# GENERALIZED BENSON-CARLSON DUALITY

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## 1. Introduction

**1.1. Background.** This paper deals with the landmark results of Benson and Carlson's *Projective resolutions and Poincaré duality complexes* [6]. Our goal is to provide some necessary background for that work, and to prove some of the results of [6] in a more general setting. In particular, we analyze what happens when we replace Benson and Carlson's complex $C_\zeta$ (in the notation of [6]) with an arbitrary Yoneda extension representing $\zeta$.

Let $\Lambda$ be a finite-dimensional cocommutative Hopf algebra over an algebraically closed field $k$. Because $\Lambda$ is cocommutative, $H^*(\Lambda, k)$ is a graded-commutative $k$-algebra (i.e., $xy = (-1)^{\deg(x)\deg(y)}yx$ for homogeneous $x, y$). Throughout this paper we will also assume that $\Lambda$ has the following *finiteness property in cohomology*: $\mathrm{Ext}^*_\Lambda(k,k)$ is a finitely generated $k$-algebra, and for any $\Lambda$-modules $M$ and $N$, $\mathrm{Ext}^*_\Lambda(M,N)$ is finitely generated as an $\mathrm{Ext}^*_\Lambda(k,k)$-module. (By $\Lambda$-*module* we always mean *finitely generated left* $\Lambda$-*module*.) The second condition is equivalent to requiring $H^*(\Lambda, M)$ to be finitely generated over $H^*(\Lambda, k)$ for any $\Lambda$-module $M$, because of the isomorphism $\mathrm{Ext}_\Lambda(M,N) \cong H^*(\Lambda, \mathrm{Hom}_k(M,N))$.

If $\Lambda$ is the group algebra of a finite group then $\Lambda$ certainly has the finiteness property in cohomology, by a theorem of Evens (cf. [7, Theorem 7.4.1]). The finiteness property also holds if $\Lambda$ is a finite-dimensional cocommutative *connected* Hopf algebra, e.g. the restricted enveloping algebra of a $p$-restricted Lie algebra (Bajer and Sadofsky [1, Lemma 6.2], Wilkerson [10]). In fact, at present we know of no example of a finite-dimensional Hopf algebra without this property. It is known that any finite-dimensional Hopf algebra is a Frobenius algebra (Larson and Sweedler [9]), which implies that a $\Lambda$-module is projective if and only if it is injective, and this property is fundamental for the proofs given here.

**1.2. Preliminaries.** We use the following notation for chain complexes of modules over a $k$-algebra $R$. If $C$ is a complex of left or right $R$-modules, let $Z_nC = \mathrm{Ker}(\partial : C_n \to C_{n-1})$, $B_nC = \mathrm{Im}(\partial : C_{n-1} \to C_n)$, and $H_nC = Z_nC/B_nC$. If $C, D$ are complexes of right, left R-modules respectively, let $C \otimes_R D$ denote the complex with $(C \otimes_R D)_n = \bigoplus_{p+q=n} C_p \otimes_R C_q$, with differential given by $\partial(x \otimes y) = \partial(x) \otimes y + (-1)^p x \otimes \partial(y)$ for $x \in C_p, y \in C_q$. If $C$ and $D$ are both complexes of left (resp. right) $R$-modules, then let $\mathrm{Hom}_R(C, D)$ denote the complex with $\mathrm{Hom}_R(C, D)_n = \prod_{q=p+n} \mathrm{Hom}_R(C_p, D_q)$, with differential given by $\partial(f) = \partial \circ f - (-1)^n f \circ \partial$, for $f \in \mathrm{Hom}_R(C, D)_n$. If $f - g \in B_n\mathrm{Hom}_R(C, D)$ we write $f \simeq g$ ($f$ and $g$ are *chain homotopic*). For any integer $r$, $C[r]$ denotes the complex with $C[r]_p = C_{p-r}$, and with differential $(-1)^r\partial$. The dual complex $C^*$ is defined to be $\mathrm{Hom}_k(C, k)$, where $k$ is considered to be a complex concentrated in degree 0; if $f \in \mathrm{Hom}_R(C, D)_r$ then $f^* \in \mathrm{Hom}_R(D^*, C^*)_r$ is defined by $f^*(\alpha) = (-1)^{rs}\alpha \circ f$ for $\alpha \in C_s^*$. We write $C|D$ if $C$ is isomorphic to a summand of $D$ (as complexes). Finally, any chain complex may be considered a cochain complex, and *vice-versa*, by setting $C^n = C_{-n}$.

Let $\mathcal{C}$ denote the category of complexes of $\Lambda$-modules. Let $\mathcal{C}_b$, ${}_b\mathcal{C}$, $\mathcal{C}^p$, ${}^e\mathcal{C}$ denote, respectively, the full subcategories of $\mathcal{C}$ of complexes bounded below, complexes bounded above, complexes where each $C_n$ is $\Lambda$-projective, and exact complexes. These adornments on the symbol $\mathcal{C}$ may be combined, so that, for example, the objects of ${}^e_b\mathcal{C}^p_b$ are bounded (above and below) exact complexes of projectives. The following are fundamental; for proofs, see Benson [4, Lemma 1.4.4, Theorem 1.4.3], replacing "module" with "bounded complex" throughout.

**Lemma 1 (Fitting).** *Let $M \in {}_b\mathcal{C}_b$ and $f \in \mathrm{End}_{\mathcal{C}}(M)$. Then $M = \mathrm{Im}(f^n) \oplus \mathrm{Ker}(f^n)$ for $n$ sufficiently large.*

**Theorem 2 (Krull-Schmidt).** *Let $M \in {}_b\mathcal{C}_b$ and let $M = \bigoplus_{i=1}^m M^{(i)}$, where each $M^{(i)}$ is an indecomposable subcomplex of $M$. If $M = \bigoplus_{i=1}^n \tilde{M}^{(i)}$ for some other indecomposable subcomplexes $\tilde{M}^{(i)}$ of $M$, then $n = m$, and after renumbering if necessary, $\tilde{M}^{(i)} \cong M^{(i)}$ for each $i$.*

## 2. Projective resolutions of complexes

**2.1. Definition and basic properties.** Let $C \in \mathcal{C}_b$. A *projective resolution of $C$* is a complex $P \in \mathcal{C}^p_b$ together with a quasi-isomorphism $\epsilon : P \to C$.

**Proposition 3.** *If $C \in \mathcal{C}_b$ then $C$ has a projective resolution.*

*Proof.* We construct $P_n$ and $\epsilon_n$ by induction on $n$. These may be taken to be 0 for $n$ sufficiently small since $C$ is bounded below. So assume that $P_m$ and $\epsilon_m$ have been constructed and satisfy $\partial\epsilon_m = \epsilon_{m-1}\partial$ for $m \leq n$. We refer to Diagram 1, leaving off subscripts of maps whenever this is unambiguous. Let $PB_{n+1}$ be the pullback of $(\delta, \theta)$ in the diagram, and let $\alpha_{n+1}$ be a surjection from a projective module $P_{n+1}$

$$
\begin{tikzcd}
P_{n+1} \arrow[rrr, "\partial"] \arrow[dr, two heads, "\alpha_{n+1}"] \arrow[ddd, "\epsilon_{n+1}"'] & & & P_n \arrow[rrr, "\partial"] \arrow[dr, two heads, "\alpha_n"] \arrow[ddd, "\epsilon_n"'] & & & P_{n-1} \arrow[ddd, "\epsilon_{n-1}"] \\
& PB_{n+1} \arrow[r, "\gamma_{n+1}"] \arrow[d, "\beta_{n+1}"'] & Z_nP \arrow[ur, hook, "\iota_n"] \arrow[d, "\theta_n"] & & PB_n \arrow[r, "\gamma_n"] \arrow[d, "\beta_n"'] & Z_{n-1}P \arrow[ur, hook, "\iota_{n-1}"] \arrow[d, "\theta_{n-1}"] & \\
& \frac{C_{n+1}}{B_{n+1}C} \arrow[r, "\delta_{n+1}"] & Z_nC \arrow[dr, hook, "\kappa_n"] & & \frac{C_n}{B_nC} \arrow[r, "\delta_n"] & Z_{n-1}C \arrow[dr, hook, "\kappa_{n-1}"] & \\
C_{n+1} \arrow[rrr, "\partial"] \arrow[ur, two heads, "\eta_{n+1}"] & & & C_n \arrow[rrr, "\partial"] \arrow[ur, two heads, "\eta_n"] & & & C_{n-1}
\end{tikzcd}
$$

Diagram 1. Construction of a projective resolution.

to $PB_{n+1}$. By projectivity there is a map $\epsilon_{n+1} : P_{n+1} \to C_{n+1}$ making the diagram commute. Define $\partial : P_{n+1} \to P_n$ to be $i\gamma\alpha$; the diagram shows that $\partial\epsilon_{n+1} = \epsilon_n\partial$.

