $m + 7$ in degree 1, and 8 in each degree $k \ge 2$ is

$$\frac{(m+3) + 4t + (1-m)t^2}{1-t}.$$

We conclude that the dimension of $A$ in each degree is less than or equal to that given by the series

$$\frac{(m+3) + 4t + (1-m)t^2}{(1-t)(1-t^2)} = \frac{(m+3) + (1-m)t}{(1-t)^2}.$$

But this is the Poincaré series of $H^*(G, \mathbb{F}_2G)$ obtained from the additive decomposition. Therefore the homomorphism from $A$ onto $H^*(G, \mathbb{F}_2G)$ is an isomorphism, and Proposition 9.1 is, at long last, proved.

## 10. Structure theorems

Our goal in this section is to obtain some general structure theorems for the rings $H^*(H, RG)$.

**The $p$-group case.** First we give a result regarding the Hochschild cohomology rings of group algebras of $p$-groups in characteristic $p$.

**Theorem 10.1.** *Let $\mathbb{F}$ be a field of characteristic $p$, and $G$ a $p$-group. Then the map of algebras $\gamma_1 : H^*(G, \mathbb{F}) \to H^*(G, \mathbb{F}G)$ induces an isomorphism modulo radicals.*

*Proof.* Let $I$ be the augmentation ideal of $\mathbb{F}G$. Then $\mathbb{F}G$, as an $\mathbb{F}G$-module under conjugation, is the direct sum of its submodules $\mathbb{F}1$ and $I$. Applying the cohomology functor to this splitting, we see that there is an algebra homomorphism $\pi$ from the Hochschild ring to the ordinary ring satisfying $\pi\gamma_1 = 1$, and we may identify the kernel of $\pi$ with $H^*(G, I)$.

It suffices to show that $H^*(G, I)$ is a nilpotent ideal. But from the definition of the cup product, for all $n > 0$ the multiplication map from $H^*(G, I)^{\otimes n}$ to $H^*(G, I)$ factors through the map $H^*(G, I^{\otimes n}) \to H^*(G, I)$ induced by multiplication in $I$. As $G$ is a $p$-group, $I$ is a nilpotent ideal, so for $n$ sufficiently large the map from $I^{\otimes n}$ to $I$ is 0. Hence $H^*(G, I)^n = 0$. $\square$

**A Green functor.** We next generalize a weaker consequence of Proposition 3.2 to a group $H$ acting nontrivially on a group $G$. Combined with a general result of Thévenaz about Green functors, this leads to a structure theorem for $H^*(H, RG)$ in Corollary 10.4.

We introduce the following Green functor. Assign to any subgroup $K$ of $H$ the ring $H^*(K, RG)$, and consider the usual maps conjugation $h^* : H^*(K, RG) \to H^*({}^hK, RG)$ for $h \in H$, and $\mathrm{res}^K_L$ and $\mathrm{cor}^K_L$ for $L \leq K \leq H$. The various properties required of a Green functor in this situation follow from [10, Ch. 4], and will be used in the remainder of this section. In particular, the image of corestriction from a subgroup $L$ of $K$ is an ideal of $H^*(K, RG)$.

Let

$$\overline{H}^*(H, RG) = H^*(H, RG) / \sum_{K<H} \mathrm{cor}^H_K\big(H^*(K, RG)\big)$$
$$\overline{H}^*(H) = H^*(H) / \sum_{K<H} \mathrm{cor}^H_K\big(H^*(K)\big).$$

Let $G^H$ denote the subgroup of $G$ consisting of all elements fixed by $H$.

**Theorem 10.2.** *Let $H$ and $G$ be finite groups with $H$ acting as automorphisms on $G$. Then*

$$\overline{H}^*(H, RG) \cong R(G^H) \otimes_R \overline{H}^*(H)$$

*as graded $R$-algebras.*

In order to prove the Theorem, we will first define a map from $H^*(H, RG)$ to $R(G^H) \otimes_R H^*(H)$ and examine some of its properties. Consider the $RH$-homomorphism $RG \to R(G^H)$ defined on $G$ by sending $g$ to $g$ (if $g \in G^H$) and 0 (otherwise). This induces a map from $H^*(H, RG)$ to $H^*(H, R(G^H))$. The latter ring is isomorphic to $R(G^H) \otimes H^*(H)$ by Proposition 3.2, and so we obtain a map $\psi : H^*(H, RG) \to R(G^H) \otimes H^*(H)$. This map is given explicitly by

$$\psi(\zeta) = \sum_{g \in G^H} g \otimes \pi_g^*(\zeta). \tag{10.1}$$

In general, $\psi$ is not an algebra homomorphism.

