THE HOCHSCHILD COHOMOLOGY RING OF A CYCLIC BLOCK

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Abstract. Suppose B is a block of a group algebra kG with cyclic defect group. We calculate the Hochschild cohomology ring of B, giving a complete set of generators and relations. We then show that if B is the principal block, the canonical map from $H^*(G, k)$ to the Hochschild cohomology ring of B induces an isomorphism modulo radicals.

1. Introduction

The representation theory of cyclic blocks (that is, blocks with cyclic defect groups) plays an important role in the representation theory of finite groups. The principal result states that a cyclic block is a Brauer tree algebra (see Alperin's book [1] for this and other results from the cyclic theory). These algebras are complicated enough to be interesting but simple enough so that many aspects of the theory afford an elegant combinatorial description. For this reason, the cyclic theory has provided an important testing ground for new developments in representation theory.

In this note we calculate the Hochschild cohomology ring of a cyclic block. Specifically, let G be a finite group, p a prime, and k an algebraically closed field of characteristic p. Suppose kG has a cyclic block B. Then Theorem 1 below gives a complete set of generators and relations for $H^*(B, B)$ as a commutative graded k-algebra.

Earlier, T. Holm [6] calculated the even subring of $H^*(B, B)$. His approach, which we share, used the fact that B is derived equivalent to kT, where T is a split extension of a cyclic p-group by a cyclic p'-group. This follows from a theorem of J. Rickard [7, Thm. 4.2], which states that two Brauer tree algebras are derived equivalent if, and only if, the trees have the same number of edges and the same multiplicity. Because algebras which are derived equivalent have isomorphic Hochschild cohomology rings ([8, Prop. 2.5]), it suffices to calculate $H^*(kT, kT)$.

At this point we part with the methods of [6] (which involve extensive calculations in the category of kT-bimodules) and instead exploit the isomorphism of $H^*(kT, kT)$ with $H^*(T, kT)$ ([10, Prop. 3.1]). The latter ring denotes the ordinary cohomology of T with coefficients in kT *considered as a kT-module by conjugation.* The ring structure is provided by the composition of the cup product with the map on cohomology

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induced by multiplication $kT \otimes kT \to kT$ ($\otimes = \otimes_k$). It turns out that this ring is quite easy to describe, using some elementary results from group cohomology.

We should point out that this result is a special case of more recent work by Erdmann-Holm [4], in which they calculate the Hochschild cohomology of a more general class of algebras, the self-injective Nakayama algebras. For the cyclic block case, however, the approach here does have the advantage that the proof, and the relations arrived at, are particularly simple.

In Section 3, we further prove that in case the principal block B_0 is cyclic, the canonical map from $H^*(G,k)$ to $H^*(B_0,B_0)$ induces an isomorphism modulo radicals. This question was raised for principal blocks in general in [10] and answered positively in the cases where G is a p-group, G is Abelian, and in a few other specific cases. This result for cyclic blocks provides further evidence for a positive answer to this question in general, and so we have elevated the question to the status of "conjecture."

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2. Generators and Relations

Suppose kG has a block B with defect group cyclic of order p^n and inertial index e. Let $m = (p^n - 1)/e$.

Theorem 1. *If $e > 1$ then $H^*(B,B)$ is generated as a commutative k-algebra by elements z, $p_1, \ldots, p_{e-1}$ of degree 0, and elements y_1, y_2, y_{2e-1}, y_{2e}, where $\deg(y_r) = r$, subject to the relations*

$$z^{m+1} = z^m y_1 = z^m y_2 = y_1 y_{2e-1} = y_1^2 = y_{2e-1}^2 = 0,$$
$$y_1 y_2^{e-1} = z y_{2e-1}, \quad y_2^e = z y_{2e}, \quad y_2 y_{2e-1} = y_1 y_{2e},$$
$$z p_i = y_1 p_i = y_2 p_i = y_{2e-1} p_i = y_{2e} p_i = 0 \qquad (1 \le i < e),$$
$$p_i p_j = 0 \qquad (1 \le i, j < e).$$