To show that $(\epsilon_*)_n$ is surjective, let $x \in Z_nC$. Then $\eta(x) \in \mathrm{Ker}(\delta)$, so it follows from the universal property of the pullback that there exists $y \in PB_n$ such that $\beta(y) = \eta(x)$ and $\gamma(y) = 0$. Choose $z \in P_n$ such that $\alpha(z) = y$. Then $\partial(z) = \iota\gamma\alpha(z) = 0$, so $z \in Z_nP$, and $\eta\epsilon_n(z) = \beta\alpha(z) = \eta(x)$, so $\epsilon_n(z) - x \in B_nC$.

To show that $(\epsilon_*)_n$ is injective, let $x \in Z_nP$ and suppose $\epsilon_n(x)$ is a boundary, say $\epsilon_n(x) = \partial(y)$, $y \in C_{n+1}$. Then $\theta(x) = \delta\eta(y)$, so there is a $z \in PB_{n+1}$ such that $\beta(z) = \eta(y)$ and $\gamma(z) = x$. Choose $w \in P_{n+1}$ such that $\alpha(w) = z$; it follows that $\partial(w) = \iota\gamma\alpha(w) = x$, i.e., $x$ is a boundary. □

*Remark* 1. Any projective resolution of $C$ must fit into a diagram like Diagram 1, for the universal property of pullbacks implies that there is a map $\alpha$ making the diagram commute at each stage. We need only check that such an $\alpha$ must be surjective. To see this, given any $x \in PB_n$, choose $y \in C_n$ such that $\eta(y) = \beta(x)$. Then by commutativity, $\partial(y) = \epsilon\iota\gamma(x)$. Since $\epsilon_*$ is an isomorphism, this implies $\iota\gamma(x)$ is a boundary, say $\iota\gamma(x) = \partial(w)$, $w \in P_n$. Now $\gamma\alpha(w) = \gamma(x)$ and

$$\partial\epsilon(w) = \epsilon\iota\gamma\alpha(w) = \epsilon\iota\gamma(x) = \partial(y).$$

So $\epsilon(w) - y \in Z_nC$, and since $\epsilon_*$ is an isomorphism, there is a $w' \in Z_nP$ such that $\eta\epsilon(w') = \eta(\epsilon(w) - y) = \eta\epsilon(w) - \beta(x)$. Now $\beta\alpha(w - w') = \eta\epsilon(w - w') = \beta(x)$ and $\gamma\alpha(w - w') = \gamma\alpha(w) = \gamma(x)$, so $\alpha(w - w') = x$.

*Remark* 2. Observe that $\partial(P) = \mathrm{Ker}(\alpha)$. For $\iota\gamma\alpha\partial = \partial^2 = 0$, hence $\gamma\alpha\partial = 0$. Moreover $\beta\alpha\partial = \eta\epsilon\partial = \eta\partial\epsilon = 0$. So by the universal property of pullbacks, $\alpha\partial = 0$, i.e., $\partial(P) \subseteq \mathrm{Ker}(\alpha)$. Conversely, if $\alpha(y) = 0$ for $y \in P$, then $y$ is a cycle and

$$
\begin{tikzcd}
 & P_{n+1} \arrow[rr] \arrow[d, "g_{n+1}"] & & P_n \arrow[rr] \arrow[d, "g_n"] \arrow[dll, dashed, "s_n"'] \arrow[ddl, dashed, "f_n"'] & & P_{n-1} \arrow[rr] \arrow[d, "g_{n-1}"] \arrow[dll, "s_{n-1}"'] \arrow[ddl, "f_{n-1}"'] & & P_{n-2} \arrow[d, "g_{n-2}"] \arrow[dll, "s_{n-2}"'] \arrow[ddl, "f_{n-2}"'] \\
 & C_{n+1} \arrow[rr] & & C_n \arrow[rr] & & C_{n-1} \arrow[rr] & & C_{n-2} \\
X_{n+1} \arrow[rr] \arrow[ur, "\epsilon_{n+1}"] & & X_n \arrow[rr] \arrow[ur, "\epsilon_n"] & & X_{n-1} \arrow[rr] \arrow[ur, "\epsilon_{n-1}"] & & X_{n-2} \arrow[ur, "\epsilon_{n-2}"] &
\end{tikzcd}
$$

Diagram 2. Proof of the existence part of the Comparison Theorem

$\eta\epsilon(y) = 0$, so $\epsilon(y)$ is a boundary, and since $\epsilon_*$ is an isomorphism, $y$ is a boundary; hence $\mathrm{Ker}(\alpha) \subseteq \partial(P)$.

**Theorem 4 (Comparison Theorem).** *Suppose $C, X \in \mathcal{C}_b$, $P \in \mathcal{C}_b^p$, and $\epsilon : X \to C$ is a quasi-isomorphism. Then the map*

$$
\begin{aligned}
H_*(\mathrm{Hom}_\Lambda(P, X)) &\longrightarrow H_*(\mathrm{Hom}_\Lambda(P, C)) \\
[f] &\longmapsto [\epsilon f]
\end{aligned}
$$

*is an isomorphism.*

*Remark* 3. For $* = 0$, the conclusion of the theorem may be expressed in the following form: given any $\Lambda$-chain map $g : P \to C$, there exists a $\Lambda$-chain map $f : P \to X$ such that $\epsilon f \simeq g$; moreover if $f' : P \to X$ is another such map, then $f \simeq f'$. In particular, if $X \xrightarrow{\epsilon} C$ and $P \xrightarrow{\tilde{\epsilon}} C$ are both projective resolutions, then there is a chain map $f : P \to X$, unique up to chain homotopy, such that $\epsilon f \simeq \tilde{\epsilon}$. Notice that the "comparison" $f$ must also be a quasi-isomorphism since $\epsilon_* f_* = \tilde{\epsilon}_*$, and both $\epsilon_*$ and $\tilde{\epsilon}_*$ are isomorphisms.

*Proof.* Suppose $g \in Z_r\mathrm{Hom}_\Lambda(P, C)$. By shifting the indices if necessary we may assume $r = 0$. We construct, by induction, maps $f_i : P_i \to X_i$ and $s_i : P_i \to X_{i+1}$ satisfying $\partial f_i = f_{i-1}\partial$ and $\epsilon f_i - g_i = \partial s_i + s_{i-1}\partial$, for all $i \leq n$. For $n$ sufficiently small these maps may all be taken to be 0, so we may assume that we have constructed them for $i < n$ and complete the inductive step.

First note that $f_{n-1}\partial(P_n) \subseteq B_{n-1}X$. For given $x \in P_n$, $\partial f_{n-1}\partial(x) = f_{n-2}\partial^2(x) = 0$, so $f_{n-1}\partial(x) \in Z_{n-1}X$, and

$$\epsilon f_{n-1}\partial(x) = g_{n-1}\partial(x) + \partial s_{n-1}\partial(x) = \partial g_n(x) + \partial s_{n-1}\partial(x)$$

is a boundary, so since $\epsilon_*$ is an isomorphism, $f_{n-1}\partial(x)$ is also a boundary.

