THE COHOMOLOGY OF SPLIT EXTENSIONS OF ELEMENTARY ABELIAN 2-GROUPS AND TOTARO'S EXAMPLE

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Abstract. In a previous paper we derived an expression for the differentials in the Lyndon-Hochschild-Serre spectral sequence of a split extension $G = H \rtimes Q$ of finite groups with coefficients in a field. Here we apply that result to the case where H and Q are elementary abelian 2-groups and char $k = 2$. We then work out a special case, in which H has rank 4 and Q has rank 2, from a class of examples constructed by Burt Totaro. Totaro proved that the spectral sequence arising from this extension could not collapse, but using our methods we are able to obtain complete information on the spectral sequence.

1. Introduction

1.1. **Background.** Most spectral sequences that have been used in group cohomology calculations arise from central group extensions. Recently, however, there has been growing interest in spectral sequences arising from non-central extensions. Benson-Feshbach [3], for example, raised some questions about the vanishing of differentials in split extensions with abelian kernel. (See also [8].) Burt Totaro [11] then constructed an interesting family of counterexamples to answer all of these questions negatively. In particular, for each prime p, Totaro constructs a split group extension in which both kernel and quotient are elementary abelian p-groups, such that the differential d_p in the Lyndon-Hochschild-Serre (LHS) spectral sequence with mod-p coefficients is non-zero. To our knowledge, this was the first example of that phenomenon, at any prime.

Totaro's argument is simple and elegant, and manages to show $d_p \neq 0$ for very general reasons. Yet it gives almost no other information about the spectral sequence, and in particular it does not show what d_p, or any other differential, actually is. It therefore might be interesting to examine one of these examples in greater detail, in the hopes of gaining greater insight into the mystery of p-group cohomology.

We do this for the smallest of Totaro's examples, in which $p = 2$ and the split extension is an elementary abelian 2-group of rank 2 acting on one of rank 4. We calculate the E_2-page (which has a very rich structure), and the differentials d_2. We then show that $d_3 = 0$, and, by the position of the generators, it follows that $E_3 = E_\infty$.

The calculation of the differentials is achieved by applying the main theorem of Siegel [9]. In that paper, we gave a general method for calculating differentials in the case of a split extension of finite groups. As we show in Section 2, that method

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works out particularly elegantly in the case where $p = 2$ and the quotient group is an elementary abelian 2-group. Section 3 deals with the E_2-page of Totaro's extension, and Section 4 applies the method of Section 2 to this case. That method requires the calculation of a certain twisting cochain, and an algorithm for doing this in the case where both quotient and kernel are elementary abelian 2-groups is explained in Section 5. The output of that algorithm for our example is given in Figures 2 and 3. Though it is on the borderline of something which can be carried out by hand, we used a computer to implement the algorithm.

1.2. **Preliminaries.** For this paper, by a complex of modules over a ring R we will mean a left $\mathbb{Z}$-graded R-module $C = \bigoplus_{n\in\mathbb{Z}} C_n$ together with an endomorphism ∂ satisfying $\partial(C_n) \subseteq C_{n-1}$ and $\partial^2 = 0$. Hom and tensor products of complexes are defined as in Brown [4, Ch. I §0]. By *bigraded complex* we will mean an R-module $E = \bigoplus_{r,s\in\mathbb{Z}} E^{r,s}$ together with an endomorphism δ satisfying $\delta(E^{r,s}) \subseteq \sum_{i\geq r} E^{i,r+s-i+1}$. The bigraded complex yields a spectral sequence arising from the filtration $F^pE = \sum_{r\geq p, s\in\mathbb{Z}} E^{r,s}$.

Let k be a field. If V is a k-module then $\langle v_1, \ldots, v_n\rangle$ denotes the subspace spanned by $v_1, \ldots, v_n \in V$. An unlabeled tensor product will always mean "$\otimes_k$". If U and V are graded k-submodules of a graded k-algebra A, then UV denotes the graded k-submodule spanned by all uv, $u \in U$, $v \in V$. If $W = UV$ and the natural map $U \otimes V \to W$ is an isomorphism then we will also write $W = U \otimes V$ (in analogy with an internal direct sum). If G is a group, σ an automorphism of G, and U and V are kG-modules, then we let $\mathrm{Hom}^{\sigma}_{kG}(U, V)$ denote the set of f in $\mathrm{Hom}_k(U, V)$ satisfying $f(gx) = \sigma(g)f(x)$ for all $g \in G$, $x \in U$. Finally, we use the notation common in computer science, where for a proposition $\mathcal{P}$, $(\mathcal{P}) = 1$ if $\mathcal{P}$, else $(\mathcal{P}) = 0$.

2. Elementary abelian 2-groups

2.1. **Notation.** Let $E = \langle g_1, \ldots, g_r\rangle$ be an elementary abelian 2-group of rank r, and k a field of characteristic 2. Let $X_i = g_i + 1 \in kE$ for each i. We construct the minimal resolution $P \to k$ as follows. Let $P = kE[\mathsf{x}_1, \ldots, \mathsf{x}_r]$, the polynomial algebra in r indeterminates over the commutative ring kE. The grading determined by degree gives P the structure of a graded kE-module which is finitely-generated and free in each degree. Now for any r-tuple $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}^r$ ($\mathbb{N} = \{0, 1, 2, \ldots\}$), let $|\alpha| = \sum_{i=1}^r \alpha_i$, $g^\alpha = g_1^{\alpha_1} \cdots g_r^{\alpha_r}$, and $\mathsf{x}^\alpha = \mathsf{x}_1^{\alpha_1} \cdots \mathsf{x}_r^{\alpha_r}$. Let $\varepsilon(i) \in \mathbb{N}^r$ be the r-tuple which is 1 in position i and 0 elsewhere. Define the differential on P by $\partial(\mathsf{x}^\alpha) = \sum X_i \mathsf{x}^{\alpha-\varepsilon(i)}$, where the sum is taken over all $1 \leq i \leq r$ such that $\alpha_i \neq 0$, and define the augmentation by $\epsilon(1) = 1$, and $\epsilon(\mathsf{x}^\alpha) = 0$ if $\mathsf{x}^\alpha \neq 1$.

Now suppose A is a graded commutative k-algebra on which E acts as graded algebra automorphisms. As a graded k-module, the cochains $C(E, A) = \mathrm{Hom}_{kE}(P, A)$ may be identified with $A[x_1, \ldots, x_r]$, where ax^α corresponds to the map which takes x^α to a and vanishes on x^β for $\beta \neq \alpha$. Now the action of kE on A induces an action of $kE[x_1, \ldots, x_r]$ on $A[x_1, \ldots, x_r]$, and it is not difficult to see that the differential in $C(E, A)$ is given by multiplication by $\sum_{i=1}^r X_i x_i$. Moreover,

Lemma 2.1. *The cup product in $H^*(E, A)$ is induced by the associative product on $A[x_1, \ldots, x_r]$ defined by $ax^\alpha \smile bx^\beta = ag^\alpha(b)x^{\alpha+\beta}$.*

Proof. There is a diagonal approximation map $P \to P \otimes P$ defined by $\mathsf{x}^\alpha \mapsto \sum_{\beta+\gamma=\alpha} \mathsf{x}^\beta \otimes g^\beta(\mathsf{x}^\gamma)$. One can see this by checking the case $r = 1$ and then using the fact that the tensor product of chain maps is again a chain map. By applying $\mathrm{Hom}_{kE}(P,-)$ one obtains the product described in the lemma. $\square$

2.2. **Differentials.** Let $G = H \rtimes Q$ be a split group extension, where for now H is any finite group, and $Q = \langle g_1, \dots, g_s\rangle$ is an elementary abelian 2-group of rank s. Let $Y = kQ[\mathsf{y}_1, \dots, \mathsf{y}_s]$ be the minimal kQ-resolution, and P the minimal kH-resolution. Our method for calculating differentials will require us to find a *twisting system for the extension*, which is a collection of maps $f_\alpha \in \mathrm{Hom}^\sigma_{kH}(P,P)_{|\alpha|-1}$, where σ is the automorphism of kH induced by g^α, $(0 \neq \alpha \in \mathbb{N}^s)$ satisfying

$$\begin{aligned} &\text{(i)}\quad \epsilon \circ f_\alpha = \epsilon \quad \text{if } |\alpha| = 1 \\ &\text{(ii)}\quad \partial \circ f_\alpha + f_\alpha \circ \partial = 1(\alpha = 2\varepsilon(i) \text{ for some } i) + \sum_{\beta+\gamma=\alpha} f_\beta \circ f_\gamma. \end{aligned} \tag{2.1}$$

