GENERALIZED BENSON-CARLSON DUALITY

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1. Introduction

1.1. Background. This paper deals with the landmark results of Benson and Carlson's *Projective resolutions and Poincaré duality complexes* [6]. Our goal is to provide some necessary background for that work, and to prove some of the results of [6] in a more general setting. In particular, we analyze what happens when we replace Benson and Carlson's complex C_ζ (in the notation of [6]) with an arbitrary Yoneda extension representing ζ.

Let Λ be a finite-dimensional cocommutative Hopf algebra over an algebraically closed field k. Because Λ is cocommutative, $H^*(\Lambda, k)$ is a graded-commutative k-algebra (i.e., $xy = (-1)^{\deg(x)\deg(y)}yx$ for homogeneous x, y). Throughout this paper we will also assume that Λ has the following *finiteness property in cohomology*: $\mathrm{Ext}^*_\Lambda(k, k)$ is a finitely generated k-algebra, and for any Λ-modules M and N, $\mathrm{Ext}^*_\Lambda(M, N)$ is finitely generated as an $\mathrm{Ext}^*_\Lambda(k, k)$-module. (By Λ-*module* we always mean *finitely generated left* Λ-*module*.) The second condition is equivalent to requiring $H^*(\Lambda, M)$ to be finitely generated over $H^*(\Lambda, k)$ for any Λ-module M, because of the isomorphism $\mathrm{Ext}_\Lambda(M, N) \cong H^*(\Lambda, \mathrm{Hom}_k(M, N))$.

If Λ is the group algebra of a finite group then Λ certainly has the finiteness property in cohomology, by a theorem of Evens (cf. [7, Theorem 7.4.1]). The finiteness property also holds if Λ is a finite-dimensional cocommutative *connected* Hopf algebra, e.g. the restricted enveloping algebra of a p-restricted Lie algebra (Bajer and Sadofsky [1, Lemma 6.2], Wilkerson [10]). In fact, at present we know of no example of a finite-dimensional Hopf algebra without this property. It is known that any finite-dimensional Hopf algebra is a Frobenius algebra (Larson and Sweedler [9]), which implies that a Λ-module is projective if and only if it is injective, and this property is fundamental for the proofs given here.

1.2. Preliminaries. We use the following notation for chain complexes of modules over a k-algebra R. If C is a complex of left or right R-modules, let $Z_nC = \mathrm{Ker}(\partial : C_n \to C_{n-1})$, $B_nC = \mathrm{Im}(\partial : C_{n-1} \to C_n)$, and $H_nC = Z_nC/B_nC$. If C, D are complexes of right, left R-modules respectively, let $C \otimes_R D$ denote the complex with $(C \otimes_R D)_n = \bigoplus_{p+q=n} C_p \otimes_R C_q$, with differential given by $\partial(x \otimes y) = \partial(x) \otimes y + (-1)^p x \otimes \partial(y)$ for $x \in C_p, y \in C_q$. If C and D are both complexes of left (resp. right) R-modules, then let $\mathrm{Hom}_R(C, D)$ denote the complex with $\mathrm{Hom}_R(C, D)_n = \prod_{q=p+n} \mathrm{Hom}_R(C_p, D_q)$, with differential given by $\partial(f) = \partial \circ f - (-1)^n f \circ \partial$, for $f \in \mathrm{Hom}_R(C, D)_n$. If $f - g \in B_n\mathrm{Hom}_R(C, D)$ we write $f \simeq g$ (f and g are *chain homotopic*). For any integer r, $C[r]$ denotes the complex with $C[r]_p = C_{p-r}$, and with differential $(-1)^r\partial$. The dual complex C^* is defined to be $\mathrm{Hom}_k(C, k)$, where k is considered to be a complex concentrated in degree 0; if $f \in \mathrm{Hom}_R(C, D)_r$ then $f^* \in \mathrm{Hom}_R(D^*, C^*)_r$ is defined by $f^*(\alpha) = (-1)^{rs}\alpha \circ f$ for $\alpha \in C^*_s$. We write $C|D$ if C is isomorphic to a summand of D (as complexes). Finally, any chain complex may be considered a cochain complex, and *vice-versa*, by setting $C^n = C_{-n}$.

Let $\mathcal{C}$ denote the category of complexes of Λ-modules. Let $\mathcal{C}_b$, ${}_b\mathcal{C}$, $\mathcal{C}^p$, ${}^e\mathcal{C}$ denote, respectively, the full subcategories of $\mathcal{C}$ of complexes bounded below, complexes bounded above, complexes where each C_n is Λ-projective, and exact complexes. These adornments on the symbol $\mathcal{C}$ may be combined, so that, for example, the objects of ${}^e_b\mathcal{C}^p_b$ are bounded (above and below) exact complexes of projectives. The following are fundamental; for proofs, see Benson [4, Lemma 1.4.4, Theorem 1.4.3], replacing "module" with "bounded complex" throughout.

Lemma 1 (Fitting). *Let $M \in {}_b\mathcal{C}_b$ and $f \in \mathrm{End}_{\mathcal{C}}(M)$. Then $M = \mathrm{Im}(f^n) \oplus \mathrm{Ker}(f^n)$ for n sufficiently large.*

Theorem 2 (Krull-Schmidt). *Let $M \in {}_b\mathcal{C}_b$ and let $M = \bigoplus_{i=1}^m M^{(i)}$, where each $M^{(i)}$ is an indecomposable subcomplex of M. If $M = \bigoplus_{i=1}^n \tilde{M}^{(i)}$ for some other indecomposable subcomplexes $\tilde{M}^{(i)}$ of M, then $n = m$, and after renumbering if necessary, $\tilde{M}^{(i)} \cong M^{(i)}$ for each i.*

2. Projective resolutions of complexes

2.1. Definition and basic properties. Let $C \in \mathcal{C}_b$. A *projective resolution of C* is a complex $P \in \mathcal{C}^p_b$ together with a quasi-isomorphism $\epsilon : P \to C$.

Proposition 3. *If $C \in \mathcal{C}_b$ then C has a projective resolution.*

Proof. We construct P_n and ϵ_n by induction on n. These may be taken to be 0 for n sufficiently small since C is bounded below. So assume that P_m and ϵ_m have been constructed and satisfy $\partial\epsilon_m = \epsilon_{m-1}\partial$ for $m \leq n$. We refer to Diagram 1, leaving off subscripts of maps whenever this is unambiguous. Let PB_{n+1} be the pullback of (δ, θ) in the diagram, and let α_{n+1} be a surjection from a projective module P_{n+1}

$$\begin{tikzcd}
P_{n+1} \arrow[rrr, "\partial"] \arrow[dr, "\alpha_{n+1}"] \arrow[ddd, "\epsilon_{n+1}"'] & & & P_n \arrow[rrr, "\partial"] \arrow[dr, "\alpha_n"] \arrow[ddd, "\epsilon_n"] & & & P_{n-1} \arrow[ddd, "\epsilon_{n-1}"] \\
& PB_{n+1} \arrow[r, "\gamma_{n+1}"] \arrow[d, "\beta_{n+1}"'] & Z_nP \arrow[ur, hook, "\iota_n"] \arrow[d, "\theta_n"] & & PB_n \arrow[r, "\gamma_n"] \arrow[d, "\beta_n"'] & Z_{n-1}P \arrow[ur, hook, "\iota_{n-1}"] \arrow[d, "\theta_{n-1}"] & \\
& \dfrac{C_{n+1}}{B_{n+1}C} \arrow[r, "\delta_{n+1}"] & Z_nC \arrow[dr, hook, "\kappa_n"] & & \dfrac{C_n}{B_nC} \arrow[r, "\delta_n"] & Z_{n-1}C \arrow[dr, hook, "\kappa_{n-1}"] & \\
C_{n+1} \arrow[rrr, "\partial"] \arrow[ur, "\eta_{n+1}"] & & & C_n \arrow[rrr, "\partial"] \arrow[ur, "\eta_n"] & & & C_{n-1}
\end{tikzcd}$$

Diagram 1. Construction of a projective resolution.

to PB_{n+1}. By projectivity there is a map $\epsilon_{n+1} : P_{n+1} \to C_{n+1}$ making the diagram commute. Define $\partial : P_{n+1} \to P_n$ to be $i\gamma\alpha$; the diagram shows that $\partial\epsilon_{n+1} = \epsilon_n\partial$.