Hence by the projectivity of $P_n$, there is a map $f'_n : P_n \to X_n$ such that $\partial f'_n = f_{n-1}\partial$. Now

$$\partial \circ (\epsilon f'_n - g_n) = (\epsilon f_{n-1} - g_{n-1})\partial = \partial s_{n-1}\partial,$$

so $\epsilon f'_n - g_n - s_{n-1}\partial$ has image contained in $Z_nC$. By considering the diagram

$$\begin{tikzcd} & & P_n \arrow[d, "\epsilon f'_n - g_n - s_{n-1}\partial"] \arrow[dll, dashed] \\ & & Z_nC \arrow[d] \\ Z_nX \arrow[r, "\epsilon|_{Z_nX}"'] & Z_nC \arrow[r, two heads] & H_nC \end{tikzcd}$$

we see there is a map $f''_n : P_n \to X_n$ with $\mathrm{Im}(f''_n) \subseteq Z_nX$ and

$$\mathrm{Im}(\epsilon f'_n + \epsilon f''_n - g_n - s_{n-1}\partial) \subseteq B_nC. \tag{1}$$

Let $f_n = f'_n + f''_n$. Then $\partial f_n = f_{n-1}\partial$, as $\partial f'' = 0$. Moreover, (1) guarantees the existence of a map $s_n$ making

$$\begin{tikzcd} & P_n \arrow[d, "\epsilon f_n - g_n - s_{n-1}\partial"] \arrow[dl, dashed, "s_n"'] \\ C_{n+1} \arrow[r, "\partial"'] & C_n \end{tikzcd}$$

commute, completing the inductive step.

We must next show that if $f : P \to X$ is a chain map and $\epsilon f \simeq 0$ then $f \simeq 0$. So suppose $\epsilon f = \partial s + s\partial$, where $s \in \mathrm{Hom}_\Lambda(P, C)_1$. We show by induction on $n$ that there are maps $t_i : P_i \to X_{i+1}$ and $w_i : P_i \to C_{i+2}$ such that

$$f_i = \partial t_i + t_{i-1}\partial \tag{2}$$

$$\epsilon t_i - s_i = \partial w_i + w_{i-1}\partial. \tag{3}$$

Assume we have such maps for $i < n$. Then

$$\partial \circ (f_n - t_{n-1}\partial) = (f_{n-1} - \partial t_{n-1})\partial = (\partial t_{n-1} + t_{n-2}\partial - \partial t_{n-1})\partial = 0,$$

so $\mathrm{Im}(f_n - t_{n-1}\partial) \subseteq Z_nX$. Moreover,

$$\epsilon \circ (f_n - t_{n-1}\partial) = \partial s_n + s_{n-1}\partial - s_{n-1}\partial + \partial w_{n-1}\partial = \partial \circ (s_n + w_{n-1}\partial),$$

hence $\mathrm{Im}(f_n - t_{n-1}\partial) \subseteq B_nX$, as $\epsilon_*$ is an isomorphism. So by the projectivity of $P_n$, there is a map $t_n : P_n \to X_{n+1}$ such that (2) holds with $i = n$.

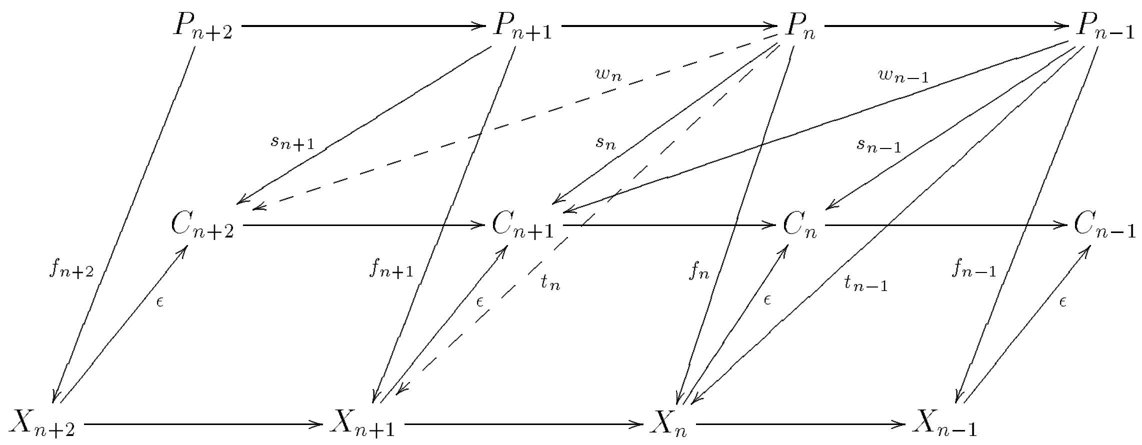


Diagram 3. Proof of the uniqueness part of the Comparison Theorem

We must next "adjust" $t_n$ to show there is a map $w_n$ satisfying (3) with $i = n$. Observe

$$\begin{aligned}\partial \circ (\epsilon t_n - s_n - w_{n-1}\partial) &= \epsilon \circ (f_n - t_{n-1}\partial) - \partial s_n - \partial w_{n-1}\partial \\ &= \partial s_n + s_{n-1}\partial - s_{n-1}\partial + \partial w_{n-1}\partial + w_{n-2}\partial^2 - \partial s_n - \partial w_{n-1}\partial \\ &= 0,\end{aligned}$$

i.e., $\mathrm{Im}(\epsilon t_n - s_n - w_{n-1}\partial) \subseteq Z_{n+1}C$. So arguing as before, there is a map $t'_n : P_n \to Z_{n+1}X$ such that $\mathrm{Im}(\epsilon t'_n + \epsilon t_n - s_n - w_{n-1}\partial) \subseteq B_{n+1}C$. Replace $t_n$ with $t_n + t'_n$ (this does not affect (2) as $\partial t'_n = 0$). We then have, by the projectivity of $P_n$, a map $w_n : P_n \to C_{n+2}$ such that $\partial w_n = \epsilon t_n - s_n - w_{n-1}\partial$, completing the inductive step. □

**2.2. Minimal resolutions.** A projective resolution $P \xrightarrow{\epsilon} C$ is said to be *minimal* if for any projective resolution $X \xrightarrow{\epsilon'} C$ of $C$, $P|X$.

**Theorem 5.** *Let $C \in \mathcal{C}_b$. Then*

(i) *$C$ has a minimal projective resolution.*
(ii) *Let $X \to C$ be an arbitrary projective resolution of $C$. Then the following are equivalent: (a) $X \to C$ is minimal, (b) $\partial(X) \subseteq \mathrm{rad}(X)$, and (c) $X$ has no nontrivial exact summands.*

Because of Theorem 5(i), for any $C \in \mathcal{C}_b$ we may let $\mathrm{MPR}(C)$ denote a minimal projective resolution of $C$. Note that $\mathrm{MPR}(C)$ is unique up to isomorphism of complexes, for if $P$ and $P'$ are both minimal, we have $P|P'|P$, and since each of these is finite-dimensional in each degree, $P \cong P'$. The proof of the theorem requires two lemmas: the first is interesting in its own right, the second is a rather technical fact from homological algebra.

**Lemma 6.** *Let $C \in \mathcal{C}_b$. Suppose $\partial(C) \subseteq \mathrm{rad}(C)$ and $f : C \to C$ is a quasi-isomorphism. Then $f$ is an isomorphism.*

*Proof.* Show by induction on $n$ that $f_n$ is an isomorphism. Since $C_n = 0$ for $n$ sufficiently small, the initial step is trivial. So assume $f_{n-1}$ is an isomorphism. By Fitting's Lemma there is an $m > 0$ such that if we set $g = f^m$, $A = \mathrm{Ker}(g)$, and $B = \mathrm{Im}(g)$, then $C_i = A_i \oplus B_i$ for $i \leq n+1$. Clearly $g_n$ is an isomorphism iff $f_n$ is. Now $g$ induces an automorphism of $H_nC = H_nA \oplus H_nB$, but $g$ is trivial on $A$, hence $H_nA = 0$, i.e., $A_{n+1} \to A_n \to A_{n-1}$ is exact. But by the inductive hypothesis, $A_{n-1} = 0$. Therefore $A_n \subseteq \partial(C_{n+1}) \subseteq \mathrm{rad}(C_n)$. On the other hand, $A_n | C_n$. Hence $A_n = 0$. $\square$

**Lemma 7.** *Let $U$, $V$, $X$, and $Y$ be $\Lambda$-modules, and $f : X \to Y$ and $g : U \oplus V \to Y$ maps of $\Lambda$-modules. Let $i_U : U \to U \oplus V$ and $i_V : V \to U \oplus V$ denote the inclusions. Suppose $V$ is projective and $gi_V(V) \subseteq f(X)$. Let $\tau : V \to X$ be a map satisfying $f\tau = gi_V$, and let*

$$\begin{tikzcd}
M \arrow[r, "\phi"] \arrow[dd, "\sigma"'] & U \arrow[d, hook, "i_U"] \\
& U \oplus V \arrow[d, "g"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

*be a pullback of $(f, gi_U)$. Then*

$$\begin{tikzcd}
M \oplus V \arrow[r, "\phi \oplus \mathrm{id}_V"] \arrow[d, "\sigma+\tau"'] & U \oplus V \arrow[d, "g"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

*is a pullback of $(f, g)$.*

Note that such a map $\tau$ exists by the definition of *projective*.