Now in the LHS spectral sequence $\{E_r, d_r\}$ we have

$$E_1 = \mathrm{Hom}_{kQ}(Y, \mathrm{Hom}_{kH}(P,k)) \approx H^*(H,k)[y_1, \dots, y_s].$$

For $\gamma \in \mathbb{N}^s$ with $r = |\gamma| \geq 1$, let $\hat{f}_\gamma = \mathrm{Hom}_{kH}(f_\gamma, k) \in \mathrm{End}_k(H^*(H,k))_{1-r}$, and set $\chi_r = \sum_{|\gamma|=r} \hat{f}_\gamma y^\gamma \in \mathrm{End}_k(H^*(H,k))[y_1, \dots, y_s]$. This last ring acts on E_1, and we let $\delta_r \colon E_1^{p,q} \to E_1^{p+r,q-r+1}$ denote multiplication by χ_r.

Theorem 2.2. *There exists a twisting system for the extension $G = H \rtimes Q$. Moreover, given a twisting system, set $\delta = \sum_{r\geq 1} \delta_r$; we then have $\delta^2 = 0$, and the spectral sequence arising from the bigraded complex (E_1, δ) equals the LHS spectral sequence.*

Proof. Define $t \colon k \otimes_{kQ} Y \to \mathrm{Hom}_{kG}(P{\uparrow}_H^G, P{\uparrow}_H^G) \approx \bigoplus_{\sigma \in Q} \sigma \otimes \mathrm{Hom}^\sigma_{kH}(P,P)$ by

$$t(\mathsf{y}^\alpha) = 1(|\alpha| = 1) + g^\alpha \otimes f_\alpha.$$

The conditions on the f_α are seen to be exactly equivalent to the condition that t be a twisting cochain for the group extension $1 \to H \to G \to Q \to 1$, as defined in [9, Section 5]. So existence follows from [9, Theorem 5.1] and the spectral sequence statement follows from [9, Theorem 8.1]. $\square$

3. Totaro's example: E_2-page

Let $Q = \langle g_1, g_2\rangle$, $H = \langle h_1, \dots, h_4\rangle$ be elementary abelian 2-groups of ranks 2 and 4, respectively. Let Q act on H according to

$$T_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix},$$

where T_j is the matrix of the linear transformation induced by g_j on the $\mathbb{F}_2$-vector-space H with respect to the basis $h_1, \dots, h_4$. We let G be the semidirect product of H and Q defined by this action; this is the smallest of the class of examples considered by Totaro in [11]. We continue with the notation of the previous section; in particular $P = \mathbb{F}_2 H[\mathsf{x}_1, \dots, \mathsf{x}_4]$ and $Y = \mathbb{F}_2 Q[\mathsf{y}_1, \mathsf{y}_2]$ are the minimal resolutions of $\mathbb{F}_2$ over $\mathbb{F}_2 H$ and $\mathbb{F}_2 Q$, respectively. Let $x_1, \dots, x_4 \in H^1(H, \mathbb{F}_2) = \mathrm{Hom}(H, \mathbb{F}_2)$ denote the dual basis to $h_1, \dots, h_4$. Let $A = H^*(H, \mathbb{F}_2) = \mathbb{F}_2[x_1, \dots, x_4]$. As a graded kQ-module, $A = \mathrm{Sym}(M)$, where M is spanned by the x_i and the action of Q is given by the transposes of T_1 and T_2. Let $\{E_r, d_r\}$ denote the LHS spectral sequence of the split extension.

We seek to describe E_2. But first we note that, though M is an indecomposable $\mathbb{F}_2Q$-module, if we let $k = \mathbb{F}_4$ then $k \otimes_{\mathbb{F}_2} M = M_1 \oplus M_2$, where $M_1 = \langle v_1, v_2 \rangle$, $M_2 = \langle v_3, v_4 \rangle$, and

$$v_1 = \omega x_1 + x_2, \quad v_2 = \bar{\omega} x_3 + x_4, \quad v_3 = \bar{\omega} x_1 + x_2, \quad v_4 = \omega x_3 + x_4.$$

Q fixes v_1 and v_3, and $g_1 v_2 = \omega v_1 + v_2$, $g_1 v_4 = \bar{\omega} v_3 + v_4$, $g_2 v_2 = v_1 + v_2$, $g_2 v_4 = v_3 + v_4$. The Frobenius automorphism $\lambda \mapsto \overline{\lambda}$ of k over $\mathbb{F}_2$ extends to a ring-automorphism of A, defined by $\overline{\lambda x^\alpha} = \overline{\lambda} x^\alpha$, which is of order 2 and commutes with the action of Q. This automorphism exchanges M_1 and M_2; specifically, $\overline{v_1} = v_3$ and $\overline{v_2} = v_4$. Setting $\mathcal{S} = k[v_1, v_2] = \mathrm{Sym}(M_1)$, we therefore have $k \otimes A = \mathcal{S} \otimes \overline{\mathcal{S}}$. Hence we will deal primarily with $k \otimes_{\mathbb{F}_2} E_2$ instead of E_2. Let $\gamma_i = [y_i] \in E_2^{1,0}$ $(i = 1, 2)$.

Proposition 3.1. *The following hold:*

(i) *For each indecomposable $\mathbb{F}_2Q$-module U let $p_U(t) = \sum_{n \geq 0} a_n t^n$, where a_n is the multiplicity of U as a summand of $H^n(H, \mathbb{F}_2)$. Then*

$$p_{\mathbb{F}_2}(t) = \frac{1}{(1-t^4)^2}, \qquad p_M(t) = \frac{t+t^3}{(1-t^4)^2}, \qquad p_{\Omega\mathbb{F}_2}(t) = \frac{2t^2}{(1-t^4)^2},$$
$$p_{\Omega^2\mathbb{F}_2}(t) = \frac{t^4}{(1-t^4)^2}, \qquad p_{\mathbb{F}_2Q}(t) = \frac{t^2(1+2t+2t^3-t^4)}{(1-t)^2(1-t^4)^2},$$

and $p_U = 0$ for U not isomorphic to one of the five modules above.

(ii) *$H^*(H, \mathbb{F}_2)^Q$ is generated by x_1, x_2,*

$$\begin{aligned}
q_1 &= x_1^2 x_3 + x_1 x_3^2 + x_1^2 x_4 + x_1 x_4^2 + x_2^2 x_4 + x_2 x_4^2, \\
q_2 &= x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 + x_2^2 x_4 + x_2 x_4^2, \\
z_1 &= x_4(x_1^2 x_2 + x_1^2 x_4 + x_1 x_2^2 + x_1 x_2 x_4 + x_2^2 x_4 + x_4^3) = \prod_{\sigma \in Q} \sigma(x_4), \\
z_2 &= x_3(x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_1 x_2 x_3 + x_2^2 x_3 + x_3^3) = \prod_{\sigma \in Q} \sigma(x_3),
\end{aligned}$$

subject to the relations

$$\begin{aligned}
0 &= q_1^2 + x_1^2 z_1 + x_1^2 z_2 + x_2^2 z_1 + x_1^2 x_2 q_1 + x_1 x_2^2 q_1, \\
0 &= q_2^2 + x_1^2 z_2 + x_2^2 z_1 + x_2^2 z_2 + x_1^2 x_2 q_2 + x_1 x_2^2 q_2,
\end{aligned}$$

and has Poincaré series $(1+t^3)^2/(1-t)^2(1-t^4)^2$.