If $e = 1$ and $p^n \neq 2$ then

$$H^*(B,B) = k[z, y_1, y_2 \mid \deg(z) = 0, \deg(y_r) = r, z^{p^n} = 0 = y_1^2],$$

while if $e = 1$ and $p^n = 2$,

$$H^*(B,B) = k[z, y_1 \mid \deg(z) = 0, \deg(y_1) = 1, z^2 = 0].$$

The rest of this section is devoted to the proof of Theorem 1. First we note that by [3, Prop. 62.35], e divides $p-1$. We consider the group algebra kT, where

$$T = \langle a, b \mid a^{p^n} = 1 = b^e, bab^{-1} = a^s \rangle,$$

and $s + p^n\mathbb{Z}$ is an element of order e in $(\mathbb{Z}/p^n\mathbb{Z})^\times$, the group of units of $\mathbb{Z}/p^n\mathbb{Z}$. By [3, Lemma 60.9], kT has only one block, and so by [1, p. 123], kT is a Brauer tree algebra for the star with e edges and exceptional multiplicity m. As we have seen, we may conclude from this that $H^*(B,B)$ is isomorphic as a graded algebra to $H^*(T,kT)$, where kT is considered a kT-module under conjugation. It is this algebra to which we now turn.

If $e=1$ then T is cyclic of order p^n. Since T is Abelian, $H^*(T,kT) \cong kT \otimes H^*(T,k)$ ([2, Thm. 2.1] or [10, Prop. 3.2]), and $H^*(T,k)$ is well-known (see [5, §3.2]). This finishes the proof for $e=1$. From this point on, we will assume $e>1$.

Our first task is to understand the conjugacy classes of T. For this we need

Lemma 2. *Suppose $[r] = r + p^n\mathbb{Z}$ is a non-identity element of $(\mathbb{Z}/p^n\mathbb{Z})^\times$ of order dividing $p-1$. Then $[r-1]$ is also a unit.*

Proof. We may assume $1 < r < p^n$. By hypothesis, $[r]^{p-1} = [1]$. Hence

$$[0] = [r - r^p] = [r - ((r-1)+1)^p] = \left[r - 1 - \sum_{i=1}^{p} \binom{p}{i} (r-1)^i\right] = [r-1][u],$$

where

$$u = 1 - \sum_{i=1}^{p} \binom{p}{i} (r-1)^{i-1}.$$

If $p | (r-1)$ then $[u]$ is a unit, which forces $[r] = [1]$, a contradiction. $\square$

Let $H = \langle a \rangle$ and $K = \langle b \rangle$. Now if $[i] \neq [0]$ then $C_T(a^i) = H$. For if

$$b^j a^i b^{-j} = a^{is^j} = a^i$$

then $[i][s^j - 1] = [0]$, which, by the Lemma, implies $j \in e\mathbb{Z}$. Hence H is the union of the class $\{1\}$ and m classes each of size e.

We claim that the remaining classes are the sets Hb^j, $(1 \le j < e)$. Indeed,

$$a^i b^j a^{-i} = a^{i(1-s^j)} b^j.$$

The Lemma implies $[1 - s^j]$ is a unit, so letting i run from 1 to p^n, we see that Hb^j is contained in the conjugacy class of b^j. On the other hand, $C_T(b^j)$ has order at least e (as it contains K), so the class can have no more than p^n elements, and the containment is an equality.

From the above remarks, we have a direct sum decomposition of kT-modules (under the conjugation action)

$$kT = kH \oplus \bigoplus_{j=1}^{e-1} kHb^j.$$

Hence as graded k-modules (but not necessarily as rings), we have

$$H^*(T,kT) = H^*(T,kH) \oplus \bigoplus_{j=1}^{e-1} H^*(T,kHb^j).$$

By the Eckmann-Shapiro Lemma,

$$H^*(T, kHb^j) \cong H^*(T, k{\uparrow}_K^T) \cong H^*(K, k) = k,$$

concentrated in degree 0, as $|K|$ is relatively prime to p. Using the identification of $H^0(T, kT)$ with the center $Z(kT)$, we set