*Proof.* Suppose $W$ is a $\Lambda$-module and $\delta : W \to X$ and $\epsilon : W \to U \oplus V$ are maps satisfying $f\delta = g\epsilon$. Then there is a commutative diagram

$$\begin{tikzcd}
W \arrow[rr, "\epsilon"] \arrow[dr, dashed, "\gamma_M"] \arrow[ddr, "\delta-\tau\pi_V\epsilon"'] & & U \oplus V \arrow[d, "\pi_U"] \\
& M \arrow[r, "\phi"] \arrow[d, "\sigma"'] & U \arrow[d, "gi_U"] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}$$

as

$$\begin{aligned} f \circ (\delta - \tau\pi_V\epsilon) &= g\epsilon - f\tau\pi_V\epsilon = g\epsilon - gi_V\pi_V\epsilon \\ &= g \circ (\mathrm{id}_V - i_V\pi_V)\epsilon = gi_U\pi_U\epsilon. \end{aligned}$$

So by the definition of pullback there is a unique map $\gamma_M$ making the diagram commute. Let $\gamma_V = \pi_V\epsilon$, and define $\gamma : W \to M \oplus V$ by $\gamma(w) = (\gamma_M(w), \gamma_V(w))$. We then have

$$(\sigma + \tau)\gamma = \sigma\gamma_M + \tau\gamma_V = \delta - \tau\pi_V\epsilon + \tau\pi_V\epsilon = \delta$$

and

$$(\phi \oplus \mathrm{id}_V)\gamma(w) = (\phi\gamma_M(w), \gamma_V(w)) = (\pi_U\epsilon(w), \pi_V\epsilon(w)) = \epsilon(w),$$

as required.

Suppose $\gamma' : W \to M \oplus V$ also satisfies $(\sigma + \tau)\gamma' = \delta$ and $(\phi + \mathrm{id}_V)\gamma' = \epsilon$. Write $\gamma'(w) = (\gamma'_M(w), \gamma'_V(w))$. From

$$(\phi + \mathrm{id}_V)\gamma = \epsilon = (\phi + \mathrm{id}_V)\gamma'$$

we get $\phi\gamma_M = \phi\gamma'_M$ and $\gamma'_V = \gamma_V$. Similarly, from

$$(\sigma + \tau)\gamma = \delta = (\sigma + \tau)\gamma'$$

we get $\sigma\gamma_M = \sigma\gamma'_M$. So by the uniqueness of $\gamma_M$, $\gamma'_M = \gamma_M$, and therefore $\gamma = \gamma'$. □

*Proof of Theorem* 5. Let $X \xrightarrow{\epsilon} C$ be a projective resolution of $C$, and fix a diagram like Diagram 1 for $X$. We first show there is a decomposition $X = P \oplus W$, with $W$ exact and $\partial(P) \subseteq \mathrm{rad}(P)$.

To show this we produce, by induction on $n$, a decomposition $X_n = P_n \oplus W_n$ such that $\partial(W_n) = Z_{n-1}W$, $\epsilon(Z_nW) \subseteq B_nC$, $\alpha|_{P_n}$ is a projective cover of $\alpha(P_n)$, and $\partial(P_n) \subseteq \mathrm{rad}(P_{n-1})$. Since $X_n = 0$ for $n$ sufficiently small, the initial step is trivial. So assume decompositions with these properties have been constructed through degree $n - 1$.

Now $W_{n-1} \to \cdots \to W_0 \to 0$ is an exact sequence of projectives, so $Z_{n-1}W$ is projective. Moreover, $Z_{n-1}X = Z_{n-1}P \oplus Z_{n-1}W$, and $\theta(Z_{n-1}W) \subseteq \delta(C_n/B_nC)$ (as $\epsilon_{n-1}(Z_{n-1}W) \subseteq B_{n-1}C$). Hence by Lemma 7 and the uniqueness of pullbacks, there is a decomposition $PB_n = PB'_n \oplus V$ such that $\gamma = \gamma' \oplus \gamma''$ and $\beta = \beta' + \mu$, where $\gamma'' : V \to Z_{n-1}W$ is an isomorphism and

$$\begin{tikzcd} PB'_n \arrow[r, "\gamma'"] \arrow[d, "\beta'"'] & Z_{n-1}P \arrow[d, "\theta' := \theta|_{Z_{n-1}P}"] \\ \dfrac{C_n}{B_nC} \arrow[r, "\delta"'] & Z_{n-1}C \end{tikzcd}$$

is a pullback.

Since $\alpha : X_n \to PB'_n \oplus V$ is surjective, there is a decomposition $X_n = P_n \oplus W_n$ such that $\alpha' := \alpha|_{P_n}$ is a projective cover of $PB'_n$ and $\alpha(W_n) = V$. Hence $\partial(W_n) = Z_{n-1}W$. To see that $\partial(P_n) \subseteq \mathrm{rad}(P_{n-1})$, recall from the remarks following Proposition 3 that $\mathrm{Ker}(\alpha_{n-1}) = \partial(X_n)$, so in particular the composite

$$P_n \xrightarrow{\alpha'_n} PB'_n \xrightarrow{\gamma'} Z_{n-1}P \xhookrightarrow{\iota'} P_{n-1} \xrightarrow{\alpha'_{n-1}} \alpha(P_{n-1})$$

is trivial, so it certainly induces the trivial map modulo radicals. But $\alpha'_{n-1}$ is a projective cover, and therefore induces an isomorphism modulo radicals, whence we conclude $\partial(P_n) = \iota'\gamma'\alpha'(P_n) \subseteq \mathrm{rad}(P_{n-1})$.

To complete the inductive step we must show $\epsilon(Z_nW) \subseteq B_nC$. But $\iota\gamma''\alpha(Z_nW) = \partial(Z_nW) = 0$, so since $\iota$ and $\gamma''$ are monomorphisms, $\alpha(Z_nW) = 0$. Hence $\eta\epsilon(Z_nW) = \beta\alpha(Z_nW) = 0$, as required.

We now claim that $P \overset{\epsilon|_P}{\to} C$ is a minimal projective resolution. It is certainly a projective resolution, since $H_*W = 0$. Suppose $\tilde{X} \overset{\tilde{\epsilon}}{\to} C$ is any projective resolution of $C$. Then we have just shown that $\tilde{X} = \tilde{P} \oplus \tilde{W}$, where $\partial(\tilde{P}) \subseteq \mathrm{rad}(\tilde{P})$ and $\tilde{P} \overset{\tilde{\epsilon}|_{\tilde{P}}}{\to} C$ is a projective resolution. By the Comparison Theorem, there are quasi-isomorphisms $f : P \to \tilde{P}$ and $g : \tilde{P} \to P$. So $fg$ and $gf$ are quasi-isomorphisms, and by Lemma 6, $fg$ and $gf$ are isomorphisms. Hence $f$ and $g$ are isomorphisms, i.e., $P \cong \tilde{P}$. So $P|\tilde{X}$, establishing the claim, and completing the proof of (i).