(iii) *$H^*(H, k)^Q$ is generated by v, $\bar{v}$, u, $\bar{u}$, s, $\bar{s}$, where $v = v_1$, $u = v_1 v_4^2 + v_2 v_3^2$, and $s = v_1^3 v_2 + v_2^4 = \prod_{\sigma \in Q} \sigma(v_2)$, subject to the relations*

$$\begin{aligned}
0 &= \omega u^2 + \bar{u}^2 + \omega v^2 \bar{s} + \bar{v}^2 s + v^3 u + \omega \bar{v}^3 \bar{u}, \\
0 &= u^2 + \omega \bar{u}^2 + v^2 \bar{s} + \omega \bar{v}^2 s + \omega v^3 u + \bar{v}^3 \bar{u}.
\end{aligned}$$

(iv) *As a bigraded k-module we have $k \otimes_{\mathbb{F}_2} E_2 = k[s, \bar{s}] \otimes \tilde{E}_2$, where*

$$\tilde{E}_2 = k[v, \bar{v}] \otimes \langle 1, u, \bar{u}, u\bar{u} \rangle \oplus k[\gamma_1, \gamma_2] \otimes \langle \gamma_2, \zeta, \bar{\zeta}, \zeta\bar{\zeta} \rangle \oplus k[\gamma_1] \otimes \langle \gamma_1, \gamma_1 v, \gamma_1 \bar{v}, \gamma_1 u, \gamma_1 \bar{u} \rangle \oplus \langle \gamma_1 v u \rangle,$$

where $\zeta = [(\bar{\omega} v_1 v_2 + \omega v_2^2) y_1 + (v_1 v_2 + v_2^2) y_2]$.

Note. $\tilde{E}_2$ is depicted in Figure 1. In the leftmost column we give the kQ-module structure of $(W\overline{W})^q$, and at each point (p, q) we give the dimension of and a basis for $\tilde{E}_2^{p,q}$.

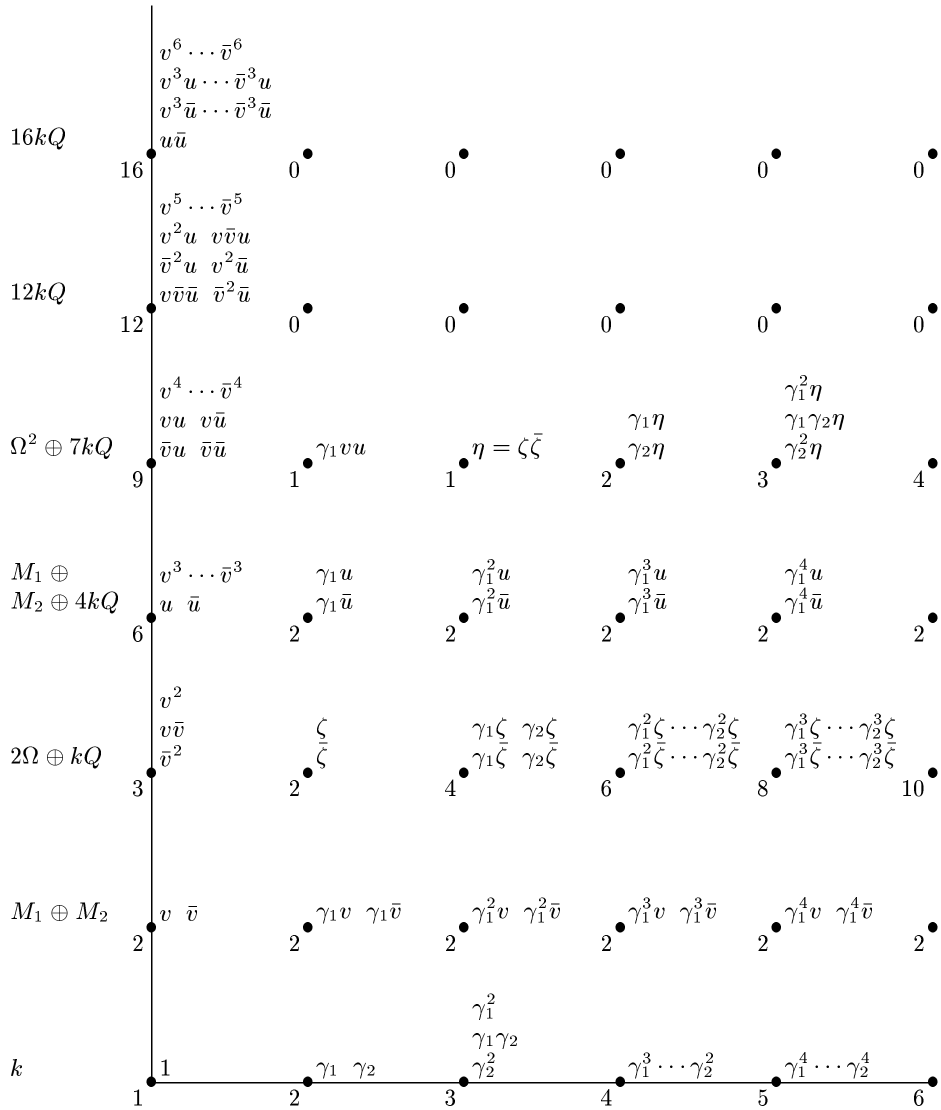

FIGURE 1. $\tilde{E}_2^{p,q} = H^p(Q, (W\overline{W})^q)$

The remainder of this section will consist of a proof of this proposition. We begin by analyzing the structure of $\mathcal{S}$. The following is well-known:

Lemma 3.2. *Let k be a field of characteristic 2 and let σ be the algebra automorphisms of $k[x,y]$ defined by $\sigma(x) = x$, $\sigma(y) = x + y$. Then $k[x,y]^\sigma$ is a polynomial ring in the two generators x and $y(x+y)$.*

By Lemma 3.2, $\mathcal{S}^{g_2} = k[v_1, w]$, where $w = v_2(v_1 + v_2)$. We have $g_1 w = v_1^2 + w$. Now multiplication by $v_1 \colon (\mathcal{S}^{2n})^{g_2} \to (\mathcal{S}^{2n+1})^{g_2}$ is an isomorphism of $k\langle g_1\rangle$-modules, so $\mathcal{S}^{g_2} = (\mathcal{S}^{2*})^{g_2} \oplus v_1(\mathcal{S}^{2*})^{g_2}$. Again, Lemma 3.2 implies $(\mathcal{S}^{2*})^{Q} = k[v_1^2, s]$, where $s = w(v_1^2 + w)$, so has Poincaré series $1/(1-t^2)(1-t^4)$. By multiplying this by $1+t$ we see that the Poincaré series of $\mathcal{S}^Q$ is $1/(1-t)(1-t^4)$, and hence $\mathcal{S}^Q = k[v_1, s]$.

We have $\mathcal{S}^0 \cong k$, $\mathcal{S}^1 \cong M_1$, $\mathcal{S}^2 \cong \Omega k$ as it is 3-dimensional with 1-dimensional socle, and $\mathcal{S}^3 \cong kQ$ as it is 4-dimensional with 1-dimensional socle. Let $W = k[v]\sum_{i=0}^{3} \mathcal{S}^i$. We claim that $\mathcal{S} = k[s] \otimes W$. To prove this it suffices to show that s, v is a regular sequence in $\mathcal{S}$. For then multiplication by v_1 induces an injective kQ-homomorphism from $\mathcal{S}^n/s\mathcal{S}^{n-4}$ to $\mathcal{S}^{n+1}/s\mathcal{S}^{n-3}$ for all $n \geq 4$. Since both domain and range have dimension 4 it is therefore an isomorphism. It follows that $\mathcal{S} = k[s]W$. Hence the natural map $k[s] \otimes W \to \mathcal{S}$ is surjective, and by comparing Poincaré series we see it is an isomorphism. Hence the claim follows from the following, with $R = k[v_1]$ and $x = v_2$:

Lemma 3.3. *Let k be a field and $R = \oplus_{n\geq 0} R_n$ a graded commutative k-algebra with $R_0 = k$, and assume that R is an integral domain. Let $\mathfrak{m} = \oplus_{n>0} R_n$. Suppose f is a monic polynomial of positive degree in $R[x]$, and let $0 \neq a \in \mathfrak{m}$. Then f, a is a regular sequence in $R[x]$.*