To show that $(\epsilon_*)_n$ is surjective, let $x \in Z_nC$. Then $\eta(x) \in \mathrm{Ker}(\delta)$, so it follows from the universal property of the pullback that there exists $y \in PB_n$ such that $\beta(y) = \eta(x)$ and $\gamma(y) = 0$. Choose $z \in P_n$ such that $\alpha(z) = y$. Then $\partial(z) = \iota\gamma\alpha(z) = 0$, so $z \in Z_nP$, and $\eta\epsilon_n(z) = \beta\alpha(z) = \eta(x)$, so $\epsilon_n(z) - x \in B_nC$.

To show that $(\epsilon_*)_n$ is injective, let $x \in Z_nP$ and suppose $\epsilon_n(x)$ is a boundary, say $\epsilon_n(x) = \partial(y)$, $y \in C_{n+1}$. Then $\theta(x) = \delta\eta(y)$, so there is a $z \in PB_{n+1}$ such that $\beta(z) = \eta(y)$ and $\gamma(z) = x$. Choose $w \in P_{n+1}$ such that $\alpha(w) = z$; it follows that $\partial(w) = \iota\gamma\alpha(w) = x$, i.e., x is a boundary. $\square$

Remark 1. Any projective resolution of C must fit into a diagram like Diagram 1, for the universal property of pullbacks implies that there is a map α making the diagram commute at each stage. We need only check that such an α must be surjective. To see this, given any $x \in PB_n$, choose $y \in C_n$ such that $\eta(y) = \beta(x)$. Then by commutativity, $\partial(y) = \epsilon\iota\gamma(x)$. Since ϵ_* is an isomorphism, this implies $\iota\gamma(x)$ is a boundary, say $\iota\gamma(x) = \partial(w)$, $w \in P_n$. Now $\gamma\alpha(w) = \gamma(x)$ and

$$\partial\epsilon(w) = \epsilon\iota\gamma\alpha(w) = \epsilon\iota\gamma(x) = \partial(y).$$

So $\epsilon(w) - y \in Z_nC$, and since ϵ_* is an isomorphism, there is a $w' \in Z_nP$ such that $\eta\epsilon(w') = \eta(\epsilon(w) - y) = \eta\epsilon(w) - \beta(x)$. Now $\beta\alpha(w - w') = \eta\epsilon(w - w') = \beta(x)$ and $\gamma\alpha(w - w') = \gamma\alpha(w) = \gamma(x)$, so $\alpha(w - w') = x$.

Remark 2. Observe that $\partial(P) = \mathrm{Ker}(\alpha)$. For $\iota\gamma\alpha\partial = \partial^2 = 0$, hence $\gamma\alpha\partial = 0$. Moreover $\beta\alpha\partial = \eta\epsilon\partial = \eta\partial\epsilon = 0$. So by the universal property of pullbacks, $\alpha\partial = 0$, i.e., $\partial(P) \subseteq \mathrm{Ker}(\alpha)$. Conversely, if $\alpha(y) = 0$ for $y \in P$, then y is a cycle and

$$\begin{tikzcd}
 & P_{n+1} \arrow[rr] \arrow[d, "g_{n+1}"] & & P_n \arrow[rr] \arrow[d, "g_n"] \arrow[dll, dashed, "s_n"'] \arrow[ddl, dashed, "f_n"'] & & P_{n-1} \arrow[rr] \arrow[d, "g_{n-1}"] \arrow[dll, "s_{n-1}"'] \arrow[ddl, "f_{n-1}"'] & & P_{n-2} \arrow[d, "g_{n-2}"] \arrow[dll, "s_{n-2}"'] \arrow[ddl, "f_{n-2}"'] \\
 & C_{n+1} \arrow[rr] & & C_n \arrow[rr] & & C_{n-1} \arrow[rr] & & C_{n-2} \\
X_{n+1} \arrow[rr] \arrow[ur, "\epsilon_{n+1}"] & & X_n \arrow[rr] \arrow[ur, "\epsilon_n"] & & X_{n-1} \arrow[rr] \arrow[ur, "\epsilon_{n-1}"] & & X_{n-2} \arrow[ur, "\epsilon_{n-2}"] &
\end{tikzcd}$$

Diagram 2. Proof of the existence part of the Comparison Theorem

$\eta\epsilon(y) = 0$, so $\epsilon(y)$ is a boundary, and since ϵ_* is an isomorphism, y is a boundary; hence $\mathrm{Ker}(\alpha) \subseteq \partial(P)$.

Theorem 4 (Comparison Theorem). *Suppose $C, X \in \mathcal{C}_b$, $P \in \mathcal{C}_b^p$, and $\epsilon : X \to C$ is a quasi-isomorphism. Then the map*

$$\begin{aligned} H_*(\mathrm{Hom}_\Lambda(P, X)) &\longrightarrow H_*(\mathrm{Hom}_\Lambda(P, C)) \\ [f] &\longmapsto [\epsilon f] \end{aligned}$$

is an isomorphism.

Remark 3. For $* = 0$, the conclusion of the theorem may be expressed in the following form: given any Λ-chain map $g : P \to C$, there exists a Λ-chain map $f : P \to X$ such that $\epsilon f \simeq g$; moreover if $f' : P \to X$ is another such map, then $f \simeq f'$. In particular, if $X \xrightarrow{\epsilon} C$ and $P \xrightarrow{\tilde{\epsilon}} C$ are both projective resolutions, then there is a chain map $f : P \to X$, unique up to chain homotopy, such that $\epsilon f \simeq \tilde{\epsilon}$. Notice that the "comparison" f must also be a quasi-isomorphism since $\epsilon_* f_* = \tilde{\epsilon}_*$, and both ϵ_* and $\tilde{\epsilon}_*$ are isomorphisms.

Proof. Suppose $g \in Z_r \mathrm{Hom}_\Lambda(P, C)$. By shifting the indices if necessary we may assume $r = 0$. We construct, by induction, maps $f_i : P_i \to X_i$ and $s_i : P_i \to X_{i+1}$ satisfying $\partial f_i = f_{i-1}\partial$ and $\epsilon f_i - g_i = \partial s_i + s_{i-1}\partial$, for all $i \leq n$. For n sufficiently small these maps may all be taken to be 0, so we may assume that we have constructed them for $i < n$ and complete the inductive step.

First note that $f_{n-1}\partial(P_n) \subseteq B_{n-1}X$. For given $x \in P_n$, $\partial f_{n-1}\partial(x) = f_{n-2}\partial^2(x) = 0$, so $f_{n-1}\partial(x) \in Z_{n-1}X$, and

$$\epsilon f_{n-1}\partial(x) = g_{n-1}\partial(x) + \partial s_{n-1}\partial(x) = \partial g_n(x) + \partial s_{n-1}\partial(x)$$

is a boundary, so since ϵ_* is an isomorphism, $f_{n-1}\partial(x)$ is also a boundary.