The proof of (ii) is now immediate. For if $X \to C$ is minimal then by uniqueness $X \cong P$, and $P$ has property (b). Hence (a) $\Rightarrow$ (b). If (b) holds and $X = C \oplus D$, with $D$ exact, then $X \twoheadrightarrow C \hookrightarrow X$ is a quasi-isomorphism, so by Lemma 6 it is an isomorphism, and therefore $D = 0$. Hence (b) $\Rightarrow$ (c). Finally, the statement (c) $\Rightarrow$ (a) follows directly from the definition of *minimal*. $\square$

If $D \in {}_b\mathcal{C}$, an *injective resolution of* $D$ is a complex $I \in {}_b\mathcal{C}^p$ together with a quasi-isomorphism $\eta : D \to I$ (recall that a $\Lambda$-module is projective iff it is injective). Since $\eta : D \to I$ is an injective resolution iff $\eta^* : I^* \to D^*$ is a projective resolution, all of the statements above concerning projective resolutions have dual versions concerning injective resolutions.

For $C \in \mathcal{C}_b, D \in {}_b\mathcal{C}$, we now define $\mathrm{Ext}^*_\Lambda(C, D) = H^*\mathrm{Hom}_\Lambda(P, D)$, where $P$ is any projective resolution of $C$. By Theorem 4, this is independent of the projective resolution chosen, in the usual sense. We also have the usual canonical isomorphisms $\mathrm{Ext}^*_\Lambda(C, D) \cong H^*\mathrm{Hom}_\Lambda(P, I) \cong H^*\mathrm{Hom}_\Lambda(C, I)$, where $I$ is any injective resolution of $D$. If $P$ is minimal and $D$ is a complex with trivial differential and $\mathrm{rad}(D) = 0$, then by Theorem 5(ii)(b), $\mathrm{Ext}^*_\Lambda(C, D) = \mathrm{Hom}_\Lambda(P, D)$, which is one of the advantages of using minimal resolutions.

**2.3. The hypercohomology spectral sequence.** Recall that a doubly-indexed collection of modules $E_0^{p,q}$ together with maps $d' : E_0^{p,q} \to E_0^{p+1,q}, d'' : E_0^{p,q} \to E_0^{p,q+1}$ forms a *double complex* if $(d')^2 = (d'')^2 = d'd'' + d''d' = 0$. A double complex yields a spectral sequence $\{E_r, d_r\}$ in which the differentials are easy to describe explicitly: $x \in E_0^{p,q}$ lives to $E_n^{p,q}$ iff there exist $x_i \in E_0^{p+i,q-i}$ $(0 \le i < n)$ with $x_0 = x$, $d''(x_0) = 0$, and $d'(x_{i-1}) + d''(x_i) = 0$ for $1 \le i < n$. If this is the case and $x$ represents $\zeta \in E_n^{p,q}$ then $d_n(\zeta)$ is represented by $d'(x_{n-1})$.

Now let $C \in \mathcal{C}_b, D \in {}_b\mathcal{C}$, and let $I_D$ be an injective resolution of $D$. The hypercohomology spectral sequence arises from the double complex defined by $E_0^{p,q} = \mathrm{Hom}_\Lambda(C_q, I_D^p), d'(f) = \partial \circ f, d''(f) = (-1)^{p+q+1} f \circ \partial$. We have

$$E_2^{p,q} \cong \mathrm{Ext}^p_\Lambda(H_q(C), D) \Rightarrow \mathrm{Ext}^{p+q}_\Lambda(C, D).$$

Moreover, if $I$ is an injective resolution of $k$ then the tensor product induces a map of double complexes

$$\mathrm{Hom}_\Lambda(k, I_D) \otimes \mathrm{Hom}_\Lambda(C_\zeta, I) \to \mathrm{Hom}_\Lambda(C_\zeta, I_D \otimes I),$$

and since $I_D \otimes I$ is also an injective resolution of $D$, this map yields a pairing of spectral sequences

$$\begin{array}{ccccc}
\mathrm{Ext}^p_\Lambda(k, D) \otimes & \mathrm{Ext}^{p'}_\Lambda(H_q(C_\zeta), k) & \longrightarrow & \mathrm{Ext}^{p+p'}_\Lambda(H_q(C_\zeta), D) \\
\| & \Downarrow & & \Downarrow \\
\mathrm{Ext}^p_\Lambda(k, D) \otimes & \mathrm{Ext}^{p'+q}_\Lambda(C_\zeta, k) & \longrightarrow & \mathrm{Ext}^{p+p'+q}_\Lambda(C_\zeta, D).
\end{array}$$

We could also start with the double complex $\mathrm{Hom}_\Lambda(P_p, D^q)$, where $P$ is a projective resolution of $C$, to obtain the second hypercohomology spectral sequence

$$E_2^{p,q} \cong \mathrm{Ext}^p_\Lambda(C, H^q(D)) \Rightarrow \mathrm{Ext}^{p+q}_\Lambda(C, D).$$

As a first consequence of these two spectral sequences, we obtain

**Lemma 8.** *Suppose $C, D \in {}_b\mathcal{C}_b$. Then $\mathrm{Ext}^*_\Lambda(C, D)$ is a finitely generated $\mathrm{Ext}^*_\Lambda(k, k)$-module.*

*Proof.* We first reduce to the case where $C$ is a module concentrated in degree 0 as follows. In the hypercohomology spectral sequence we have $E_2$ is a finite direct sum $\bigoplus_q \operatorname{Ext}^*_\Lambda(H_q(C), D)$, so $E_2$ is finitely generated over $\operatorname{Ext}^*_\Lambda(k,k)$. The spectral sequence stops (it has a finite number of non-zero rows), so $E_\infty$ is also finitely generated over $\operatorname{Ext}^*_\Lambda(k,k)$, and this implies that $\operatorname{Ext}^*_\Lambda(C,D)$ is as well by [7, Lemma 7.4.5]. Now if $C$ is a module concentrated in degree 0 then apply the second spectral sequence to reduce to the case where both $C$ and $D$ are modules. $\square$

**2.4. Complexes of finite projective dimension.** Let $C \in \mathcal{C}_b$. We say that $C$ has *finite projective dimension* if $C$ has a bounded projective resolution, i.e., a projective resolution $P \to C$ with $P \in {}_b\mathcal{C}^p_b$. Dually, for $C \in {}_b\mathcal{C}$, we say that $C$ has *finite injective dimension* if $C$ has a bounded injective resolution.

**Lemma 9.** *Let $C \in {}_b\mathcal{C}_b$. Then the following are equivalent:*

(i) *$C$ has finite projective dimension.*
(ii) *$C$ has finite injective dimension.*
(iii) $\operatorname{Ext}^n_\Lambda(C,C) = 0$ *for $n$ sufficiently large.*
(iv) *$C = C' \oplus C''$, where $C' \in {}_b\mathcal{C}^p_b$, $C'' \in {}^e_b\mathcal{C}_b$, and $C'$ has no exact summands.*

Note that for such a $C$ it follows from Theorem 5(ii) that the inclusion $C' \hookrightarrow C$ is a minimal projective resolution of $C$.

*Proof.* It is clear that (i)$\Rightarrow$(iii). Conversely, if (iii) holds, then for any simple $\Lambda$-module $S$, Lemma 8 implies $\operatorname{Ext}^*_\Lambda(C,S)$ is finitely generated over $\operatorname{Ext}^*_\Lambda(k,k)$, so it is certainly finitely generated over $\operatorname{Ext}^*_\Lambda(C,C)$ (here we are using the fact that the cup and Yoneda products are compatible, cf. [6, Lemma 2.2]). Hence $\operatorname{Ext}^*_\Lambda(C,S)$ is finite-dimensional, i.e., bounded. Let $P \to C$ be a minimal resolution of $C$. Since there are only finitely many simple $\Lambda$-modules, this means that there is an integer $N$ such that $\operatorname{Hom}_\Lambda(P_n, S) = \operatorname{Ext}^n_\Lambda(C,S) = 0$ for all simples $S$ and $n > N$. Hence $P_n = 0$ for all $n > N$, i.e. (i) holds. Proceeding dually, we obtain (ii)$\Leftrightarrow$(iii).