Proof. Given an element of $R[x]/R[x]f$ we may choose $g \in R[x]$ representing that element with $\deg(g) < \deg(f)$. Thus $\deg(ag) = \deg(g) < \deg(f)$, while every non-zero element of $R[x]f$ has degree at least $\deg(f)$. Hence $a + R[x]f$ is not a zero-divisor in $R[x]/R[x]f$. □

We thus have the following Poincaré series for the multiplicities of the 4 indecomposables which occur as summands of $\mathcal{S}$:

$$p_k(t) = \frac{1}{1-t^4}, \quad p_{M_1}(t) = \frac{t}{1-t^4}, \quad p_{\Omega k}(t) = \frac{t^2}{1-t^4}, \quad p_{kQ}(t) = \frac{t^3}{(1-t)(1-t^4)}.$$

The same statement holds replacing M_1 with M_2 and $\mathcal{S}$ with $\overline{\mathcal{S}}$. Since $k \otimes_{\mathbb{F}_2} A \cong \mathcal{S} \otimes \overline{\mathcal{S}}$, we need to know the tensor products of these indecomposables. But the tensor product of a free module with any module is free, and furthermore we claim

$$M_1 \otimes M_2 \cong kQ, \;\; M_1 \otimes \Omega k \cong M_1 \oplus kQ, \;\; \Omega k \otimes \Omega k \cong \Omega^2 k \oplus kQ.$$

The first isomorphism can be seen using varieties (cf. Evens [7, Theorem 10.1.1]), as

$$\begin{aligned} V_Q(\tilde{k} \otimes M_1 \otimes M_2) &= V_Q(\tilde{k} \otimes M_1) \cap V_Q(\tilde{k} \otimes M_2) \\ &= V((y_1 + \omega y_2)) \cap V((y_1 + \bar{\omega} y_2)) = \{0\}, \end{aligned}$$

where $\tilde{k}$ is an algebraic closure of k, so $\tilde{k} \otimes M_1 \otimes M_2$, and therefore $M_1 \otimes M_2$, is free. The other two follow from the facts that for any f.g. kQ-module U, $\Omega k \otimes U$ is isomorphic to the direct sum of ΩU and a projective module (cf. [1, Corollary 3.1.6]), and $\Omega M_i \cong M_i$ because M_i is periodic and hence has period 1, by Benson-Carlson [2, Proposition 2.2]. From this we can determine the kQ-module structure of $k \otimes_{\mathbb{F}_2} A$. To obtain the $\mathbb{F}_2 Q$-module structure of A, use the Noether-Deuring Theorem [6, Theorem 29.7], which says that for any f.g. $\mathbb{F}_2 Q$-modules U, V, $U \cong V \Leftrightarrow k \otimes_{\mathbb{F}_2} U \cong k \otimes_{\mathbb{F}_2} V$. So the multiplicities of $\mathbb{F}_2$, $\Omega\mathbb{F}_2$, $\Omega^2\mathbb{F}_2$, $\mathbb{F}_2 Q$, and M in A equal the respective multiplicities of k, Ωk, $\Omega^2 k$, kQ, and M_1 in $k \otimes_{\mathbb{F}_2} A$. Hence

$$p_{\mathbb{F}_2} = p_k^2, \quad p_M = p_k p_{M_1} + p_{\Omega k} p_{M_1}, \quad p_{\Omega \mathbb{F}_2} = 2p_k p_{\Omega k}, \quad p_{\Omega^2 \mathbb{F}_2} = p_{\Omega k}^2,$$
$$p_{\mathbb{F}_2 Q} = 2p_k p_{kQ} + p_{M_1}^2 + 2p_{\Omega k} p_{M_1} + 4p_{M_1} p_{kQ} + p_{\Omega^2 k} + 6p_{\Omega k} p_k + 4p_{kQ}^2,$$

from which we obtain part (i) of the proposition.

We now turn to (ii). The Poincaré series for A^Q is as claimed since it equals $\sum_U \dim_k(U^Q) p_U(t)$, the sum taken over representatives U of the isomorphism classes of indecomposable kQ-modules. We must next check the elements are invariant and satisfy the relations. One is helped here by the automorphism ϕ of M of order 3, defined by $x_1 \mapsto x_2$, $x_2 \mapsto x_1 + x_2$, $x_3 \mapsto x_3 + x_4$, $x_4 \mapsto x_3$. We have $\phi(q_1) = q_2$, $\phi(q_2) = q_1 + q_2$, and similarly for z_i replacing q_i. So if q_1 is invariant, so is q_2. Since z_1 is a norm, it is invariant, and therefore z_2 is as well, since the norm commutes with ϕ. Finally ϕ takes the first relation to the second, so only the first must be checked.

Let B be the subring of A^Q generated by the x_i, q_i, z_i. We will show that $B = A^Q$. First we claim that the subring T of B generated by x_1, x_2, z_1, z_2 is a polynomial ring in those 4 generators. That follows from the following lemma (with $R = k[x_1, x_2]$, $S = R[x_4, x_3]$), the proof of which is an easy exercise:

Lemma 3.4. *Let R be an integral domain, n and d positive integers, and $S = R[w_1, \ldots, w_n]$, the polynomial ring in n variables. Suppose $z_1, \ldots, z_n \in S$ and for each i there exists $0 \neq r_i \in R$ such that* $\deg(z_i - r_i w_i^d) < d$. *Then $z_1, \ldots, z_n$ are algebraically independent over R.*

In particular, T has Poincaré series $1/(1-t)^2(1-t^4)^2$. Now consider the subspace $T + q_1 T + q_2 T + q_1 q_2 T$ of B. We claim that this sum is direct. To see this, suppose

$$f_0 + q_1 f_1 + q_2 f_2 + q_1 q_2 f_3 = 0 \qquad f_i \in T \tag{3.1}$$

Consider these as polynomials in x_4 with coefficients in $\mathbb{F}_2[x_1, x_2, x_3]$. Write $q_i = \sum_{j=0}^{2} a_{i,j} x_4^j$, $q_1 q_2 = \sum_{j=0}^{4} b_j x_4^j$, with $a_{i,j}, b_j$ of degree 0. We have $a_{1,1} = x_1^2 + x_2^2$, $a_{1,2} = x_1 + x_2$, $a_{2,1} = x_2^2$, $a_{2,2} = x_2$, $b_3 \neq 0$, $b_4 \neq 0$. For each i such that $f_i \neq 0$, write $f_i = \sum_{j=0}^{m_i} c_{i,j} z_1^j = c_{i,m_i} x_4^{4m_i} + g_i$, where $\deg(f_i) = 4m_i$, $\deg(c_{i,j}) = 0$ and $\deg(g_i) \leq 4m_i - 2$. We may do this since the coefficient of x_4^3 in z_4 is 0. Assuming not all the f_i are 0, let n be the maximum degree of $f_0, q_1 f_1, q_2 f_2, q_1 q_2 f_3$. There are two possibilities: either $n = \deg(f_0) = \deg(q_1 q_2 f_3) \equiv 0 \pmod 4$, or $n = \deg(q_1 f_1) = \deg(q_2 f_2) \equiv 2 \pmod 4$. In the first case, say $n = 4m$. Consideration of the x_4^n- and x_4^{n-1}-terms of (3.1) yields

$$c_{0,m} x_4^n + 0 x_4^{n-1} = b_4 c_{3,m-1} x_4^n + b_3 c_{3,m-1} x_4^{n-1},$$

which implies $c_{3,m-1} = 0$, contradicting $n = \deg(q_1 q_2 f_3)$. In the second case, say $n = 4m + 2$. A similar consideration shows $(x_1 + x_2) c_{1,m} = x_2 c_{2,m}$ and $(x_1^2 + x_2^2) c_{1,m} = x_2^2 c_{2,m}$, from which it is easy to see $c_{2,m} = 0$, contradicting $n = \deg(q_2 f_2)$. Hence all the f_i are 0, so the sum is direct and therefore has Poincaré series $(1 + 2t^3 + t^6)/(1-t)^2(1-t^4)^2$, which equals the Poincaré series of A^Q. Hence $B = A^Q$.