Hence by the projectivity of P_n, there is a map $f'_n : P_n \to X_n$ such that $\partial f'_n = f_{n-1}\partial$. Now

$$\partial \circ (\epsilon f'_n - g_n) = (\epsilon f_{n-1} - g_{n-1})\partial = \partial s_{n-1}\partial,$$

so $\epsilon f'_n - g_n - s_{n-1}\partial$ has image contained in Z_nC. By considering the diagram

$$\begin{tikzcd}
& & P_n \arrow[d, "\epsilon f'_n - g_n - s_{n-1}\partial"] \arrow[ddll, dashed] \\
& & Z_nC \arrow[d] \\
Z_nX \arrow[r, "\epsilon|_{Z_nX}"'] & Z_nC \arrow[r, two heads] & H_nC
\end{tikzcd}$$

we see there is a map $f''_n : P_n \to X_n$ with $\mathrm{Im}(f''_n) \subseteq Z_nX$ and

$$\mathrm{Im}(\epsilon f'_n + \epsilon f''_n - g_n - s_{n-1}\partial) \subseteq B_nC. \tag{1}$$

Let $f_n = f'_n + f''_n$. Then $\partial f_n = f_{n-1}\partial$, as $\partial f'' = 0$. Moreover, (1) guarantees the existence of a map s_n making

$$\begin{tikzcd}
& P_n \arrow[d, "\epsilon f_n - g_n - s_{n-1}\partial"] \arrow[dl, dashed, "s_n"'] \\
C_{n+1} \arrow[r, "\partial"'] & C_n
\end{tikzcd}$$

commute, completing the inductive step.

We must next show that if $f : P \to X$ is a chain map and $\epsilon f \simeq 0$ then $f \simeq 0$. So suppose $\epsilon f = \partial s + s\partial$, where $s \in \mathrm{Hom}_\Lambda(P, C)_1$. We show by induction on n that there are maps $t_i : P_i \to X_{i+1}$ and $w_i : P_i \to C_{i+2}$ such that

$$f_i = \partial t_i + t_{i-1}\partial \tag{2}$$

$$\epsilon t_i - s_i = \partial w_i + w_{i-1}\partial. \tag{3}$$

Assume we have such maps for $i < n$. Then

$$\partial \circ (f_n - t_{n-1}\partial) = (f_{n-1} - \partial t_{n-1})\partial = (\partial t_{n-1} + t_{n-2}\partial - \partial t_{n-1})\partial = 0,$$

so $\mathrm{Im}(f_n - t_{n-1}\partial) \subseteq Z_nX$. Moreover,

$$\epsilon \circ (f_n - t_{n-1}\partial) = \partial s_n + s_{n-1}\partial - s_{n-1}\partial + \partial w_{n-1}\partial = \partial \circ (s_n + w_{n-1}\partial),$$

hence $\mathrm{Im}(f_n - t_{n-1}\partial) \subseteq B_nX$, as ϵ_* is an isomorphism. So by the projectivity of P_n, there is a map $t_n : P_n \to X_{n+1}$ such that (2) holds with $i = n$.

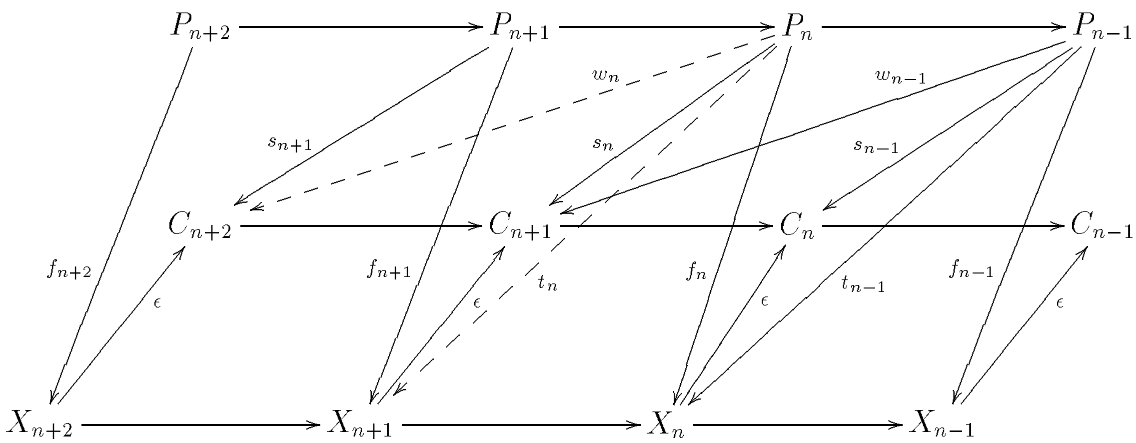

Diagram 3. Proof of the uniqueness part of the Comparison Theorem

We must next "adjust" t_n to show there is a map w_n satisfying (3) with $i = n$. Observe

$$\begin{aligned}
\partial \circ (\epsilon t_n - s_n - w_{n-1}\partial) &= \epsilon \circ (f_n - t_{n-1}\partial) - \partial s_n - \partial w_{n-1}\partial \\
&= \partial s_n + s_{n-1}\partial - s_{n-1}\partial + \partial w_{n-1}\partial + w_{n-2}\partial^2 - \partial s_n - \partial w_{n-1}\partial \\
&= 0,
\end{aligned}$$

i.e., $\mathrm{Im}(\epsilon t_n - s_n - w_{n-1}\partial) \subseteq Z_{n+1}C$. So arguing as before, there is a map $t'_n : P_n \to Z_{n+1}X$ such that $\mathrm{Im}(\epsilon t'_n + \epsilon t_n - s_n - w_{n-1}\partial) \subseteq B_{n+1}C$. Replace t_n with $t_n + t'_n$ (this does not affect (2) as $\partial t'_n = 0$). We then have, by the projectivity of P_n, a map $w_n : P_n \to C_{n+2}$ such that $\partial w_n = \epsilon t_n - s_n - w_{n-1}\partial$, completing the inductive step. $\square$

2.2. Minimal resolutions. A projective resolution $P \xrightarrow{\epsilon} C$ is said to be *minimal* if for any projective resolution $X \xrightarrow{\epsilon'} C$ of C, $P|X$.

Theorem 5. *Let $C \in \mathcal{C}_b$. Then*

(i) *C has a minimal projective resolution.*
(ii) *Let $X \to C$ be an arbitrary projective resolution of C. Then the following are equivalent: (a) $X \to C$ is minimal, (b) $\partial(X) \subseteq \mathrm{rad}(X)$, and (c) X has no nontrivial exact summands.*

Because of Theorem 5(i), for any $C \in \mathcal{C}_b$ we may let $\mathrm{MPR}(C)$ denote a minimal projective resolution of C. Note that $\mathrm{MPR}(C)$ is unique up to isomorphism of complexes, for if P and P' are both minimal, we have $P|P'|P$, and since each of these is finite-dimensional in each degree, $P \cong P'$. The proof of the theorem requires two lemmas: the first is interesting in its own right, the second is a rather technical fact from homological algebra.

Lemma 6. *Let $C \in \mathcal{C}_b$. Suppose $\partial(C) \subseteq \mathrm{rad}(C)$ and $f : C \to C$ is a quasi-isomorphism. Then f is an isomorphism.*

Proof. Show by induction on n that f_n is an isomorphism. Since $C_n = 0$ for n sufficiently small, the initial step is trivial. So assume f_{n-1} is an isomorphism. By Fitting's Lemma there is an $m > 0$ such that if we set $g = f^m$, $A = \mathrm{Ker}(g)$, and $B = \mathrm{Im}(g)$, then $C_i = A_i \oplus B_i$ for $i \leq n+1$. Clearly g_n is an isomorphism iff f_n is. Now g induces an automorphism of $H_nC = H_nA \oplus H_nB$, but g is trivial on A, hence $H_nA = 0$, i.e., $A_{n+1} \to A_n \to A_{n-1}$ is exact. But by the inductive hypothesis, $A_{n-1} = 0$. Therefore $A_n \subseteq \partial(C_{n+1}) \subseteq \mathrm{rad}(C_n)$. On the other hand, $A_n | C_n$. Hence $A_n = 0$. □

Lemma 7. *Let U, V, X, and Y be Λ-modules, and $f : X \to Y$ and $g : U \oplus V \to Y$ maps of Λ-modules. Let $i_U : U \to U \oplus V$ and $i_V : V \to U \oplus V$ denote the inclusions. Suppose V is projective and $gi_V(V) \subseteq f(X)$. Let $\tau : V \to X$ be a map satisfying $f\tau = gi_V$, and let*

$$\begin{tikzcd} M \arrow[r, "\phi"] \arrow[dd, "\sigma"'] & U \arrow[d, hook, "i_U"] \\ & U \oplus V \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

be a pullback of (f, gi_U). Then

$$\begin{tikzcd} M \oplus V \arrow[r, "\phi \oplus \mathrm{id}_V"] \arrow[d, "\sigma+\tau"'] & U \oplus V \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

is a pullback of (f, g).