Now suppose (i), (ii), and (iii) hold. Let $P \xrightarrow{\epsilon} C$ be a minimal projective resolution of $C$, $C \xrightarrow{\eta} I$ a minimal injective resolution, and let $f = \eta\epsilon$. Then $f : P \to I$ is a quasi-isomorphism, and to prove (iv) it suffices to show that $f$ is an isomorphism. But one may extend $f$ to a surjective map $P \oplus W \to I$, for some $W \in {}^e_b\mathcal{C}^p_b$. Let $K$ be the kernel of this map. Then $K$ is a bounded complex of projectives, and by considering the long exact sequence in homology arising from the short exact sequence

$$0 \to K \to P \oplus W \to I \to 0,$$

we conclude that $H_*K = 0$, i.e. $K \in {}^e_b\mathcal{C}^p_b$. This means that $K$ is isomorphic to the direct sum of complexes of the form $\cdots \to 0 \to N \xrightarrow{=} N \to 0 \to \cdots$, where $N$ is an injective (i.e. projective) $\Lambda$-module, each of which is easily seen to be an injective object in $\mathcal{C}$. Hence $K$ is an injective object in $\mathcal{C}$, so the short exact sequence above splits, and $P \oplus W \cong K \oplus I$. Now $K$ and $W$ contain only exact summands, while $P$

and $I$ have no exact summands, so by the Krull-Schmidt Theorem, $P \cong I$. Fix an isomorphism $\theta : I \to P$. Then $\theta f : P \to P$ is a quasi-isomorphism, so by Lemma 6, $\theta f$, and therefore $f$, is an isomorphism, and we have (iv). Finally, (iv)$\Rightarrow$(i), as $C' \hookrightarrow C$ is a (minimal) projective resolution of $C$. $\square$

The following is a direct generalization of Lemma 5.1 of [6].

**Proposition 10.** *Suppose $C, D \in {}_b\mathcal{C}_b$ have finite projective dimension, and $f : C \to D$ is a quasi-isomorphism. Then there are decompositions $C = C' \oplus C''$ and $D = D' \oplus D''$, with $C', D' \in {}_b\mathcal{C}_b^p$ and $C'', D'' \in {}_b^e\mathcal{C}_b$, such that $f|_{C'}$ is an isomorphism onto $D'$.*

*Proof.* Write $C = C' \oplus C'', D = D' \oplus D''$ as in Lemma 9 (iv), and let $i : C' \hookrightarrow C$ denote inclusion and $\pi : D \to D'$ the projection. It suffices to show there exists a map $g : D \to C'$ such that $gfi = \mathrm{id}_{C'}$ (by replacing $D'$ with $fi(C')$ and $D''$ with $\mathrm{Ker}(g)$). Now $h = \pi fi : C' \to D'$ is a quasi-isomorphism. Moreover, $C'$ and $D'$ are both minimal projective resolutions of $D$, and are therefore isomorphic. So we may apply Lemma 6 to conclude that $h$ is an isomorphism. Let $g = h^{-1}\pi$. Then $gfi = h^{-1}\pi fi = h^{-1}h = \mathrm{id}_{C'}$. $\square$

Now suppose $C, D \in {}_b\mathcal{C}_b$. We write $C \sim_d D$ if there is a sequence of quasi-isomorphisms

$$C \to X_1 \leftarrow X_2 \to \cdots \leftarrow X_n \to D$$

with each $X_i \in {}_b\mathcal{C}_b$. (This just means that $C$ and $D$ are isomorphic objects in the derived category.) It is clear that $\sim_d$ is an equivalence relation.

**Proposition 11.** *Suppose $C, D \in {}_b\mathcal{C}_b$ and $C \sim_d D$. If $C$ has finite projective dimension then so does $D$, and* $\mathrm{MPR}(C) \cong \mathrm{MPR}(D)$.

*Proof.* If $X \to Y$ is a quasi-isomorphism and $P \to X$ is a bounded projective resolution of $X$ then $P \to X \to Y$ is a bounded projective resolution of $Y$, and by Proposition 10, $\mathrm{MPR}(X) \cong \mathrm{MPR}(Y)$. Dually, if $X \leftarrow Y$ is a quasi-isomorphism and $I \leftarrow X$ is a bounded injective resolution of $X$, then $I \leftarrow X \leftarrow Y$ is a bounded injective resolution of $Y$, and by Proposition 10, $\mathrm{MPR}(X) \cong \mathrm{MPR}(Y)$. So the proof follows by induction on the length $n$ of a chain of quasi-isomorphisms joining $C$ and $D$. $\square$

## 3. Generalized Benson-Carlson duality

**3.1. The main theorem.** Let $U$ and $V$ be $\Lambda$-modules and $n$ a positive integer. An $n$-extension of $U$ by $V$ is a complex $e \in {}_b^e\mathcal{C}_b$ with $e_{-1} = U, e_n = V$, and $e_r = 0$ if $r > n$ or $r < -1$. If $\tilde{e}$ is also an $n$-extension of $U$ by $V$ we write $e \rightsquigarrow \tilde{e}$ if there exists $f \in Z_0\mathrm{Hom}_\Lambda(e, \tilde{e})$ with $f_{-1} = \mathrm{id}_U, f_n = \mathrm{id}_V$. We complete $\rightsquigarrow$ to an equivalence relation and let $\mathrm{YExt}^n_\Lambda(U, V)$ denote the set of equivalence classes. There is a functorial bijection $\mathrm{YExt}^n_\Lambda(U,V) \longrightarrow \mathrm{Ext}^n_\Lambda(U,V)$ (cf. Hilton and Stammbach [8, IV.9]).

Let $n \geq 2$. A *truncated $n$-extension of $U$ by $V$* is a complex $C \in {}_b\mathcal{C}_b$ with $C_i = 0$ if $i > n-1$ or $i < 0$, and with $H_i(C)$ isomorphic to $U$ if $i = 0$, $V$ if $i = n-1$, and 0 otherwise. If $U$ and $V$ are simple, then it is easily seen that the bijection referred to above induces a bijection between equivalence classes of truncated $n$-extensions of $U$ by $V$ under $\sim_d$ and $P(\mathrm{Ext}^n_\Lambda(U,V))$, the projective space of $\mathrm{Ext}^n_\Lambda(U,V)$. This follows from Schur's Lemma, which states that the endomorphism ring of $U$ or $V$ consists of scalar multiples of the identity, which implies that a truncated $n$-extension determines an $n$-extension only up to non-0 scalar multiple.

We can now state the main theorem, which generalizes Theorem 5.5 of [6]. Let $D \in {}_b\mathcal{C}_b$ and let $R$ be the quotient of $\mathrm{Ext}^*_\Lambda(k,k)$ by the annihilator of $\mathrm{Ext}^*_\Lambda(D,D)$. Recall that non-zero homogeneous elements of positive degree $\zeta_1, \ldots, \zeta_d \in \mathrm{Ext}^*_\Lambda(k,k)$ are said to form a *homogeneous system of parameters* (h.s.o.p.) for $\mathrm{Ext}^*_\Lambda(D,D)$ if their images in $R$ generate a subring over which $R$ is finitely generated as a module.

**Theorem 12.** *Let $k$ be an algebraically closed field and $\Lambda$ a finite-dimensional co-commutative Hopf algebra over $k$ with the finiteness property in cohomology. Let $D \in {}_b\mathcal{C}_b$, and $\zeta_1, \ldots, \zeta_r$ a h.s.o.p. for $\mathrm{Ext}^*_\Lambda(D,D)$. Assume $n_i = \deg(\zeta_i) \geq 2$ for all $i$. Let $C_{\zeta_i}$ be a truncated $n_i$-extension representing $[\zeta_i]$, and let $C = \bigotimes_i C_{\zeta_i} \otimes D$. Then*

(i) *There is a decomposition $C = N \oplus Q$, where $N \in {}_b\mathcal{C}^p_b$, $Q \in {}^e_b\mathcal{C}_b$, and $N$ has no exact summands.*
(ii) *If $D^*[t] \sim_d D$ for an integer $t$, then $N^*[s] \cong N$, where $s = t + \sum_{i=1}^r (n_i - 1)$.*
(iii) *Up to isomorphism of complexes, $N$ is independent of the choices $C_{\zeta_i}$ of truncated $n_i$-extension representing $[\zeta_i]$*

A key observation is the following Lemma, which is essentially [6, Proposition 5.2]. Given $\zeta \in \mathrm{Ext}^n_\Lambda(k,k)$, let $e_\zeta$ be an $n$-extension of $k$ by $k$ representing $\zeta$. Then $e^*_\zeta[n-1]$ is also an $n$-extension of $k$ by $k$, and therefore represents some element $\zeta^*$ of $\mathrm{Ext}^n_\Lambda(k,k)$. Since equivalent extensions are taken to equivalent extensions by this operation, we have a well-defined operation $\zeta \mapsto \zeta^*$ on $\mathrm{Ext}^n_\Lambda(k,k)$.