We now turn to the relations. Let $\tilde{B} = \mathbb{F}_2[\tilde{x}_1, \tilde{x}_2, \tilde{q}_1, \tilde{q}_2, \tilde{z}_1, \tilde{z}_2]$. We have a map of $\mathbb{F}_2$-algebras $\tilde{B} \twoheadrightarrow B$ which has kernel containing (r_1, r_2), where the r_i are defined as in (ii), and we wish to show this is exactly the kernel. But Lemma 3.3 (with $R = \mathbb{F}_2[\tilde{x}_1, \tilde{x}_2, \tilde{q}_2, \tilde{z}_1, \tilde{z}_2]$, $x = \tilde{q}_1$) shows that r_1, r_2 is a regular sequence, so the Poincaré series of $\tilde{B}/(r_1, r_2)$ is $(1-t^6)^2/(1-t)^2(1-t^3)^2(1-t^4)^2$, which equals the Poincaré series of B. So (r_1, r_2) is the kernel, proving (ii).

Now for (iii). We have $x_1 = v + \bar{v}$, $x_2 = \bar{\omega}v + \omega\bar{v}$, $q_1 = \omega u + \bar{\omega}\bar{u}$, $q_2 = \bar{\omega}u + \omega\bar{u}$, $z_1 = \omega s + \bar{\omega}\bar{s} + \bar{\omega}\bar{u}v + \omega u\bar{v}$, $z_2 = s + \bar{s} + \bar{u}v + u\bar{v}$. So by (ii), the alleged generators do generate $k \otimes_{\mathbb{F}_2} B$. The relations are the same as those in (ii) multiplied by $\bar{\omega}$, proving (iii).

To prove (iv), recall that $\mathcal{S} = k[s] \otimes W$. Applying the Frobenius, we have $\overline{\mathcal{S}} = k[\bar{s}] \otimes \overline{W}$, so $k \otimes_{\mathbb{F}_2} A = \mathcal{S} \otimes \overline{\mathcal{S}} = k[s, \bar{s}] \otimes W \otimes \overline{W}$, and therefore $k \otimes_{\mathbb{F}_2} E_2 = k[s, \bar{s}] \otimes \tilde{E}_2$, where $\tilde{E}_2 = H^*(Q, W\overline{W})$. Consider the natural map from $k[v, \bar{v}] \otimes \langle 1, u, \bar{u}, u\bar{u}\rangle$ to $(W\overline{W})^Q$. As $v, \bar{v}, u, \bar{u}$ are algebraically independent (by Lemma 3.4, with $R = k[v, \bar{v}]$, $S = R[v_4, v_2]$), the map is certainly injective, and a comparison of the Poincaré series shows it is an isomorphism. That describes $\tilde{E}_2^{0*}$. To get the rest of $\tilde{E}_2$ we must describe the cohomology of the relevant indecomposables:

Lemma 3.5. *For the following kQ-modules U, we give a presentation for U in diagram form, $B(Q,U) = B\operatorname{Hom}_{kQ}(Y, U)$, and a complement $\tilde{H}(Q,U)$ of $B(Q,U)$ in $Z(Q,U)$.*

U	diagram	$B(Q,U)$	$\tilde{H}(Q,U)$
M_1	a_2 a_1 ωa_1	$k[y_1, y_2]\langle \omega a_1 y_1 + a_1 y_2\rangle$	$k[y_1]a_1$
Ωk	b c a	$k[y_1, y_2]\langle a y_1, a y_2\rangle$	$\langle a\rangle \oplus k[y_1, y_2]\langle c y_1 + b y_2\rangle$
$\Omega^2 k$	c_3 c_4 c_5 c_1 c_2	$\langle c_2 y_1 + c_1 y_2\rangle \oplus$ $k[y_1, y_2]\langle c_1 y_1, c_1 y_2^2, c_2 y_1^2, c_2 y_2\rangle$	$\langle c_1, c_2, c_2 y_1\rangle \oplus$ $k[y_1, y_2]\langle c_5 y_1^2 + c_4 y_1 y_2 + c_3 y_2^2\rangle$

In the diagrams, (resp.) signifies the action of $g_1 + 1$ (resp. $g_2 + 1$). The proofs are routine. Next, we have the following explicit decompositions of the $W^i\overline{W}^j$, where the action of Q on the basis elements corresponds to the labeling given in Lemma 3.5:

module	summand	basis
$W^2\overline{W}^0$	Ωk	$a = v_1^2,\ b = v_1 v_2 + v_2^2,\ c = \bar{\omega} v_1 v_2 + \omega v_2^2$
$W^1\overline{W}^2$	M_1	$a_1 = v_1 v_4^2 + v_2 v_3^2,\ a_2 = v_1 v_3 v_4 + v_2 v_4^2,$
	kQ	$v_1 v_3^2,\ v_1 v_3 v_4,\ v_2 v_3^2,\ v_2 v_3 v_4$
$W^2\overline{W}^2$	$\Omega^2 k$	$c_1 = a\bar{b} + b\bar{a},\ c_2 = a\bar{a} + c\bar{a} + a\bar{c},\ c_3 = b\bar{a} + b\bar{b},$ $c_4 = b\bar{a} + c\bar{b} + b\bar{c},\ c_5 = c\bar{c}$
	kQ	$a\bar{a},\ c\bar{a},\ a\bar{b},\ c\bar{b}$

Using this and the Frobenius, we have explicit decompositions of $(W\overline{W})^q$ for $0 \le q \le 4$. Hence by Lemma 3.5, $\tilde{E}_2^{*,1} = H^*(Q, M_1 \oplus M_2) = k[\gamma_1] \otimes \langle v, \bar{v}\rangle$, and therefore $\tilde{E}_2^{+,1} = k[\gamma_1] \otimes \langle \gamma_1 v, \gamma_1 \bar{v}\rangle$. Similarly,

$$\tilde{E}_2^{+,2} = H^+(Q, W^2) \oplus H^+(Q, \overline{W}^2) = k[\gamma_1, \gamma_2] \otimes \langle \zeta, \bar{\zeta}\rangle,$$
$$\tilde{E}_2^{+,3} = H^+(Q, \langle a_1, a_2\rangle) \oplus H^+(Q, \langle \bar{a}_1, \bar{a}_2\rangle) = k[\gamma_1] \otimes \langle \gamma_1 u, \gamma_1 \bar{u}\rangle,$$
$$\tilde{E}_2^{+,4} = H^+(Q, \langle c_1, \ldots, c_5\rangle) = \langle \gamma_1 c_2\rangle \oplus k[\gamma_1, \gamma_2]\eta,$$

where $\eta = [c_5y_1^2 + c_4y_1y_2 + c_3y_2^2]$. It remains to show $[\gamma_1 c_2] = [\gamma_1 vu]$ and $\eta = \zeta\bar{\zeta}$. The first is straightforward, and Lemma 2.1 implies $\zeta\bar{\zeta}$ is represented by

$$\begin{aligned}(cy_1 + by_2) \smile (\bar{c}y_1 + \bar{b}y_2) &= c\overline{g_1(c)}y_1^2 + (c\overline{g_1(b)} + b\overline{g_2(c)})y_1y_2 + b\overline{g_2(b)}y_2^2\\ &= c\bar{c}y_1^2 + (c(\bar{a}+\bar{b}) + b(\bar{a}+\bar{c}))y_1y_2 + b\bar{b}y_2^2\\ &= c_5y_1^2 + (c_4 + c\bar{a})y_1y_2 + (c_1 + c_3 + a\bar{b})y_2^2\\ &\sim c_5y_1^2 + c_4y_1y_2 + c_3y_2^2,\end{aligned}$$

as $c\bar{a}$ and $a\bar{b}$ are in the kQ-summand and $c_1y_2^2$ is a coboundary by Lemma 3.5. This completes the proof of Proposition 3.1.