$$p_j = \sum_{i=1}^{p^n} a^i b^j \quad (1 \le j < e),$$

so that $H^0(T, kHb^j)$ is spanned by p_j. It is clear that $p_i p_j = 0$. Furthermore, if $\alpha \in H^r(T, kH)$ with $r > 0$, then $\alpha p_j = 0$. This is because

$$H^r(T, kH) H^0(T, kHb^j) \subseteq H^r(T, (kH)(kHb^j)) = H^r(T, kHb^j) = 0.$$

We now turn to the kT-module structure of kH. For any integer j, let V_j be the eigenspace in kH, with eigenvalue s^j, for the linear transformation induced by conjugation by b. First note that the s^j $(0 \le j < e)$ are distinct as elements of k: by the Lemma, $s^i - 1$ $(1 \le i < e)$ is a unit in $\mathbb{Z}/p^n\mathbb{Z}$, and so cannot be a multiple of p. We claim that kH is the direct sum of $V_0, \ldots, V_{e-1}$. To see this, we must first describe the simple kT-modules. As H is a normal subgroup of T, any simple kT-module restricts to a semisimple kH-module by Clifford's Theorem [1, Thm. 3.4], and as H is a p-group, kH thus acts trivially on any simple kT-module. Therefore the simple kT-modules are just the simple kK-modules considered as kT-modules with H acting trivially. Since e is relatively prime to p, the simples are, up to isomorphism, $S_0, \ldots, S_{e-1}$, where each S_j is one-dimensional and b acts on S_j as multiplication by s^j. Hence to establish our claim we must show that kH is a semisimple kT-module. But from our analysis of the conjugacy classes we see that kH is isomorphic to the direct sum of k and m copies of

$$k{\uparrow}_H^T \cong kK \cong S_0 \oplus \cdots \oplus S_{e-1},$$

and that establishes the claim. It also shows that $\dim(V_0) = m+1$, while $\dim(V_j) = m$ for $1 \le j < e$.

As H is the Sylow p-subgroup of G, restriction from $H^*(T, kH)$ to $H^*(H, kH)$ is injective [5, Prop. 4.2.2]. By [5, Cor. 4.2.7], the image is $H^*(H, kH)^K$, and therefore res_H^T provides an isomorphism of graded algebras

$$H^*(T, kH) \cong H^*(H, kH)^K.$$

Since H is Abelian, [10, Prop. 3.2] provides an isomorphism of graded algebras

$$H^*(H, kH) \xrightarrow{\cong} kH \otimes H^*(H, k).$$

This is also a map of kK-modules, where K acts diagonally on the tensor product. Composing these two isomorphisms, we have an isomorphism of graded algebras

$$H^*(T, kH) \cong (kH \otimes H^*(H, k))^K, \tag{1}$$

where K acts by conjugation on kH. We claim that the action of K on

$$H^*(H,k) = k[x,y \mid \deg(x)=1, \deg(y)=2, x^2=0].$$

is given by $bx = s^{-1}x$, $by = s^{-1}y$. To see this, let $\mathbf{P} \xrightarrow{\epsilon} k$ be the standard (minimal) kH-resolution, i.e., each P_n is a free kH-module on one generator and the differentials alternate between multiplication by $a-1$ and multiplication by $\sum_{h\in H} h$. Let t be the integer between 1 and p^n-1 satisfying $[t]=[s]^{-1}$ in $(\mathbb{Z}/p^n\mathbb{Z})^\times$. We may define a chain map θ from $\mathbf{P}$ to itself, commuting with ϵ and satisfying $\theta(hx) = b^{-1}hb\theta(x)$ ($h \in H$, $x \in \mathbf{P}$), such that in degrees 1 and 2, θ multiplies the generator by $1+a+\cdots+a^{t-1}$. On the level of cocycles, the action of b is given by precomposing with θ, so in degrees 1 and 2 this just multiplies the cocycle by t.