**Lemma 13.** *For $n \geq 1$ and $\zeta \in \mathrm{Ext}^n_\Lambda(k,k)$, $\zeta^* = \pm\zeta$. In particular, if $C_\zeta$ is a truncated $n$-extension representing $[\zeta]$, then $(C_\zeta)^*[n-1] \sim_d C_\zeta$.*

*Proof.* Let $P \to k$ be a projective resolution of $k$. After identifying $P$ with $P^{**}$ in the usual way, there is an endomorphism $\theta$ of $\mathrm{Hom}_\Lambda(P,P^*)$ defined by $\theta(f) = f^*$. We claim that for $f \in Z\mathrm{Hom}_\Lambda(P,P^*)$, $f \simeq f^*$. To see this, reason as follows: $\mathrm{Hom}_\Lambda(P,P^*) \cong \mathrm{Hom}_\Lambda(P \otimes_k P, k)$, and $\theta$ is induced by the twisting endomorphism $\tau$ of $P \otimes_k P$. By the Comparison Theorem, $\tau$ is homotopic to the identity on $P \otimes_k P$, hence $\theta$ is homotopic to the identity on $\mathrm{Hom}_\Lambda(P,P^*)$. Say $\theta - 1 = \partial s + s\partial$. Then given $f \in Z\mathrm{Hom}_\Lambda(P,P^*)$, $\partial(f) = 0$, hence $\theta(f) - f = \partial(s(f))$, i.e., $\theta(f) \simeq f$.

Now $\mathrm{Ext}^*_\Lambda(k,k) \cong H^*(\mathrm{Hom}_\Lambda(P,P^*))$, as $k \to P^*$ is an injective resolution of $k$. Fix an $n$-extension $e$ representing $\zeta$. Then by projectivity and injectivity, there exist maps $f_i, g_i$ making the following diagram, in which the middle row is $e$, commute:

$$\begin{tikzcd}
P_{n+1} \arrow[r] \arrow[d] & P_n \arrow[r] \arrow[d, "f_n"'] & P_{n-1} \arrow[r] \arrow[d, "f_{n-1}"'] & P_{n-2} \arrow[r] \arrow[d, "f_{n-2}"'] & \cdots \arrow[r] & P_1 \arrow[r] \arrow[d, "f_1"'] & P_0 \arrow[r, "\epsilon"] \arrow[d, "f_0"'] & k \arrow[r] \arrow[d, equal] & 0 \arrow[d] \\
0 \arrow[r] \arrow[d] & k \arrow[r] \arrow[d, equal] & M_{n-1} \arrow[r] \arrow[d, "g_{n-1}"'] & M_{n-2} \arrow[r] \arrow[d, "g_{n-2}"'] & \cdots \arrow[r] & M_1 \arrow[r] \arrow[d, "g_1"'] & M_0 \arrow[r] \arrow[d, "g_0"'] & k \arrow[r] \arrow[d, "g_{-1}"'] & 0 \arrow[d] \\
0 \arrow[r] & k \arrow[r, "\epsilon^*"] & P^*_0 \arrow[r] & P^*_1 \arrow[r] & \cdots \arrow[r] & P^*_{n-2} \arrow[r] & P^*_{n-1} \arrow[r] & P^*_n \arrow[r] & P^*_{n+1}
\end{tikzcd}$$

Let $f = f_n, g = g_{-1}$. Now $\epsilon^* f \in Z^n\mathrm{Hom}_\Lambda(P,P^*)$ represents $\zeta$. But the diagram shows that $\pm g\epsilon \simeq \epsilon^* f$, hence $\pm g\epsilon$ also represents $\zeta$. Taking the dual diagram, we see that $\epsilon^* g^*$ represents $\pm\zeta^*$. But as was shown in the previous paragraph, $\epsilon^* g^* = \pm(g\epsilon)^* \simeq g\epsilon$. □

The next lemma involves the hypercohomology spectral sequence (see Section 2.3). Let $n \geq 2$, $\zeta \in \mathrm{Ext}^n_\Lambda(k,k)$, and $C_\zeta$ a truncated $n$-extension representing $[\zeta]$. Extend $C_\zeta$ to an $n$-extension representing $\zeta$ with maps $\eta : k \to (C_\zeta)_{n-1}$ and $\epsilon : (C_\zeta)_0 \to k$. This yields identifications $H_{n-1}(C_\zeta) = k$ and $H_0(C_\zeta) = k$ and therefore also $E_2^{0,n-1} = \mathrm{Ext}^0_\Lambda(H_{n-1}(C_\zeta), k) = k$ and $E_2^{n,0} = \mathrm{Ext}^n_\Lambda(H_0(C_\zeta), k) = \mathrm{Ext}^n_\Lambda(k,k)$. Define $\tilde{\zeta} \in E_2^{0,n-1}$ to be the element corresponding to $1 \in k$.

**Lemma 14.** *Let $D \in {}_b\mathcal{C}$. Then in the two-row spectral sequence*

$$E_2^{pq}(\zeta) = \mathrm{Ext}^p_\Lambda(H_q(C_\zeta), D) \Rightarrow \mathrm{Ext}^{p+q}_\Lambda(C_\zeta, D)$$

*the differential $d_n$ is given by*

$$d_n(\alpha\tilde{\zeta}) = \pm\alpha\zeta \in E_n^{p+n,0}(\zeta) = \mathrm{Ext}^{p+n}_\Lambda(k, D).$$

*Proof.* If we can show that $d_n(\tilde{\zeta}) = \zeta$ in $E_n(k)$ then we are done, using the fact that $d_n$ is a derivation. So without loss of generality we assume $D = k$. By injectivity there are maps $f_i, \hat{\zeta}$ making the following diagram commute:

$$\begin{tikzcd}
0 \arrow[r] & k \arrow[r, "\eta"] \arrow[d, equal] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"'] & C_{n-2} \arrow[r] \arrow[d, "f_{n-2}"'] & \cdots \arrow[r] & C_1 \arrow[r] \arrow[d, "f_1"'] & C_0 \arrow[r, "\epsilon"] \arrow[d, "f_0"'] & k \arrow[r] \arrow[d, "\hat{\zeta}"] & 0 \arrow[d] & \\
0 \arrow[r] & k \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & \cdots \arrow[r] & I^{n-2} \arrow[r] & I^{n-1} \arrow[r] & I^n \arrow[r] & I^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

Now $\hat{\zeta}$ represents $\zeta$ (this is just the bijection between $\mathrm{YExt}^n_\Lambda(k,k)$ and $\mathrm{Ext}^n_\Lambda(k,k)$) and $\tilde{\zeta}$ is represented in $E_0^{0,n-1}$ by $f_{n-1}$. By definition of the differential, $\pm d_n(\tilde{\zeta})$ is represented by $\pm\partial \circ f_0$. By commutativity this is $\hat{\zeta} \circ \epsilon$, which represents $\zeta$ under the identification $\mathrm{Ext}^n_\Lambda(k,k) = \mathrm{Ext}^n_\Lambda(H_0(C), k)$. □

*Proof of Theorem* 12. To prove (i), show $\mathrm{Ext}^n_\Lambda(C,C)=0$ for $n$ sufficiently large and apply Lemma 9. To do this proceed as in the proof of Theorem 4.1 of [6] and show that the $E_\infty$-term of the appropriate spectral sequence is a finite module over $\mathrm{Ext}^*_\Lambda(k,k)/(\zeta_1,\dots,\zeta_d)$, using Lemma 14.