4. Totaro's example: differentials

We now apply Theorem 2.2 to find the differentials in this spectral sequence. The Theorem requires us to find a twisting system $\{f_\alpha\}$ for the group extension, and the algorithm used for doing this is described in the following section. The values $\hat{f}_\alpha(v^\beta)$ (for $|\alpha| \in \{2,3\}$, $|\beta| \le 4$) appear in Figures 2 and 3. For readability we have abbreviated v^γ by γ in the tables.

Proposition 4.1. *We have $d_2(\zeta) = \gamma_1^3\bar{v}$, $d_2(\bar{\zeta}) = \gamma_1^3 v$, and d_2 vanishes on γ_1, γ_2, v, $\bar{v}$, u, $\bar{u}$, s, $\bar{s}$. Hence $k\otimes_{\mathbb{F}_2} E_3 = k[s,\bar{s}]\otimes\tilde{E}_3$, where*

$$\begin{aligned}\tilde{E}_3 = k[v,\bar{v}]\otimes\langle 1,u,\bar{u},u\bar{u}\rangle \oplus k[\gamma_1,\gamma_2]\otimes\langle\gamma_2,\xi,\bar{\xi},\eta\rangle \oplus&\\ k[\gamma_1]\otimes\langle\gamma_1,\gamma_1 u,\gamma_1\bar{u}\rangle \oplus \langle\gamma_1 v,\gamma_1\bar{v},\gamma_1^2 v,\gamma_1^2\bar{v},\gamma_1 vu\rangle&,\end{aligned}$$

$\eta = \zeta\bar{\zeta}$, and $\xi = (\gamma_1 + \omega\gamma_2)\zeta$. Moreover, $E_3 = E_\infty$, so $H^*(G,\mathbb{F}_2)$ has Poincaré series

$$\frac{1+2t+t^2+2t^4+t^5+t^7}{(1-t^4)^2(1-t)^2}.$$

Proof. Let $b_1 = \bar{\omega}v_1v_2 + \omega v_2^2$, $b_2 = v_1v_2 + v_2^2$. By Theorem 2.2, $d_2(\zeta)$ is represented by

$$\begin{aligned}(\hat{f}_{20}y_1^2 + \hat{f}_{11}y_1y_2 + \hat{f}_{02}y_2^2)(b_1y_1 + b_2y_2) &= v_3y_1^3 + \bar{\omega}v_3y_1^2y_2 + \omega v_3y_1y_2^2 + v_3y_2^3\\ &\sim v_3y_1^3,\end{aligned}$$

which represents $\gamma_1^3\bar{v}$. Here we also used Lemma 3.5, which implies $v_3y_2 \sim \bar{\omega}v_3y_1$. Similarly, $d_2(u)$ is represented by

$$\hat{f}_{20}(u)y_1^2 + \hat{f}_{11}(u)y_1y_2 + \hat{f}_{02}(u)y_2^2 = \omega ay_1^2 + ay_2^2 \sim 0.$$

Applying the Frobenius gives the desired values of d_2 on $\bar{\zeta}$ and $\bar{u}$. We also know all differentials vanish on s and $\bar{s}$ as $s = \mathrm{res}_H^G \mathrm{norm}_H^G(v_2)$ by [7, Theorem 6.1.1 (N4)], where norm_H^G denotes the Evens norm map. To show that $d_2(\eta) = 0$ we could use the fact that d_2 is a derivation, but we will do this using Theorem 2.2 as follows. First, we may choose a representative $x^{2,4} \in \tilde{E}_1^{2,4}$ of η such that the coefficients of $x^{2,4}$ lie in $W^2\overline{W}^2$. From Figure 2, we see that $\hat{f}_\alpha(W^2\overline{W}^2) \subseteq W^3 \oplus \overline{W}^3 \oplus \langle v_1^2v_3\rangle \oplus \langle v_1v_3^2\rangle$ whenever $|\alpha| = 2$. In particular, the coefficients of $\delta_2(x^{2,4})$ lie in the sum of the four kQ-summands of $(W\overline{W})^3$, so $d_2(\eta) = 0$. Finally, d_2 vanishes on the other generators because of their positions and the fact that all differentials into the horizontal edge vanish in the case of a split extension (cf. [7, Proposition 7.3.2]).

$\beta\backslash\alpha$	20	11	02
1^2	0	0	0
12	0	0	0
13	0	0	0
14	0	0	0
2^2	$\bar\omega 3$	0	3
23	0	0	0
24	$\omega 1 + \bar\omega 3$	$\omega 1 + \bar\omega 3$	$\omega 1 + \bar\omega 3$
3^2	0	0	0
34	0	0	0
4^2	$\omega 1$	0	1
1^3	0	0	0
$1^2 2$	0	0	0
$1^2 3$	0	0	0
$1^2 4$	0	0	0
12^2	$\bar\omega 13$	0	13
123	0	0	0
124	$\omega 1^2 + \bar\omega 13$	$\omega 1^2 + \bar\omega 13$	$\omega 1^2 + \bar\omega 13$
13^2	0	0	0
134	0	0	0
14^2	$\omega 1^2$	0	1^2
2^3	$\bar\omega 23$	0	23
$2^2 3$	$\bar\omega 3^2$	0	3^2
$2^2 4$	$\bar\omega 34$	0	34
23^2	0	0	0
234	$\omega 13 + \bar\omega 3^2$	$\omega 13 + \bar\omega 3^2$	$\omega 13 + \bar\omega 3^2$
24^2	$\omega 12$	0	12
3^3	0	0	0
$3^2 4$	0	0	0
34^2	$\omega 13$	0	13
4^3	$\omega 14$	0	14
1^4	0	0	0
$1^3 2$	0	0	0
$1^3 3$	0	0	0
$1^3 4$	0	0	0
$1^2 2^2$	$\bar\omega 1^2 3$	0	$1^2 3$
$1^2 23$	0	0	0
$1^2 24$	$\omega 1^3 + \bar\omega 1^2 3$	$\omega 1^3 + \bar\omega 1^2 3$	$\omega 1^3 + \bar\omega 1^2 3$
$1^2 3^2$	0	0	0
$1^2 34$	0	0	0
$1^2 4^2$	$\omega 1^3$	0	1^3
12^3	$\bar\omega 123$	0	123
$12^2 3$	$\bar\omega 13^2$	0	13^2
$12^2 4$	$\bar\omega 134$	0	134
123^2	0	0	0
1234	$\omega 1^2 3 + \bar\omega 13^2$	$\omega 1^2 3 + \bar\omega 13^2$	$\omega 1^2 3 + \bar\omega 13^2$
124^2	$\omega 1^2 2$	0	$1^2 2$
13^3	0	0	0
$13^2 4$	0	0	0
134^2	$\omega 1^2 3$	0	$1^2 3$
14^3	$\omega 1^2 4$	0	$1^2 4$
2^4	0	0	0
$2^3 3$	$\bar\omega 23^2$	0	23^2
$2^3 4$	$\omega 12^2 + \bar\omega 2^2 3 + \bar\omega 234$	$\bar\omega 1^3 + 1^2 3 + \omega 12^2 + \bar\omega 2^2 3$	$\omega 12^2 + \bar\omega 2^2 3 + 234$
$2^2 3^2$	$\bar\omega 3^3$	0	3^3
$2^2 34$	$\bar\omega 3^2 4$	0	$3^2 4$
$2^2 4^2$	$\omega 12^2 + \bar\omega 34^2$	0	$12^2 + 34^2$
23^3	0	0	0
$23^2 4$	$\omega 13^2 + \bar\omega 3^3$	$\omega 13^2 + \bar\omega 3^3$	$\omega 13^2 + \bar\omega 3^3$
234^2	$\omega 123$	0	123
24^3	$\omega 124 + \omega 14^2 + \bar\omega 34^2$	$13^2 + \omega 14^2 + \omega 3^3 + \bar\omega 34^2$	$124 + \omega 14^2 + \bar\omega 34^2$
3^4	0	0	0
$3^3 4$	0	0	0
$3^2 4^2$	$\omega 13^2$	0	13^2
34^3	$\omega 134$	0	134
4^4	0	0	0