It follows from equation (1) that $H^*(T,kH)$ is isomorphic to the subring

$$V_0 \otimes 1 \quad \oplus \quad \bigoplus_{i=1}^{\infty} V_i \otimes \langle xy^{i-1}, y^i\rangle.$$

We now turn to the problem of finding generators and relations for this subring. Let

$$w = \sum_{j=0}^{e-1} s^{-j} a^{s^j}.$$

Clearly $w \in V_1$. We will show shortly that $w \in J \setminus J^2$, where J is the radical of the algebra kH, that is the ideal generated by $1-a$. It will follow that $w = u(1-a)$, where u is an invertible element of kH. Hence $w^r \in J^r \setminus J^{r+1}$ for $0 \le r < p^n$. In particular the w^r form a basis for kH. In fact, since $V_qV_r \subseteq V_{q+r}$ for any integers q and r, we see that the w^{ei} ($0 \le i \le m$) form a basis for V_0, and the w^{ei+j} ($0 \le i < m$) form a basis for V_j ($1 \le j < e$).

To prove our claim about w, we look at its image in kH/J^2. First note that for any $r \ge 0$,

$$a^r \equiv (1+(a-1))^r \equiv 1 + r(a-1) \equiv ra + 1 - r \mod J^2.$$

Also note that $\sum_{j=0}^{e-1} s^{-j} = 0$ (multiply by $1-s^{-1}$). Hence

$$w \equiv \sum_{j=0}^{e-1} s^{-j}(s^j a + 1 - s^j) = \sum_{j=0}^{e-1}(a - 1 + s^{-j}) \equiv e(a-1) \mod J^2,$$

which establishes the claim that $w \in J \setminus J^2$.

Now define the remaining generators as follows:

$$z = w^e \otimes 1, \quad y_1 = w \otimes x, \quad y_2 = w \otimes y, \quad y_{2e-1} = 1 \otimes xy^{e-1}, \quad y_{2e} = 1 \otimes y^e.$$

It is easy to see these generate $H^*(T,kH)$. First, the z^i ($0 \le i \le m$) form a basis for $V_0 \otimes 1$. Second, if $1 \le j < e$ then the $z^i y_1 y_2^{j-1}$ ($0 \le i < m$) form a basis for $V_j \otimes xy^{j-1}$, and the $z^i y_2^j$ form a basis for $V_j \otimes y^j$. Third, the $z^i y_{2e-1}$ ($0 \le i \le m$) form a basis for $V_e \otimes xy^{e-1}$. Finally, multiplication by y_{2e} is an isomorphism from

$H^r(T,kH)$ to $H^{r+2e}(T,kH)$ for all $r \geq 0$, and using this we obtain an explicit basis for each homogeneous component of $H^*(T,kH)$.

Since $w = u(1-a)$, it is clear that $zp_j = 0$. The remaining relations are equally easy to check. We conclude that there is a surjective map of graded algebras from A^* to $H^*(T,kT)$, where A^* is the algebra defined abstractly by the generators and relations of the Theorem. We wish to show that this map is an isomorphism, and to do this it suffices to show $\dim(A^r) \leq \dim(H^r(T,kT))$ for all $r \geq 0$.

Since the product of p_j $(1 \leq j < e)$ with each generator of A^* is 0, we have $A^* = A_1 \oplus A_2$, where A_2 is spanned by the p_j and A_1 is the subalgebra generated by the remaining generators. Direct inspection of the relations shows that A_1 is spanned by the same elements described above which form a basis for $H^*(T,kH)$. Thus the dimension inequality is satisfied, and the proof is complete.