Note that such a map τ exists by the definition of *projective*.

Proof. Suppose W is a Λ-module and $\delta : W \to X$ and $\epsilon : W \to U \oplus V$ are maps satisfying $f\delta = g\epsilon$. Then there is a commutative diagram

$$\begin{tikzcd}
W \arrow[rr, "\epsilon"] \arrow[dr, dashed, "\gamma_M"] \arrow[ddr, "\delta-\tau\pi_V\epsilon"'] & & U \oplus V \arrow[d, "\pi_U"] \\
& M \arrow[r, "\phi"] \arrow[d, "\sigma"'] & U \arrow[d, "gi_U"] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}$$

as

$$\begin{aligned} f \circ (\delta - \tau\pi_V\epsilon) &= g\epsilon - f\tau\pi_V\epsilon = g\epsilon - gi_V\pi_V\epsilon \\ &= g \circ (\mathrm{id}_V - i_V\pi_V)\epsilon = gi_U\pi_U\epsilon. \end{aligned}$$

So by the definition of pullback there is a unique map γ_M making the diagram commute. Let $\gamma_V = \pi_V\epsilon$, and define $\gamma : W \to M \oplus V$ by $\gamma(w) = (\gamma_M(w), \gamma_V(w))$. We then have

$$(\sigma + \tau)\gamma = \sigma\gamma_M + \tau\gamma_V = \delta - \tau\pi_V\epsilon + \tau\pi_V\epsilon = \delta$$

and

$$(\phi \oplus \mathrm{id}_V)\gamma(w) = (\phi\gamma_M(w), \gamma_V(w)) = (\pi_U\epsilon(w), \pi_V\epsilon(w)) = \epsilon(w),$$

as required.

Suppose $\gamma' : W \to M \oplus V$ also satisfies $(\sigma + \tau)\gamma' = \delta$ and $(\phi + \mathrm{id}_V)\gamma' = \epsilon$. Write $\gamma'(w) = (\gamma'_M(w), \gamma'_V(w))$. From

$$(\phi + \mathrm{id}_V)\gamma = \epsilon = (\phi + \mathrm{id}_V)\gamma'$$

we get $\phi\gamma_M = \phi\gamma'_M$ and $\gamma'_V = \gamma_V$. Similarly, from

$$(\sigma + \tau)\gamma = \delta = (\sigma + \tau)\gamma'$$

we get $\sigma\gamma_M = \sigma\gamma'_M$. So by the uniqueness of γ_M, $\gamma'_M = \gamma_M$, and therefore $\gamma = \gamma'$. $\square$

Proof of Theorem 5. Let $X \xrightarrow{\epsilon} C$ be a projective resolution of C, and fix a diagram like Diagram 1 for X. We first show there is a decomposition $X = P \oplus W$, with W exact and $\partial(P) \subseteq \mathrm{rad}(P)$.

To show this we produce, by induction on n, a decomposition $X_n = P_n \oplus W_n$ such that $\partial(W_n) = Z_{n-1}W$, $\epsilon(Z_nW) \subseteq B_nC$, $\alpha|_{P_n}$ is a projective cover of $\alpha(P_n)$, and $\partial(P_n) \subseteq \mathrm{rad}(P_{n-1})$. Since $X_n = 0$ for n sufficiently small, the initial step is trivial. So assume decompositions with these properties have been constructed through degree $n - 1$.

Now $W_{n-1} \to \cdots \to W_0 \to 0$ is an exact sequence of projectives, so $Z_{n-1}W$ is projective. Moreover, $Z_{n-1}X = Z_{n-1}P \oplus Z_{n-1}W$, and $\theta(Z_{n-1}W) \subseteq \delta(C_n/B_nC)$ (as $\epsilon_{n-1}(Z_{n-1}W) \subseteq B_{n-1}C$). Hence by Lemma 7 and the uniqueness of pullbacks, there is a decomposition $PB_n = PB'_n \oplus V$ such that $\gamma = \gamma' \oplus \gamma''$ and $\beta = \beta' + \mu$, where $\gamma'' : V \to Z_{n-1}W$ is an isomorphism and

$$\begin{tikzcd}
PB'_n \arrow[r, "\gamma'"] \arrow[d, "\beta'"'] & Z_{n-1}P \arrow[d, "\theta' := \theta|_{Z_{n-1}P}"] \\
\dfrac{C_n}{B_nC} \arrow[r, "\delta"'] & Z_{n-1}C
\end{tikzcd}$$

is a pullback.

Since $\alpha : X_n \to PB'_n \oplus V$ is surjective, there is a decomposition $X_n = P_n \oplus W_n$ such that $\alpha' := \alpha|_{P_n}$ is a projective cover of PB'_n and $\alpha(W_n) = V$. Hence $\partial(W_n) = Z_{n-1}W$. To see that $\partial(P_n) \subseteq \mathrm{rad}(P_{n-1})$, recall from the remarks following Proposition 3 that $\mathrm{Ker}(\alpha_{n-1}) = \partial(X_n)$, so in particular the composite

$$P_n \xrightarrow{\alpha'_n} PB'_n \xrightarrow{\gamma'} Z_{n-1}P \xhookrightarrow{\iota'} P_{n-1} \xrightarrow{\alpha'_{n-1}} \alpha(P_{n-1})$$

is trivial, so it certainly induces the trivial map modulo radicals. But α'_{n-1} is a projective cover, and therefore induces an isomorphism modulo radicals, whence we conclude $\partial(P_n) = \iota'\gamma'\alpha'(P_n) \subseteq \mathrm{rad}(P_{n-1})$.

To complete the inductive step we must show $\epsilon(Z_nW) \subseteq B_nC$. But $\iota\gamma''\alpha(Z_nW) = \partial(Z_nW) = 0$, so since ι and γ'' are monomorphisms, $\alpha(Z_nW) = 0$. Hence $\eta\epsilon(Z_nW) = \beta\alpha(Z_nW) = 0$, as required.

We now claim that $P \xrightarrow{\epsilon|_P} C$ is a minimal projective resolution. It is certainly a projective resolution, since $H_*W = 0$. Suppose $\tilde{X} \xrightarrow{\tilde{\epsilon}} C$ is any projective resolution of C. Then we have just shown that $\tilde{X} = \tilde{P} \oplus \tilde{W}$, where $\partial(\tilde{P}) \subseteq \mathrm{rad}(\tilde{P})$ and $\tilde{P} \xrightarrow{\tilde{\epsilon}|_{\tilde{P}}} C$ is a projective resolution. By the Comparison Theorem, there are quasi-isomorphisms $f : P \to \tilde{P}$ and $g : \tilde{P} \to P$. So fg and gf are quasi-isomorphisms, and by Lemma 6, fg and gf are isomorphisms. Hence f and g are isomorphisms, i.e., $P \cong \tilde{P}$. So $P|\tilde{X}$, establishing the claim, and completing the proof of (i).