FIGURE 2. $\hat{f}_\alpha(v^\beta)$ for $|\alpha| = 2$

$\beta\backslash\alpha$	30	21	12	03
1^3	0	0	0	0
$1^2 2$	0	0	0	0
$1^2 3$	0	0	0	0
$1^2 4$	0	0	0	0
12^2	0	0	0	0
123	0	0	0	0
124	0	0	0	0
13^2	0	0	0	0
134	0	0	0	0
14^2	0	0	0	0
2^3	$\omega 1 + \bar\omega 3$	$1 + \omega 3$	$\bar\omega 1 + 3$	$\omega 1 + \bar\omega 3$
$2^2 3$	0	0	0	0
$2^2 4$	$\omega 1$	$\bar\omega 1$	$\bar\omega 1$	1
23^2	0	0	0	0
234	0	0	0	0
24^2	$\bar\omega 3$	$\omega 3$	$\omega 3$	3
3^3	0	0	0	0
$3^2 4$	0	0	0	0
34^2	0	0	0	0
4^3	$\omega 1 + \bar\omega 3$	$\bar\omega 1 + 3$	$1 + \omega 3$	$\omega 1 + \bar\omega 3$
1^4	0	0	0	0
$1^3 2$	0	0	0	0
$1^3 3$	0	0	0	0
$1^3 4$	0	0	0	0
$1^2 2^2$	0	0	0	0
$1^2 23$	0	0	0	0
$1^2 24$	0	0	0	0
$1^2 3^2$	0	0	0	0
$1^2 34$	0	0	0	0
$1^2 4^2$	0	0	0	0
12^3	$\omega 1^2 + \bar\omega 13$	$1^2 + \omega 13$	$\bar\omega 1^2 + 13$	$\omega 1^2 + \bar\omega 13$
$12^2 3$	0	0	0	0
$12^2 4$	$\omega 1^2$	$\bar\omega 1^2$	$\bar\omega 1^2$	1^2
123^2	0	0	0	0
1234	0	0	0	0
124^2	$\bar\omega 13$	$\omega 13$	$\omega 13$	13
13^3	0	0	0	0
$13^2 4$	0	0	0	0
134^2	0	0	0	0
14^3	$\omega 1^2 + \bar\omega 13$	$\bar\omega 1^2 + 13$	$1^2 + \omega 13$	$\omega 1^2 + \bar\omega 13$
2^4	0	0	0	0
$2^3 3$	$\omega 13 + \bar\omega 3^2$	$13 + \omega 3^2$	$\bar\omega 13 + 3^2$	$\omega 13 + \bar\omega 3^2$
$2^3 4$	$\bar\omega 1^2 + \omega 12 + 13$ $+\omega 14 + \omega 3^2 + \bar\omega 34$	$\bar\omega 1^2 + \bar\omega 12 + 13$ $+14 + \omega 3^2 + \omega 34$	$1^2 + \bar\omega 12 + \omega 13$ $+\bar\omega 14 + \bar\omega 3^2 + 34$	$1^2 + 12 + \omega 13$ $+\omega 14 + \bar\omega 3^2 + \bar\omega 34$
$2^2 3^2$	0	0	0	0
$2^2 34$	$\omega 13$	$\bar\omega 13$	$\bar\omega 13$	13
$2^2 4^2$	0	0	0	0
23^3	0	0	0	0
$23^2 4$	0	0	0	0
234^2	$\bar\omega 3^2$	$\omega 3^2$	$\omega 3^2$	3^2
24^3	$\bar\omega 1^2 + \omega 12 + 13$ $+\bar\omega 23 + \omega 3^2 + \bar\omega 34$	$\bar\omega 1^2 + \bar\omega 12 + 13$ $+23 + \omega 3^2 + \omega 34$	$\omega 1^2 + 12 + \bar\omega 13$ $+\omega 23 + 3^2 + \omega 34$	$\omega 1^2 + \omega 12 + \bar\omega 13$ $+\bar\omega 23 + 3^2 + 34$
3^4	0	0	0	0
$3^3 4$	0	0	0	0
$3^2 4^2$	0	0	0	0
34^3	$\omega 13 + \bar\omega 3^2$	$\bar\omega 13 + 3^2$	$13 + \omega 3^2$	$\omega 13 + \bar\omega 3^2$
4^4	0	0	0	0

FIGURE 3. $\hat f_\alpha(v^\beta)$ for $|\alpha| = 3$

It is straightforward to obtain the description of E_3. Since $E_3^{p,1} = 0$ for $p \geq 3$, it remains only to show that $d_3(\eta) = 0$. Let N denote the kQ summand of $W^1\overline{W}^2$ specified in Section 3, and let $J = \operatorname{rad}(kQ)$. Since $J^2N = \langle v_1 v_3^2\rangle$, it is not difficult to see we may choose $x^{3,3} \in \tilde E_1^{3,3}$ such that $\delta_1(x^{3,3}) = \delta_2(x^{2,4})$ and the coefficients of $x^{3,3}$ are in the space $W^3 \oplus \overline{W}^3 \oplus JN \oplus J\overline{N}$ (note $JN = \langle v_1 v_3^2, v_1 v_3 v_4, v_2 v_3^2\rangle$). Using Figure 2, we observe that each $\hat f_\alpha$ with $|\alpha| = 2$ takes this space into $W^1\overline{W}^1$. Similarly, Figure 3 shows that $\hat f_\beta(W^2\overline{W}^2) \subseteq W^1\overline{W}^1$ whenever $|\beta| = 3$. By Theorem 2.2,

$d_3(\eta)$ is represented by $\delta_2(x^{3,3}) + \delta_3(x^{2,4})$, which has coefficients in $W^1\overline{W}^1 \cong kQ$; hence $d_3(\eta) = 0$. $\square$

5. How to calculate a twisting cochain

5.1. **An algorithm.** We return to the notation of Section 2. Viewing (2.1), one sees that the problem of calculating a twisting system for the extension $G = H \rtimes Q$ reduces to the following two problems: (1) given $\sigma \in \operatorname{Aut}(H)$, construct a map $f \in Z_0 \operatorname{Hom}^{\sigma}_{kH}(P, P)$ satisfying $\epsilon \circ f = \epsilon$, and (2) given $d \geq 0$ and $f \in B_d \operatorname{Hom}^{\sigma}_{kH}(P, P)$, construct a map $g \in \operatorname{Hom}^{\sigma}_{kH}(P, P)_{d+1}$ satisfying $\partial g + g\partial = f$.

To solve these problems we now define the *standard contracting homotopy* $s \in \operatorname{Hom}_k(P, P)_1$. For any $S \subseteq \{1, 2, \ldots, r\}$, let $X_S = \prod_{i \in S} X_i$. Define $m(S)$ to be $\min(S)$ or ∞ if $S = \emptyset$, and define $M(S)$ to be $\max(S)$ or $-\infty$ if $S = \emptyset$. (Here ∞ is just a fixed integer greater than r.) Given $\alpha \in \mathbb{N}^r$, define $m(\alpha)$ to be the least i such that $\alpha_i \neq 0$, or ∞ if $\alpha = 0$, and define $M(\alpha)$ to be the greatest i such that $\alpha_i \neq 0$, or $-\infty$ if $\alpha = 0$. Now set

$$s(X_S \mathsf{x}^\alpha) = X_{S - \{m(S)\}} \mathsf{x}_{m(S)} \mathsf{x}^\alpha (S \neq \emptyset \text{ and } m(S) \leq m(\alpha))$$

Then $\partial s + s\partial = 1 + \eta\epsilon$, where $\eta \colon k \to P$ is the unit map, as required. So if $n \geq 1$, $\mathsf{x} \in P_n$, and $\partial(\mathsf{x}) = 0$, then $\partial s(\mathsf{x}) = \mathsf{x}$.

Problem (1) can now be solved as follows. Define $f(1) = 1$, and extend this in the unique way on P_0. Suppose f has been defined through degree n. Then define $f(\mathsf{x}^\alpha) = sf\partial(\mathsf{x}^\alpha)$ for $|\alpha| = n + 1$, and extend to P_{n+1}. Then $\partial f(\mathsf{x}^\alpha) = f\partial(\mathsf{x}^\alpha)$, and since the x^α form a kH-basis of P_{n+1} we have $\partial f = f\partial$ on all of P_{n+1}, as required.