Let $\rho\colon H^*(H) \to \overline{H}^*(H)$ denote the quotient map, and $\overline{\psi} = (\mathrm{id} \otimes \rho) \circ \psi$. In the next lemma, we show that $\overline{\psi}$ *is* an algebra homomorphism. Finally, we prove that it induces the isomorphism in the theorem.

**Lemma 10.3.** *$\overline{\psi}\colon H^*(H, RG) \to R(G^H) \otimes_R \overline{H}^*(H)$ is an algebra homomorphism.*

*Proof.* By Lemma 4.2, it suffices to show that

$$\overline{\psi}(\gamma_i(\alpha) \smile \gamma_j(\beta)) = \overline{\psi}(\gamma_i(\alpha))\overline{\psi}(\gamma_j(\beta)) \tag{10.2}$$

for all $1 \le i, j \le r$ and $\alpha \in H^*(H_i)$, $\beta \in H^*(H_j)$. By Lemma 5.2 (iii) and (iv) and the definitions of $\overline{\psi}$ and $\gamma_i$, the right side is nonzero only when both $g_i, g_j \in G^H$, and in this case the right side of equation (10.2) is $g_i g_j \otimes \rho(\alpha \smile \beta)$. On the other hand, by Theorem 5.1, the left side of equation (10.2) is equal to

$$\begin{aligned}\sum_{x\in D} \overline{\psi}\gamma_k \Big(\mathrm{cor}_W^{H_k}\big(\mathrm{res}_W^{{}^yH_i}\, y^*\alpha \smile \mathrm{res}_W^{{}^{yx}H_j}(yx)^*\beta\big)\Big) \\ = \sum_{g\in G^H}\sum_{x\in D} g \otimes \rho\pi_g^*\gamma_k\, \mathrm{cor}_W^{H_k}\big(\mathrm{res}_W^{{}^yH_i}\, y^*\alpha \smile \mathrm{res}_W^{{}^{yx}H_j}(yx)^*\beta\big),\end{aligned}$$

where $D$ is a set of double coset representatives for $H_i\backslash H/H_j$, $k = k(x)$ and $y = y(x)$ are chosen to satisfy $g_k = {}^yg_i{}^{yx}g_j$, and $W = W(x) = {}^yH_i \cap {}^{yx}H_j$. By Lemma 5.2 (iii) and (iv), for each $g \in G^H$, $\pi_g^*\gamma_k$ is the identity if $g = g_k$ and 0 otherwise. Therefore this sum is equal to

$$\sum_{g\in G^H}\sum_{x\in D} \delta_{g,g_k} g_k \otimes \rho\, \mathrm{cor}_W^H\big(\mathrm{res}_W^{{}^yH_i}\, y^*\alpha \smile \mathrm{res}_W^{{}^{yx}H_j}(yx)^*\beta\big).$$

Now, if $g_i, g_j \in G^H$, then $H_i = H_j = W = H$, and this sum contains the single term $g_k \otimes \rho(\alpha \smile \beta) = g_i g_j \otimes \rho(\alpha \smile \beta)$, so in this case we have established equation (10.2). If one of $g_i, g_j$ is *not* in $G^H$, then each $W$ appearing in the above sum is a *proper* subgroup of $H$, and so the sum is 0, as was the case for the right side of equation (10.2). Therefore $\overline{\psi}$ is a ring homomorphism. ◻

*Proof of Theorem 10.2.* We will show that the map $\overline{\psi}$ of the Lemma induces the required isomorphism. We first show that $\overline{\psi}$ is surjective. Let $\alpha \in H^*(H)$ and $h \in G^H$. If we let $\zeta = \theta_h^*(\alpha)$, then $\psi(\zeta) = \sum_{g\in G^H} g \otimes \pi_g^*\theta_h^*(\alpha) = h \otimes \alpha$ by Lemma 5.2 (iv). Therefore $\overline{\psi}$ is surjective.

Next we will show that $\psi$ factors through $\overline{H}^*(H, RG)$. This will follow once we show that $\psi$ takes the image of $\mathrm{cor}_K^H$ into $R(G^H) \otimes \mathrm{Im}(\mathrm{cor}_K^H)$ for each subgroup $K$ of $H$. Let $\zeta = \gamma_i(\alpha)$ be an element of $H^*(K, RG)$, with $\alpha \in H^*(K_i)$. If $g_i$ is not in $G^H$, then $\psi(\mathrm{cor}_K^H(\zeta)) = 0$ by Lemma 5.2 (iii) and (iv) and the definition (10.1) of $\psi$. If $g_i \in G^H$, Lemma 5.2 (iii) and (iv) show that $\psi(\mathrm{cor}_K^H(\zeta)) = g_i \otimes \mathrm{cor}_K^H \alpha$, which is in $R(G^H)\otimes \mathrm{Im}(\mathrm{cor}_K^H)$. Therefore $\psi$ takes the image of $\mathrm{cor}_K^H$ into $R(G^H)\otimes\mathrm{Im}(\mathrm{cor}_K^H)$, and so $\overline{\psi}$ factors through $\overline{H}^*(H, RG)$.