The proof of (ii) is now immediate. For if $X \to C$ is minimal then by uniqueness $X \cong P$, and P has property (b). Hence (a) $\Rightarrow$ (b). If (b) holds and $X = C \oplus D$, with D exact, then $X \twoheadrightarrow C \hookrightarrow X$ is a quasi-isomorphism, so by Lemma 6 it is an isomorphism, and therefore $D = 0$. Hence (b) $\Rightarrow$ (c). Finally, the statement (c) $\Rightarrow$ (a) follows directly from the definition of *minimal.* □

If $D \in {}_b\mathcal{C}$, an *injective resolution of* D is a complex $I \in {}_b\mathcal{C}^p$ together with a quasi-isomorphism $\eta : D \to I$ (recall that a Λ-module is projective iff it is injective). Since $\eta : D \to I$ is an injective resolution iff $\eta^* : I^* \to D^*$ is a projective resolution, all of the statements above concerning projective resolutions have dual versions concerning injective resolutions.

For $C \in \mathcal{C}_b, D \in {}_b\mathcal{C}$, we now define $\mathrm{Ext}^*_\Lambda(C, D) = H^*\mathrm{Hom}_\Lambda(P, D)$, where P is any projective resolution of C. By Theorem 4, this is independent of the projective resolution chosen, in the usual sense. We also have the usual canonical isomorphisms $\mathrm{Ext}^*_\Lambda(C, D) \cong H^*\mathrm{Hom}_\Lambda(P, I) \cong H^*\mathrm{Hom}_\Lambda(C, I)$, where I is any injective resolution of D. If P is minimal and D is a complex with trivial differential and $\mathrm{rad}(D) = 0$, then by Theorem 5(ii)(b), $\mathrm{Ext}^*_\Lambda(C, D) = \mathrm{Hom}_\Lambda(P, D)$, which is one of the advantages of using minimal resolutions.

2.3. The hypercohomology spectral sequence. Recall that a doubly-indexed collection of modules $E_0^{p,q}$ together with maps $d' : E_0^{p,q} \to E_0^{p+1,q}, d'' : E_0^{p,q} \to E_0^{p,q+1}$ forms a *double complex* if $(d')^2 = (d'')^2 = d'd'' + d''d' = 0$. A double complex yields a spectral sequence $\{E_r, d_r\}$ in which the differentials are easy to describe explicitly: $x \in E_0^{p,q}$ lives to $E_n^{p,q}$ iff there exist $x_i \in E_0^{p+i,q-i}$ $(0 \le i < n)$ with $x_0 = x$, $d''(x_0) = 0$, and $d'(x_{i-1}) + d''(x_i) = 0$ for $1 \le i < n$. If this is the case and x represents $\zeta \in E_n^{p,q}$ then $d_n(\zeta)$ is represented by $d'(x_{n-1})$.

Now let $C \in \mathcal{C}_b, D \in {}_b\mathcal{C}$, and let I_D be an injective resolution of D. The hypercohomology spectral sequence arises from the double complex defined by $E_0^{p,q} = \mathrm{Hom}_\Lambda(C_q, I_D^p), d'(f) = \partial \circ f, d''(f) = (-1)^{p+q+1} f \circ \partial$. We have

$$E_2^{p,q} \cong \mathrm{Ext}^p_\Lambda(H_q(C), D) \Rightarrow \mathrm{Ext}^{p+q}_\Lambda(C, D).$$

Moreover, if I is an injective resolution of k then the tensor product induces a map of double complexes

$$\mathrm{Hom}_\Lambda(k, I_D) \otimes \mathrm{Hom}_\Lambda(C_\zeta, I) \to \mathrm{Hom}_\Lambda(C_\zeta, I_D \otimes I),$$

and since $I_D \otimes I$ is also an injective resolution of D, this map yields a pairing of spectral sequences

$$\begin{array}{ccccc}
\mathrm{Ext}^p_\Lambda(k, D) \otimes & \mathrm{Ext}^{p'}_\Lambda(H_q(C_\zeta), k) & \longrightarrow & \mathrm{Ext}^{p+p'}_\Lambda(H_q(C_\zeta), D) \\
\| & \Downarrow & & \Downarrow \\
\mathrm{Ext}^p_\Lambda(k, D) \otimes & \mathrm{Ext}^{p'+q}_\Lambda(C_\zeta, k) & \longrightarrow & \mathrm{Ext}^{p+p'+q}_\Lambda(C_\zeta, D).
\end{array}$$

We could also start with the double complex $\mathrm{Hom}_\Lambda(P_p, D^q)$, where P is a projective resolution of C, to obtain the second hypercohomology spectral sequence

$$E_2^{p,q} \cong \mathrm{Ext}^p_\Lambda(C, H^q(D)) \Rightarrow \mathrm{Ext}^{p+q}_\Lambda(C, D).$$

As a first consequence of these two spectral sequences, we obtain

Lemma 8. *Suppose $C, D \in {}_b\mathcal{C}_b$. Then $\mathrm{Ext}^*_\Lambda(C, D)$ is a finitely generated $\mathrm{Ext}^*_\Lambda(k, k)$-module.*

Proof. We first reduce to the case where C is a module concentrated in degree 0 as follows. In the hypercohomology spectral sequence we have E_2 is a finite direct sum $\bigoplus_q \operatorname{Ext}^*_\Lambda(H_q(C), D)$, so E_2 is finitely generated over $\operatorname{Ext}^*_\Lambda(k,k)$. The spectral sequence stops (it has a finite number of non-zero rows), so E_∞ is also finitely generated over $\operatorname{Ext}^*_\Lambda(k,k)$, and this implies that $\operatorname{Ext}^*_\Lambda(C,D)$ is as well by [7, Lemma 7.4.5]. Now if C is a module concentrated in degree 0 then apply the second spectral sequence to reduce to the case where both C and D are modules. $\square$

2.4. Complexes of finite projective dimension. Let $C \in \mathcal{C}_b$. We say that C has *finite projective dimension* if C has a bounded projective resolution, i.e., a projective resolution $P \to C$ with $P \in {}_b\mathcal{C}^p_b$. Dually, for $C \in {}_b\mathcal{C}$, we say that C has *finite injective dimension* if C has a bounded injective resolution.

Lemma 9. *Let $C \in {}_b\mathcal{C}_b$. Then the following are equivalent:*

(i) *C has finite projective dimension.*
(ii) *C has finite injective dimension.*
(iii) $\operatorname{Ext}^n_\Lambda(C,C) = 0$ *for n sufficiently large.*
(iv) *$C = C' \oplus C''$, where $C' \in {}_b\mathcal{C}^p_b$, $C'' \in {}^e_b\mathcal{C}_b$, and C' has no exact summands.*

Note that for such a C it follows from Theorem 5(ii) that the inclusion $C' \hookrightarrow C$ is a minimal projective resolution of C.

Proof. It is clear that (i)$\Rightarrow$(iii). Conversely, if (iii) holds, then for any simple Λ-module S, Lemma 8 implies $\operatorname{Ext}^*_\Lambda(C,S)$ is finitely generated over $\operatorname{Ext}^*_\Lambda(k,k)$, so it is certainly finitely generated over $\operatorname{Ext}^*_\Lambda(C,C)$ (here we are using the fact that the cup and Yoneda products are compatible, cf. [6, Lemma 2.2]). Hence $\operatorname{Ext}^*_\Lambda(C,S)$ is finite-dimensional, i.e., bounded. Let $P \to C$ be a minimal resolution of C. Since there are only finitely many simple Λ-modules, this means that there is an integer N such that $\operatorname{Hom}_\Lambda(P_n, S) = \operatorname{Ext}^n_\Lambda(C,S) = 0$ for all simples S and $n > N$. Hence $P_n = 0$ for all $n > N$, i.e. (i) holds. Proceeding dually, we obtain (ii)$\Leftrightarrow$(iii).