The solution for Problem (2) is similar. First define $g(1) = sf(1)$, and extend on P_0. If $d = 0$ then since $f \sim 0$ we have $\epsilon \circ f = 0$, so $\partial g(1) = \partial sf(1) = f(1) + \eta\epsilon(f(1)) = f(1)$, as required. If $d > 0$ then $\partial f(1) = f\partial(1) = 0$ so $\partial g(1) = f(1)$ in this case as well. Now suppose g has been constructed through degree n to satisfy $\partial g + g\partial = f$. Then $\partial f - \partial g \partial$ vanishes on P_{n+1}. So define $g(\mathsf{x}^\beta) = s(f(\mathsf{x}^\beta) - g\partial(\mathsf{x}^\beta))$ for $|\beta| = n + 1$, and extend to P_{n+1}, and we have $\partial g + g\partial = f$ on P_{n+1}.

We call the result of this algorithm the *standard twisting system.* Of course, it depends on the choice of generators for H and Q, but after fixing those it is well-defined. We implemented this algorithm in MIT Scheme, and executed it on a Sun SparcStation 20 (see [10] for source code and output). For the Burt Group of order 64, the program also converted from the x^β-basis of $H^*(H, k)$ to the v^β-basis, and produced $\hat{f}_\alpha$ from f_α; those algorithms are completely standard.

5.2. **Properties.** It appears that much (if not all) of the data produced by the algorithm described above can be expressed by explicit, closed formulas. We will show how to do this for the f_α where $|\alpha| = 1$ here. Finding similar formulas for $|\alpha| > 1$ is one of the important open problems in the field, and would greatly reduce the amount of calculation required.

We will need two additional kH-algebra structures on the graded kH-module $kH[\mathsf{x}_1, \ldots, \mathsf{x}_r]$. The first of these we denote by $u \otimes v \mapsto u * v$ and is defined by

$$\mathsf{x}^\alpha * \mathsf{x}^\beta = \binom{\alpha + \beta}{\beta} \mathsf{x}^{\alpha+\beta} = \binom{\alpha_1 + \beta_1}{\beta_1} \cdots \binom{\alpha_r + \beta_r}{\beta_r} \mathsf{x}^{\alpha+\beta},$$

where $\alpha, \beta \in \mathbb{N}^r$. This is just the "divided polynomial algebra" structure, and is associative and commutative. It also satisfies $\partial(u * v) = \partial(u) * v + u * \partial(v)$, which means $(P, \partial, *)$ is a differential graded kH-algebra (cf. [5, Ch. XII §7]). The second we denote by $u \otimes v \mapsto u \circ v$ and is defined by

$$\mathsf{x}^\alpha \circ \mathsf{x}^\beta = \mathsf{x}^{\alpha+\beta}(M(\alpha) \leq m(\beta))$$

It is associative, but not commutative (unless $r = 1$). Both algebras have unit 1.

Lemma 5.1. *Let s be the standard contracting homotopy for P.*

(i) $\operatorname{Ker} s = \operatorname{Im} s \cup k \cdot 1 = \sum kX_S\mathsf{x}^\alpha$, *where the sum is taken over all subsets $S \subseteq \{1, 2, \dots, r\}$ and $\alpha \in \mathbb{N}^r$ such that $m(S) > m(\alpha)$.*
(ii) $s(u \circ v) = s(u) \circ v$ *for* $u \in P$, $v \in \operatorname{Im} s$
(iii) $u, v \in \operatorname{Im} s \Rightarrow u * v \in \operatorname{Im} s$

Proof. Each basis element $X_S\mathsf{x}^\alpha$ of P_n such that $S \neq \emptyset$ and $m(S) \leq m(\alpha)$ is taken under s to a distinct basis element $X_{S-\{m(S)\}}\mathsf{x}_{m(S)}\mathsf{x}^\alpha$ of P_{n+1}. That is all that is needed to prove (i). It suffices to prove (ii) when u and v are basis elements, since $\circ$ is bilinear and s is linear. So let $u = X_S\mathsf{x}^\alpha$, $v = X_T\mathsf{x}^\beta$, with $m(\beta) < m(T)$. The definitions show

$$\begin{aligned} s(u) \circ v &= X_{S\cup T-\{m(S)\}}\mathsf{x}_{m(S)}\mathsf{x}^{\alpha+\beta}(\mathcal{P}) \\ s(u \circ v) &= X_{S\cup T-\{m(S\cup T)\}}\mathsf{x}_{m(S\cup T)}\mathsf{x}^{\alpha+\beta}(\mathcal{Q}), \end{aligned}$$

where

$$\begin{aligned} \mathcal{P} &= S \neq \emptyset \text{ and } m(S) \leq m(\alpha) \text{ and } m(S) \leq m(\beta) \text{ and } (S - \{m(S)\}) \cap T = \emptyset \\ &\qquad \text{and } M(\alpha) \leq m(\beta), \\ \mathcal{Q} &= S \cup T \neq \emptyset \text{ and } m(S \cup T) \leq m(\alpha + \beta) \text{ and } S \cap T = \emptyset \text{ and } M(\alpha) \leq m(\beta). \end{aligned}$$

But $\mathcal{P}$ and $\mathcal{Q}$ are easily seen to be equivalent, and if they hold then $m(S) = m(S \cup T)$, establishing (ii). Finally, if $m(S) > m(\alpha)$ and $m(T) > m(\beta)$ then $m(S \cup T) > m(\alpha + \beta)$, and (iii) follows. □

Proposition 5.2. *Let $\{f_\alpha\}$ be the standard twisting system for the extension $G = H \rtimes Q$. Then for $|\alpha| = 1$ we have $f_\alpha(\mathsf{x}_1^{i_1} \dots \mathsf{x}_r^{i_r}) = f_\alpha(\mathsf{x}_1)^{\circ i_1} * \cdots * f_\alpha(\mathsf{x}_r)^{\circ i_r}$.*

Proof. Let $f = f_\alpha$, and let σ be the automorphism induced on kH by g^α. Suppose $1 \leq i \leq r$. We show by induction on n that $f(\mathsf{x}_i^n) = f(\mathsf{x}_i)^{\circ n}$ for $n \geq 1$. This is empty for $n = 1$, so suppose it holds for n. By construction, $f(\mathsf{x}_i^n) \in \operatorname{Im} s$. So by Lemma 5.1(ii),

$$\begin{aligned} f(\mathsf{x}_i^{n+1}) = sf\partial(\mathsf{x}_i^{n+1}) = sf(X_i\mathsf{x}_i^n) = s(\sigma(X_i)f(\mathsf{x}_i^n)) &= s(\sigma(X_i)) \circ f(\mathsf{x}_i^n) \\ &= f(\mathsf{x}_i)^{\circ(n+1)}. \end{aligned}$$

We next show that $f(u*v) = f(u)*f(v)$ for all $u, v \in P$, completing the proof. We do this by induction on $\deg(u)+\deg(v)$ for homogeneous u, v. The statement being trivial in total degree 0, we need only show the inductive step. By kH-linearity, we may assume $u = \mathsf{x}^\beta$ and $v = \mathsf{x}^\gamma$. Then

$$\begin{aligned} f(u * v) = sf\partial(u * v) &= sf(\partial(u) * v + u * \partial(v)) \\ &= s(f\partial(u) * f(v) + f(u) * f\partial(v)) \\ &= s(\partial f(u) * f(v) + f(u) * \partial f(v)) \\ &= s\partial(f(u) * f(v)) \\ &= f(u) * f(v). \end{aligned}$$

The last step follows because $s\partial(w) = w - \partial s(w)$ for w of positive degree, but since $f(u), f(v) \in \operatorname{Im} s$, Lemma 5.1(iii) implies $w = f(u) * f(v) \in \operatorname{Im} s$, hence $s(w) = 0$ by Lemma 5.1(i). $\square$

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