It remains to prove that the kernel of $\overline{\psi}$ is contained in $\sum_{K<H} \mathrm{Im}(\mathrm{cor}_K^H)$. Let $\zeta \in \mathrm{Ker}(\overline{\psi})$, and write $\zeta = \sum_i \gamma_i(\alpha_i)$ with $\alpha_i \in H^*(H_i)$. As we have shown that $\mathrm{Im}(\mathrm{cor}_K^H)$ is contained in the kernel of $\overline{\psi}$ for all proper subgroups $K$ of $H$, and $\gamma_i(\alpha_i) = \mathrm{cor}_{H_i}^H \theta_i^*(\alpha_i)$ by definition, we may assume that $\alpha_i = 0$ for all $i$ with $g_i \notin G^H$. Considering the definition (10.1) of $\psi$, we may further assume that $\alpha_i$ is nonzero only for a single fixed value of $i$, that is $\zeta = \theta_i^*(\alpha_i)$ with $g_i \in G^H$. By the definition of $\overline{\psi}$ and Lemma 5.2 (iv), we have $\overline{\psi}(\zeta) = g_i \otimes \overline{\alpha_i}$. As $\overline{\psi}(\zeta) = 0$, this

implies $\alpha_i \in \sum_{K<H} \mathrm{Im}(\mathrm{cor}_K^H)$. By applying Lemma 5.2 (iii) to $\zeta = \theta_i^*(\alpha_i)$, we see that $\zeta$ is in $\sum_{K<H} \mathrm{Im}(\mathrm{cor}_K^H)$. □

Next consider the ring homomorphism

$$\beta\colon H^*(H, RG) \to \prod_K \big(\overline{H}^*(K, RG)\big)^{N_H(K)},$$

where the product is over a set of representatives $K$ of conjugacy classes of subgroups of $H$, given in the $K$-component by the composition of $\mathrm{res}_K^H$ with the quotient map. The kernel of $\beta$ is nilpotent by [29, Thm. 3.2]. By Theorem 10.2, we know the structure of the factors in the above product. Thus we obtain the following corollary.

**Corollary 10.4.** *There is a ring homomorphism, with nilpotent kernel,*

$$\beta\colon H^*(H, RG) \to \prod_K \big(R(G^K) \otimes_R \overline{H}^*(K)\big)^{N_H(K)},$$

*the product taken over a set of representatives $K$ of conjugacy classes of subgroups of $H$.*

## 11. Questions

We conclude with a few questions indicated by the results above. The first is suggested by our calculations and the result on $p$-groups: suppose $\mathbb{F}$ is a field and $G$ is a finite group such that $\mathbb{F}G$ is indecomposable. Then we may ask

**Question 1.** *Does $\gamma_1\colon H^*(G,\mathbb{F}) \to H^*(G,\mathbb{F}G)$ induce an isomorphism modulo radicals?*

As we have seen, Question 1 has an affirmative answer for $S_3$ mod 3, $A_4$ mod 2, and for any $p$-group mod $p$. Of course, if $\mathbb{F}G$ has more than one block, than Question 1 has a negative answer; this is because the center of $\mathbb{F}G$ modulo its radical has dimension equal to the number of blocks, so the map fails to be an isomorphism in degree 0. However, we may refine the question as follows. Suppose now $G$ is any finite group, and consider the map $f$ which is $\gamma_1$ followed by the projection onto $H^*(B_0, B_0)$, where $B_0$ is the principal block of $\mathbb{F}G$.

**Question 2.** *Does $f\colon H^*(G,\mathbb{F}) \to H^*(B_0, B_0)$ induce an isomorphism modulo radicals?*

Question 2 has an affirmative answer for the above cases, and also for $S_3$ mod 2, as is easily verified. It also has an affirmative answer in the case where $G$ is Abelian. For in this case we may identify the Hochschild cohomology ring with $\mathbb{F}G \otimes_{\mathbb{F}} H^*(G)$, by Prop. 3.2. Under this identification, $\gamma_1$ takes $\alpha \in H^*(G)$ to $1 \otimes \alpha$. The principal block component of the Hochschild ring modulo its radical is

$$\frac{B_0 \otimes H^*(G)}{\mathrm{rad}\big(B_0 \otimes H^*(G)\big)} \cong \frac{B_0}{\mathrm{rad}(B_0)} \otimes \frac{H^*(G)}{\mathrm{rad}(H^*(G))} \cong \mathbb{F} \otimes \frac{H^*(G)}{\mathrm{rad}(H^*(G))} \cong \frac{H^*(G)}{\mathrm{rad}(H^*(G))}$$

Hence the map modulo radicals induced by $f$ is indeed an isomorphism.

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