Now suppose (i), (ii), and (iii) hold. Let $P \xrightarrow{\epsilon} C$ be a minimal projective resolution of C, $C \xrightarrow{\eta} I$ a minimal injective resolution, and let $f = \eta\epsilon$. Then $f : P \to I$ is a quasi-isomorphism, and to prove (iv) it suffices to show that f is an isomorphism. But one may extend f to a surjective map $P \oplus W \to I$, for some $W \in {}^e_b\mathcal{C}^p_b$. Let K be the kernel of this map. Then K is a bounded complex of projectives, and by considering the long exact sequence in homology arising from the short exact sequence

$$0 \to K \to P \oplus W \to I \to 0,$$

we conclude that $H_*K = 0$, i.e. $K \in {}^e_b\mathcal{C}^p_b$. This means that K is isomorphic to the direct sum of complexes of the form $\cdots \to 0 \to N \xrightarrow{=} N \to 0 \to \cdots$, where N is an injective (i.e. projective) Λ-module, each of which is easily seen to be an injective object in $\mathcal{C}$. Hence K is an injective object in $\mathcal{C}$, so the short exact sequence above splits, and $P \oplus W \cong K \oplus I$. Now K and W contain only exact summands, while P

and I have no exact summands, so by the Krull-Schmidt Theorem, $P \cong I$. Fix an isomorphism $\theta : I \to P$. Then $\theta f : P \to P$ is a quasi-isomorphism, so by Lemma 6, θf, and therefore f, is an isomorphism, and we have (iv). Finally, (iv)⇒(i), as $C' \hookrightarrow C$ is a (minimal) projective resolution of C. □

The following is a direct generalization of Lemma 5.1 of [6].

Proposition 10. *Suppose $C, D \in {}_b\mathcal{C}_b$ have finite projective dimension, and $f : C \to D$ is a quasi-isomorphism. Then there are decompositions $C = C' \oplus C''$ and $D = D' \oplus D''$, with $C', D' \in {}_b\mathcal{C}_b^p$ and $C'', D'' \in {}_b^e\mathcal{C}_b$, such that $f|_{C'}$ is an isomorphism onto D'.*

Proof. Write $C = C' \oplus C'', D = D' \oplus D''$ as in Lemma 9 (iv), and let $i : C' \hookrightarrow C$ denote inclusion and $\pi : D \to D'$ the projection. It suffices to show there exists a map $g : D \to C'$ such that $gfi = \mathrm{id}_{C'}$ (by replacing D' with $fi(C')$ and D'' with $\mathrm{Ker}(g)$). Now $h = \pi f i : C' \to D'$ is a quasi-isomorphism. Moreover, C' and D' are both minimal projective resolutions of D, and are therefore isomorphic. So we may apply Lemma 6 to conclude that h is an isomorphism. Let $g = h^{-1}\pi$. Then $gfi = h^{-1}\pi f i = h^{-1}h = \mathrm{id}_{C'}$. □

Now suppose $C, D \in {}_b\mathcal{C}_b$. We write $C \sim_d D$ if there is a sequence of quasi-isomorphisms

$$C \to X_1 \leftarrow X_2 \to \cdots \leftarrow X_n \to D$$

with each $X_i \in {}_b\mathcal{C}_b$. (This just means that C and D are isomorphic objects in the derived category.) It is clear that $\sim_d$ is an equivalence relation.

Proposition 11. *Suppose $C, D \in {}_b\mathcal{C}_b$ and $C \sim_d D$. If C has finite projective dimension then so does D, and* $\mathrm{MPR}(C) \cong \mathrm{MPR}(D)$.

Proof. If $X \to Y$ is a quasi-isomorphism and $P \to X$ is a bounded projective resolution of X then $P \to X \to Y$ is a bounded projective resolution of Y, and by Proposition 10, $\mathrm{MPR}(X) \cong \mathrm{MPR}(Y)$. Dually, if $X \leftarrow Y$ is a quasi-isomorphism and $I \leftarrow X$ is a bounded injective resolution of X, then $I \leftarrow X \leftarrow Y$ is a bounded injective resolution of Y, and by Proposition 10, $\mathrm{MPR}(X) \cong \mathrm{MPR}(Y)$. So the proof follows by induction on the length n of a chain of quasi-isomorphisms joining C and D. □

3. Generalized Benson-Carlson duality

3.1. The main theorem. Let U and V be Λ-modules and n a positive integer. An n-extension of U by V is a complex $e \in {}_b^e\mathcal{C}_b$ with $e_{-1} = U, e_n = V$, and $e_r = 0$ if $r > n$ or $r < -1$. If $\tilde{e}$ is also an n-extension of U by V we write $e \rightsquigarrow \tilde{e}$ if there exists $f \in Z_0\mathrm{Hom}_\Lambda(e, \tilde{e})$ with $f_{-1} = \mathrm{id}_U, f_n = \mathrm{id}_V$. We complete $\rightsquigarrow$ to an equivalence relation and let $\mathrm{YExt}_\Lambda^n(U, V)$ denote the set of equivalence classes. There is a functorial bijection $\mathrm{YExt}^n_\Lambda(U,V) \longrightarrow \mathrm{Ext}^n_\Lambda(U,V)$ (cf. Hilton and Stammbach [8, IV.9]).

Let $n \geq 2$. A *truncated n-extension of U by V* is a complex $C \in {}_b\mathcal{C}_b$ with $C_i = 0$ if $i > n-1$ or $i < 0$, and with $H_i(C)$ isomorphic to U if $i = 0$, V if $i = n-1$, and 0 otherwise. If U and V are simple, then it is easily seen that the bijection referred to above induces a bijection between equivalence classes of truncated n-extensions of U by V under $\sim_d$ and $P(\mathrm{Ext}^n_\Lambda(U,V))$, the projective space of $\mathrm{Ext}^n_\Lambda(U,V)$. This follows from Schur's Lemma, which states that the endomorphism ring of U or V consists of scalar multiples of the identity, which implies that a truncated n-extension determines an n-extension only up to non-0 scalar multiple.

We can now state the main theorem, which generalizes Theorem 5.5 of [6]. Let $D \in {}_b\mathcal{C}_b$ and let R be the quotient of $\mathrm{Ext}^*_\Lambda(k,k)$ by the annihilator of $\mathrm{Ext}^*_\Lambda(D,D)$. Recall that non-zero homogeneous elements of positive degree $\zeta_1, \dots, \zeta_d \in \mathrm{Ext}^*_\Lambda(k,k)$ are said to form a *homogeneous system of parameters* (h.s.o.p.) for $\mathrm{Ext}^*_\Lambda(D,D)$ if their images in R generate a subring over which R is finitely generated as a module.

Theorem 12. *Let k be an algebraically closed field and Λ a finite-dimensional cocommutative Hopf algebra over k with the finiteness property in cohomology. Let $D \in {}_b\mathcal{C}_b$, and $\zeta_1, \dots, \zeta_r$ a h.s.o.p. for $\mathrm{Ext}^*_\Lambda(D,D)$. Assume $n_i = \deg(\zeta_i) \geq 2$ for all i. Let C_{ζ_i} be a truncated n_i-extension representing $[\zeta_i]$, and let $C = \bigotimes_i C_{\zeta_i} \otimes D$. Then*

(i) *There is a decomposition $C = N \oplus Q$, where $N \in {}_b\mathcal{C}^p_b$, $Q \in {}^e_b\mathcal{C}_b$, and N has no exact summands.*
(ii) *If $D^*[t] \sim_d D$ for an integer t, then $N^*[s] \cong N$, where $s = t + \sum_{i=1}^r (n_i - 1)$.*
(iii) *Up to isomorphism of complexes, N is independent of the choices C_{ζ_i} of truncated n_i-extension representing $[\zeta_i]$*

A key observation is the following Lemma, which is essentially [6, Proposition 5.2]. Given $\zeta \in \mathrm{Ext}^n_\Lambda(k,k)$, let e_ζ be an n-extension of k by k representing ζ. Then $e^*_\zeta[n-1]$ is also an n-extension of k by k, and therefore represents some element ζ^* of $\mathrm{Ext}^n_\Lambda(k,k)$. Since equivalent extensions are taken to equivalent extensions by this operation, we have a well-defined operation $\zeta \mapsto \zeta^*$ on $\mathrm{Ext}^n_\Lambda(k,k)$.