3. The principal block case

Suppose for now that G is any finite group and let

$$kG = B_0 + \cdots + B_s$$

be the block decomposition of kG, with B_0 the principal block. Considering kG as a module under conjugation, this yields an isomorphism of graded k-algebras

$$H^*(G,kG) \cong H^*(G,B_0) \oplus \cdots \oplus H^*(G,B_s).$$

It is not hard to see that $H^*(G,B_i) \cong H^*(B_i,B_i)$ $(0 \leq i \leq s)$. Now there are maps of kG-modules

$$k \longrightarrow B_0 \longrightarrow k$$

whose composite is the identity: the first map sends 1 to the principal block idempotent, the second is the restriction of the augmentation map to B_0. (If B_0 is replaced by B_i for $i > 0$ then the restriction of the augmentation map is 0.) Both of these are also maps of k-algebras and so applying the functor $H^*(G,-)$ we obtain maps of graded algebras

$$H^*(G,k) \xrightarrow{f} H^*(G,B_0) \longrightarrow H^*(G,k)$$

whose composite is the identity. In particular f is injective, so the induced map

$$\frac{H^*(G,k)}{\operatorname{rad}(H^*(G,k))} \xrightarrow{\bar{f}} \frac{H^*(G,B_0)}{\operatorname{rad}(H^*(G,B_0))}.$$

is also injective.

Conjecture 1. *Let G be a finite group and k a field of characteristic p. Then the map $\bar{f}$ is an isomorphism.*

In [10, §§10–11], we showed that this conjecture holds when G is a p-group, when G is Abelian, when $G = A_4$ and $p = 2$, and when $G = S_3$ and p is 2 or 3. We now show that it holds whenever B_0 is cyclic:

Theorem 3. *Suppose G has cyclic Sylow p-subgroups. Then Conjecture 1 holds.*

Proof. Let P be a Sylow p-subgroup of G, and let p^n be the order of P. Let $N = N_G(P)$, so N is the semidirect product of P and a p'-group Q. Let $Q_1 = C_Q(P)$, so $\bar{Q} = Q/Q_1$ acts faithfully on P. Let $e = |\bar{Q}|$. Then e divides $p-1$ and $\bar{Q} = \langle b \rangle$ is cyclic of order e. This is because $\bar{Q}$ is isomorphic to a p'-subgroup of $\mathrm{Aut}(P)$, and $\mathrm{Aut}(P)$ is the product of a p-group and a cyclic group of order $p-1$. In fact, e is the inertial index of B_0, as can be seen from the definition [1, p. 123]: In this case the Brauer correspondent of B_0 is the principal block b_0 of N, which covers the principal block b_0' of $C_G(P) = PC_G(P)$. The block b_0' is stabilized by N, so the index of $C_G(P)$ in the stabilizer of b_0' is $[N : C_G(P)] = [Q : Q_1] = e$.

Let $T = P \rtimes \bar{Q}$. It is clear that b acts on $P = \langle a \rangle$ by sending a to a^s, where $[s]$ is a unit in $\mathbb{Z}/p^n$ of order e, as this is the only way $\bar{Q}$ can act faithfully on P. We have already seen that kT is derived equivalent to B_0, but in this case there is a more direct way to see this: a theorem of Rouquier [9, Thm. 10] implies that B_0 is derived equivalent to the principal block of kN, and since T is the quotient of N by a normal p'-subgroup, the principal block of kN is actually isomorphic to kT.

Suppose $p^n > 2$. We claim

$$H^*(G,k) = k[u, v \mid \deg(u) = 2e-1, \deg(v) = 2e, u^2 = 0].$$

To see this, consider the restriction map

$$H^*(G,k) \to H^*(P,k) = k[x, y \mid \deg(x) = 1, \deg(y) = 2, x^2 = 0].$$

This map is injective and since P is Abelian, the image is the set of fixed points under the action of N ([5, Thm. 4.2.8]). This is the same as the fixed points under the action of b, and it is not hard to see that $u = xy^{e-1}$ and $v = y^e$ generate the fixed point subring. Now from Theorem 1 we see that $H^*(B_0, B_0)$ modulo its radical is a polynomial algebra in one generator in degree $2e$. The same is true for $H^*(G,k)$ modulo its radical. Since $\bar{f}$ is injective, it must therefore be an isomorphism.

If $p^n = 2$ then both algebras modulo their radicals are polynomial algebras in one generator in degree 1, so $\bar{f}$ is an isomorphism in this case as well. □

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