Now if $A, A', B, B' \in {}_b\mathcal{C}_b$ and $A \sim_d A'$ and $B \sim_d B'$ then $A\otimes B \sim_d A'\otimes B'$. So Lemma 13 implies $C^*[s] \sim_d C$. Hence by Proposition 11, the minimal projective resolutions of $C^*[s]$ and $C$ are isomorphic. Now $C \cong N \oplus Q$, so $C^*[s] \cong N^*[s] \oplus Q^*[s]$. On the other hand, $C^*[s] \cong N \oplus Q'$, for some $Q' \in {}^e_b\mathcal{C}_b$. So by the Krull-Schmidt Theorem, $N \cong N^*[s]$, proving (ii).

If we choose different truncated extensions $C'_{\zeta_i}$ representing the $[\zeta_i]$, then $C'_{\zeta_i} \sim_d C_{\zeta_i}$ for each $i$, and therefore $\bigotimes_i C'_{\zeta_i} \sim_d C$. Hence their minimal projective resolutions are isomorphic, proving (iii). □

*Example.* Suppose $k$ has characteristic 2, $G$ is the alternating group on 4 letters, and $\Lambda = kG$. We have (cf. [3, p. 197])

$$H^*(G,k) = k[u,v,w \mid \deg(u)=2, \deg(v)=\deg(w)=3, u^3+v^2+vw+w^2=0].$$

The simple $\Lambda$-modules are $k, S, T$ where $S$ and $T$ also are one-dimensional (corresponding to the third roots of 1 in $k$) and we have $S^* \cong T$. The projective covers of the simples have Loewy structures

$$P_k = \begin{matrix} & k & \\ S & & T \\ & k & \end{matrix}, \quad P_S = \begin{matrix} & S & \\ k & & T \\ & S & \end{matrix}, \quad P_T = \begin{matrix} & T & \\ k & & S \\ & T & \end{matrix}.$$

For a h.s.o.p. we may take $u$ and $v$, and these are represented by extensions of the form

$$\begin{array}{llllllllllll} u: & & 0 & \to & k & \to & \begin{matrix} S \\ k \end{matrix} & \to & \begin{matrix} k \\ S \end{matrix} & \to k & \to & 0 \\ v: & 0 \to k & \to & \begin{matrix} S\ \ T \\ k \end{matrix} & & \to & \begin{matrix} T \\ S \end{matrix} \oplus \begin{matrix} S \\ T \end{matrix} & \to & \begin{matrix} k \\ S\ \ T \end{matrix} & \to k & \to & 0 \end{array}$$

One then gets $C = C_u \otimes C_v$ decomposes as a direct sum of a complex $N$ of projectives

$$P_k \to P_S \oplus P_T \to P_T \oplus P_S \to P_k,$$

which indeed satisfies $N^*[3] \cong N$, and an exact complex

$$\begin{matrix} T \\ S \end{matrix} \to \begin{matrix} T \\ S \end{matrix} \oplus \begin{matrix} T \\ k \end{matrix} \oplus \begin{matrix} k \\ T \end{matrix} \to \begin{matrix} T \\ k \end{matrix} \oplus \begin{matrix} k \\ T \end{matrix} \oplus \begin{matrix} S \\ T \end{matrix} \to \begin{matrix} S \\ T \end{matrix}.$$

**3.2. Further applications.** One may also apply the techniques described above to Benson and Carlson's work on the construction of projective resolutions as tensor products of periodic complexes [5]. We review this briefly here.

Let $n \geq 2$, $\zeta \in \operatorname{Ext}^n_\Lambda(k,k)$, and $P \to k$ the minimal resolution. One constructs a special $n$-extension $\acute{e}_\zeta$ representing $\zeta$ as the bottom row in the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & P_n \arrow[r] \arrow[d, "\zeta"'] & P_{n-1} \arrow[r] \arrow[d] & P_{n-2} \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[d, equal] & k \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & k \arrow[r] & M_\zeta \arrow[r] & P_{n-2} \arrow[r] & \cdots \arrow[r] & P_0 \arrow[r] & k \arrow[r] & 0
\end{tikzcd}$$

where the square with $M_\zeta$ in the bottom right is a pushout. Let $\acute{C}_\zeta$ denote the truncated $n$-extension arising from this, and let $\acute{C}^{(\infty)}_\zeta$ denote the complex in $\mathcal{C}_b$ formed by splicing together infinitely many copies of $\acute{C}_\zeta$.

Now let $M$ be a $\Lambda$-module and $\zeta_1, \ldots, \zeta_r$ a h.s.o.p. for $\operatorname{Ext}^*_\Lambda(M,M)$ with $n_i = \deg(\zeta_i) \geq 2$ for each $i$. Using variety theory, one can show that $M \otimes M_{\zeta_1} \otimes \cdots \otimes M_{\zeta_r}$ is projective. Since the tensor product of a projective $\Lambda$-module with any $\Lambda$-module is also projective, it follows that $\acute{X} = M \otimes \acute{C}^{(\infty)}_{\zeta_1} \otimes \cdots \otimes \acute{C}^{(\infty)}_{\zeta_r} \in \mathcal{C}^p_b$. Moreover, the Künneth Theorem implies that $\acute{X}$ is exact in positive degrees, and that $\acute{X}_1 \to \acute{X}_0 \to M \to 0$ is exact (where the map $\acute{X}_0 \to M$ is formed by tensoring the identity on $M$ with the augmentations from $\acute{C}_{\zeta_i} \to k$). Hence

**Theorem 15 (Benson-Carlson).** *$M \otimes \acute{C}^{(\infty)}_{\zeta_1} \otimes \cdots \otimes \acute{C}^{(\infty)}_{\zeta_r}$ is a projective resolution of $M$.*

We can generalize this as follows.

**Theorem 16.** *Let $D \in {}_b\mathcal{C}_b$, $\zeta_1, \ldots, \zeta_r$ a h.s.o.p. for $\operatorname{Ext}^*_\Lambda(D,D)$ such that $n_i = \deg(\zeta_i) \geq 2$ for all $i$, and for each $i$ let $C_{\zeta_i}$ be a truncated $n_i$-extension representing $[\zeta_i]$. Let $X = D \otimes C^{(\infty)}_{\zeta_1} \otimes \cdots \otimes C^{(\infty)}_{\zeta_r}$. Then there is a decomposition $X = Y \oplus Z$, where $Y$ is exact and $Z$ is a projective resolution of $D$.*

*Proof.* It is easily seen that the special extension $\acute{e}_\zeta$ has the following property: if $e_\zeta$ is any $n$-extension representing $\zeta$ then $\acute{e}_\zeta \rightsquigarrow e_\zeta$. On the truncated level, this means that if $C_\zeta$ is any truncated $n$-extension representing $[\zeta]$ then there is a quasi-isomorphism $\acute{C}_\zeta \to C_\zeta$. Now let $\grave{C}_\zeta = \acute{C}^*_\zeta[n-1]$. By Lemma 13, $\grave{C}_\zeta$ also represents $[\zeta]$, and it is equally easy to see that $\grave{C}_\zeta$ enjoys the following dual property: if $C_\zeta$ is any truncated $n$-extension representing $\zeta$ then there is a quasi-isomorphism $C_\zeta \to \grave{C}_\zeta$. Clearly, Theorem 15 remains true if each $\acute{C}_\zeta$ is replaced by $\grave{C}_\zeta$.

Now we may splice together quasi-isomorphisms to get quasi-isomorphisms $\acute{C}^{(\infty)}_{\zeta_i} \to C^{(\infty)}_{\zeta_i} \to \grave{C}^{(\infty)}_{\zeta_i}$. Tensoring over $i$ yields quasi-isomorphisms $\acute{X} \to X \to \grave{X}$. But both $\acute{X}$ and $\grave{X}$ are projective resolutions of $D$, so there exist quasi-isomorphisms

$P_D \to \acute{X}$ and $\grave{X} \to P_D$, where $P_D$ is the minimal resolution of $D$. Composing, we have quasi-isomorphisms $P_D \to X \to P_D$. By Lemma 6, this composition must be an isomorphism. Hence $P_D$ splits off of $X$. $\square$

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