Lemma 13. *For $n \geq 1$ and $\zeta \in \mathrm{Ext}^n_\Lambda(k,k)$, $\zeta^* = \pm\zeta$. In particular, if C_ζ is a truncated n-extension representing $[\zeta]$, then $(C_\zeta)^*[n-1] \sim_d C_\zeta$.*

Proof. Let $P \to k$ be a projective resolution of k. After identifying P with P^{**} in the usual way, there is an endomorphism θ of $\mathrm{Hom}_\Lambda(P, P^*)$ defined by $\theta(f) = f^*$. We claim that for $f \in Z\mathrm{Hom}_\Lambda(P,P^*)$, $f \simeq f^*$. To see this, reason as follows: $\mathrm{Hom}_\Lambda(P,P^*) \cong \mathrm{Hom}_\Lambda(P \otimes_k P, k)$, and θ is induced by the twisting endomorphism τ of $P \otimes_k P$. By the Comparison Theorem, τ is homotopic to the identity on $P \otimes_k P$, hence θ is homotopic to the identity on $\mathrm{Hom}_\Lambda(P,P^*)$. Say $\theta - 1 = \partial s + s\partial$. Then given $f \in Z\mathrm{Hom}_\Lambda(P,P^*)$, $\partial(f) = 0$, hence $\theta(f) - f = \partial(s(f))$, i.e., $\theta(f) \simeq f$.

Now $\mathrm{Ext}^*_\Lambda(k,k) \cong H^*(\mathrm{Hom}_\Lambda(P,P^*))$, as $k \to P^*$ is an injective resolution of k. Fix an n-extension e representing ζ. Then by projectivity and injectivity, there exist maps f_i, g_i making the following diagram, in which the middle row is e, commute:

$$\begin{tikzcd}
P_{n+1} \arrow[r] \arrow[d] & P_n \arrow[r] \arrow[d, "f_n"'] & P_{n-1} \arrow[r] \arrow[d, "f_{n-1}"'] & P_{n-2} \arrow[r] \arrow[d, "f_{n-2}"'] & \cdots \arrow[r] & P_1 \arrow[r] \arrow[d, "f_1"'] & P_0 \arrow[r, "\epsilon"] \arrow[d, "f_0"'] & k \arrow[r] \arrow[d, equal] & 0 \arrow[d] \\
0 \arrow[r] \arrow[d] & k \arrow[r] \arrow[d, equal] & M_{n-1} \arrow[r] \arrow[d, "g_{n-1}"'] & M_{n-2} \arrow[r] \arrow[d, "g_{n-2}"'] & \cdots \arrow[r] & M_1 \arrow[r] \arrow[d, "g_1"'] & M_0 \arrow[r] \arrow[d, "g_0"'] & k \arrow[r] \arrow[d, "g_{-1}"'] & 0 \arrow[d] \\
0 \arrow[r] & k \arrow[r, "\epsilon^*"] & P_0^* \arrow[r] & P_1^* \arrow[r] & \cdots \arrow[r] & P_{n-2}^* \arrow[r] & P_{n-1}^* \arrow[r] & P_n^* \arrow[r] & P_{n+1}^*
\end{tikzcd}$$

Let $f = f_n, g = g_{-1}$. Now $\epsilon^* f \in Z^n \mathrm{Hom}_\Lambda(P, P^*)$ represents ζ. But the diagram shows that $\pm g\epsilon \simeq \epsilon^* f$, hence $\pm g\epsilon$ also represents ζ. Taking the dual diagram, we see that $\epsilon^* g^*$ represents $\pm\zeta^*$. But as was shown in the previous paragraph, $\epsilon^* g^* = \pm(g\epsilon)^* \simeq g\epsilon$. $\square$

The next lemma involves the hypercohomology spectral sequence (see Section 2.3). Let $n \geq 2$, $\zeta \in \mathrm{Ext}^n_\Lambda(k,k)$, and C_ζ a truncated n-extension representing $[\zeta]$. Extend C_ζ to an n-extension representing ζ with maps $\eta : k \to (C_\zeta)_{n-1}$ and $\epsilon : (C_\zeta)_0 \to k$. This yields identifications $H_{n-1}(C_\zeta) = k$ and $H_0(C_\zeta) = k$ and therefore also $E_2^{0,n-1} = \mathrm{Ext}^0_\Lambda(H_{n-1}(C_\zeta), k) = k$ and $E_2^{n,0} = \mathrm{Ext}^n_\Lambda(H_0(C_\zeta), k) = \mathrm{Ext}^n_\Lambda(k,k)$. Define $\tilde{\zeta} \in E_2^{0,n-1}$ to be the element corresponding to $1 \in k$.

Lemma 14. *Let $D \in {}_b\mathcal{C}$. Then in the two-row spectral sequence*

$$E_2^{pq}(\zeta) = \mathrm{Ext}^p_\Lambda(H_q(C_\zeta), D) \Rightarrow \mathrm{Ext}^{p+q}_\Lambda(C_\zeta, D)$$

the differential d_n is given by

$$d_n(\alpha\tilde{\zeta}) = \pm\alpha\zeta \in E_n^{p+n,0}(\zeta) = \mathrm{Ext}^{p+n}_\Lambda(k, D).$$

Proof. If we can show that $d_n(\tilde{\zeta}) = \zeta$ in $E_n(k)$ then we are done, using the fact that d_n is a derivation. So without loss of generality we assume $D = k$. By injectivity there are maps $f_i, \hat{\zeta}$ making the following diagram commute:

$$\begin{tikzcd}
0 \arrow[r] & k \arrow[r, "\eta"] \arrow[d, equal] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"'] & C_{n-2} \arrow[r] \arrow[d, "f_{n-2}"'] & \cdots \arrow[r] & C_1 \arrow[r] \arrow[d, "f_1"'] & C_0 \arrow[r, "\epsilon"] \arrow[d, "f_0"'] & k \arrow[r] \arrow[d, "\hat{\zeta}"] & 0 \arrow[d] & \\
0 \arrow[r] & k \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & \cdots \arrow[r] & I^{n-2} \arrow[r] & I^{n-1} \arrow[r] & I^n \arrow[r] & I^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

Now $\hat{\zeta}$ represents ζ (this is just the bijection between $\mathrm{YExt}^n_\Lambda(k,k)$ and $\mathrm{Ext}^n_\Lambda(k,k)$) and $\tilde{\zeta}$ is represented in $E_0^{0,n-1}$ by f_{n-1}. By definition of the differential, $\pm d_n(\tilde{\zeta})$ is represented by $\pm\partial \circ f_0$. By commutativity this is $\hat{\zeta} \circ \epsilon$, which represents ζ under the identification $\mathrm{Ext}^n_\Lambda(k,k) = \mathrm{Ext}^n_\Lambda(H_0(C), k)$. $\square$

Proof of Theorem 12. To prove (i), show $\mathrm{Ext}^n_\Lambda(C,C)=0$ for n sufficiently large and apply Lemma 9. To do this proceed as in the proof of Theorem 4.1 of [6] and show that the E_∞-term of the appropriate spectral sequence is a finite module over $\mathrm{Ext}^*_\Lambda(k,k)/(\zeta_1,\dots,\zeta_d)$, using Lemma 14.

Now if $A,A',B,B'\in {}_b\mathcal{C}_b$ and $A\sim_d A'$ and $B\sim_d B'$ then $A\otimes B\sim_d A'\otimes B'$. So Lemma 13 implies $C^*[s]\sim_d C$. Hence by Proposition 11, the minimal projective resolutions of $C^*[s]$ and C are isomorphic. Now $C\cong N\oplus Q$, so $C^*[s]\cong N^*[s]\oplus Q^*[s]$. On the other hand, $C^*[s]\cong N\oplus Q'$, for some $Q'\in {}^e_b\mathcal{C}_b$. So by the Krull-Schmidt Theorem, $N\cong N^*[s]$, proving (ii).

If we choose different truncated extensions C'_{ζ_i} representing the $[\zeta_i]$, then $C'_{\zeta_i}\sim_d C_{\zeta_i}$ for each i, and therefore $\bigotimes_i C'_{\zeta_i}\sim_d C$. Hence their minimal projective resolutions are isomorphic, proving (iii). $\square$

Example. Suppose k has characteristic 2, G is the alternating group on 4 letters, and $\Lambda=kG$. We have (cf. [3, p. 197])

$$H^*(G,k)=k[u,v,w\mid \deg(u)=2,\deg(v)=\deg(w)=3,u^3+v^2+vw+w^2=0].$$

The simple Λ-modules are k,S,T where S and T also are one-dimensional (corresponding to the third roots of 1 in k) and we have $S^*\cong T$. The projective covers of the simples have Loewy structures

$$P_k=\begin{array}{ccc} & k & \\ S & & T\\ & k & \end{array},\quad P_S=\begin{array}{ccc} & S & \\ k & & T\\ & S & \end{array},\quad P_T=\begin{array}{ccc} & T & \\ k & & S\\ & T & \end{array}.$$

For a h.s.o.p. we may take u and v, and these are represented by extensions of the form

$$\begin{array}{lccccccccccccc} u: & & & 0 & \to & k & \to & \begin{array}{c}S\\k\end{array} & \to & \begin{array}{c}k\\S\end{array} & \to & k & \to & 0\\[2ex] v: & 0 & \to & k & \to & \begin{array}{ccc}S&&T\\&k&\end{array} & \to & \begin{array}{c}T\\S\end{array}\oplus\begin{array}{c}S\\T\end{array} & \to & \begin{array}{ccc}&k&\\S&&T\end{array} & \to & k & \to & 0\end{array}$$

One then gets $C=C_u\otimes C_v$ decomposes as a direct sum of a complex N of projectives

$$P_k\to P_S\oplus P_T\to P_T\oplus P_S\to P_k,$$

which indeed satisfies $N^*[3]\cong N$, and an exact complex

$$\begin{array}{c}T\\S\end{array}\to\begin{array}{c}T\\S\end{array}\oplus\begin{array}{c}T\\k\end{array}\oplus\begin{array}{c}k\\T\end{array}\to\begin{array}{c}T\\k\end{array}\oplus\begin{array}{c}k\\T\end{array}\oplus\begin{array}{c}S\\T\end{array}\to\begin{array}{c}S\\T\end{array}.$$

3.2. Further applications. One may also apply the techniques described above to Benson and Carlson's work on the construction of projective resolutions as tensor products of periodic complexes [5]. We review this briefly here.

Let $n \geq 2$, $\zeta \in \operatorname{Ext}^n_\Lambda(k,k)$, and $P \to k$ the minimal resolution. One constructs a special n-extension $\acute{e}_\zeta$ representing ζ as the bottom row in the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & P_n \arrow[r] \arrow[d, "\zeta"'] & P_{n-1} \arrow[r] \arrow[d] & P_{n-2} \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[d, equal] & k \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & k \arrow[r] & M_\zeta \arrow[r] & P_{n-2} \arrow[r] & \cdots \arrow[r] & P_0 \arrow[r] & k \arrow[r] & 0
\end{tikzcd}$$

where the square with M_ζ in the bottom right is a pushout. Let $\acute{C}_\zeta$ denote the truncated n-extension arising from this, and let $\acute{C}^{(\infty)}_\zeta$ denote the complex in $\mathcal{C}_b$ formed by splicing together infinitely many copies of $\acute{C}_\zeta$.

Now let M be a Λ-module and $\zeta_1, \ldots, \zeta_r$ a h.s.o.p. for $\operatorname{Ext}^*_\Lambda(M,M)$ with $n_i = \deg(\zeta_i) \geq 2$ for each i. Using variety theory, one can show that $M \otimes M_{\zeta_1} \otimes \cdots \otimes M_{\zeta_r}$ is projective. Since the tensor product of a projective Λ-module with any Λ-module is also projective, it follows that $\acute{X} = M \otimes \acute{C}^{(\infty)}_{\zeta_1} \otimes \cdots \otimes \acute{C}^{(\infty)}_{\zeta_r} \in \mathcal{C}^p_b$. Moreover, the Künneth Theorem implies that $\acute{X}$ is exact in positive degrees, and that $\acute{X}_1 \to \acute{X}_0 \to M \to 0$ is exact (where the map $\acute{X}_0 \to M$ is formed by tensoring the identity on M with the augmentations from $\acute{C}_{\zeta_i} \to k$). Hence

Theorem 15 (Benson-Carlson). *$M \otimes \acute{C}^{(\infty)}_{\zeta_1} \otimes \cdots \otimes \acute{C}^{(\infty)}_{\zeta_r}$ is a projective resolution of M.*

We can generalize this as follows.

Theorem 16. *Let $D \in {}_b\mathcal{C}_b$, $\zeta_1, \ldots, \zeta_r$ a h.s.o.p. for $\operatorname{Ext}^*_\Lambda(D,D)$ such that $n_i = \deg(\zeta_i) \geq 2$ for all i, and for each i let C_{ζ_i} be a truncated n_i-extension representing $[\zeta_i]$. Let $X = D \otimes C^{(\infty)}_{\zeta_1} \otimes \cdots \otimes C^{(\infty)}_{\zeta_r}$. Then there is a decomposition $X = Y \oplus Z$, where Y is exact and Z is a projective resolution of D.*

Proof. It is easily seen that the special extension $\acute{e}_\zeta$ has the following property: if e_ζ is any n-extension representing ζ then $\acute{e}_\zeta \rightsquigarrow e_\zeta$. On the truncated level, this means that if C_ζ is any truncated n-extension representing $[\zeta]$ then there is a quasi-isomorphism $\acute{C}_\zeta \to C_\zeta$. Now let $\grave{C}_\zeta = \acute{C}^*_\zeta[n-1]$. By Lemma 13, $\grave{C}_\zeta$ also represents $[\zeta]$, and it is equally easy to see that $\grave{C}_\zeta$ enjoys the following dual property: if C_ζ is any truncated n-extension representing ζ then there is a quasi-isomorphism $C_\zeta \to \grave{C}_\zeta$. Clearly, Theorem 15 remains true if each $\acute{C}_\zeta$ is replaced by $\grave{C}_\zeta$.

Now we may splice together quasi-isomorphisms to get quasi-isomorphisms $\acute{C}^{(\infty)}_{\zeta_i} \to C^{(\infty)}_{\zeta_i} \to \grave{C}^{(\infty)}_{\zeta_i}$. Tensoring over i yields quasi-isomorphisms $\acute{X} \to X \to \grave{X}$. But both $\acute{X}$ and $\grave{X}$ are projective resolutions of D, so there exist quasi-isomorphisms

$P_D \to \acute{X}$ and $\grave{X} \to P_D$, where P_D is the minimal resolution of D. Composing, we have quasi-isomorphisms $P_D \to X \to P_D$. By Lemma 6, this composition must be an isomorphism. Hence P_D splits off of X. $\square$

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