Projective Modules for A_9 in Characteristic Three

Stephen Siegel *

Department of Mathematics
University of Chicago
Chicago, Illinois 60637

1 Introduction

Let k be an algebraically closed field of characteristic 3, and A_9 the alternating group on 9 letters. The main result of this paper is

Theorem 1 *The Loewy structures of the principal indecomposable modules (*PIM's*) of kA_9 are:*

$$\begin{array}{c} A \\ BEF \\ AAABDEFF \\ AABBBDEEFF \\ AAABDEFF \\ BEF \\ A \end{array}, \quad \begin{array}{c} B \\ ADEF \\ ABBBDEFF \\ AAABBDDEEFF \\ ABBBDEFF \\ ADEF \\ B \end{array}, \quad \begin{array}{c} D \\ BDF \\ ABDDEF \\ ABBDEFF \\ ABDDEF \\ BDF \\ D \end{array}, \quad \begin{array}{c} E \\ ABF \\ ABDEEF \\ AABBDFF \\ ABDEEF \\ ABF \\ E \end{array},$$

$$\begin{array}{c} F \\ ABDEF \\ AABBDEFFF \\ AABBDDEEFFF \\ AABBDEFFF \\ ABDEF \\ F \end{array}, \quad \begin{array}{c} C \\ H \\ C \end{array}, \quad \begin{array}{c} H \\ C \\ H \end{array}, \textit{ and } G.$$

Here, the simple kA_9 modules are labeled as in the Brauer character table given in the Appendix. This table is easily computed from the Brauer character table of the symmetric group S_9 at the prime 3 given in [6]. The ordinary character table of A_9 is well known, and can be found in [5]. Notice that

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there are three blocks of kA_9: the principal block, a block of defect zero, and a block of defect one. The non-principal blocks are easy to describe, so we restrict ourselves here to the principal block.

The Loewy structures of the PIM's of kA_6 are well known and appear in [3]; their module diagrams can be found in [4]. The PIM structures for kA_7 and kA_8 have been calculated by J. Scopes ([8]). The structures for A_8 and A_9 have also been done in characteristic two in [1] and [2] respectively. We shall use the results of Scopes extensively, as well as some details from her proofs.

The method employed in this paper is largely diagramatic. These diagrams, the most important being the L-filtrations, are defined in Section 2. The structure of the proof can then be summarised as follows: we begin with the Loewy structures of the PIM's of kA_8. By inducing these to kA_9, we obtain L-filtrations of the PIM's of kA_9. By manipulating these diagrams, we finally prove Theorem 1.

2 Diagrams

Let G be a finite group, p a prime, and k an algebraically closed field of characteristic p. By 'kG-module' we will always mean a left kG-module which is finite dimensional over k.

Given a kG-module M, we will use various diagrams $\mathbf{D}$ to describe some aspect of the structure of M, and say "$\mathbf{D}$ describes M", "M has Loewy structure $\mathbf{D}$", or "M has L-filtration $\mathbf{D}$", as the case may be, and in any case write $M \sim \mathbf{D}$. Each of these diagrams consists of symbols representing the composition factors of M (with multiplicities), arranged in horizontal rows, and possibly some other symbols. By *the i-th row of* $\mathbf{D}$ we will mean the module that is the direct sum of the simple modules occuring in the i-th row from the top of $\mathbf{D}$. We now define these diagrams precisely.

First, the Loewy structure of M is considered a diagram. Its i-th row is

$$L_i(M) = \mathrm{rad}^{i-1}(M)/\mathrm{rad}^i(M).$$

The socle structure of M is also a diagram; its i-th row is

$$\mathrm{soc}^{n-i+1}(M)/\mathrm{soc}^{n-i}(M),$$

where n is the Loewy length of M. However, all structures considered in this paper will be Loewy structures, unless explicity stated otherwise.

Secondly, if $M \cong U \oplus V$, and $\mathbf{D}_U$ and $\mathbf{D}_V$ are diagrams describing U and V, respectively, then $\mathbf{D}_U \oplus \mathbf{D}_V$ is a diagram describing M, and the i-th row of $\mathbf{D}_U \oplus \mathbf{D}_V$ is just the direct sum of the i-th row of $\mathbf{D}_U$ and the i-th row of $\mathbf{D}_V$.

Finally, if U is a submodule of $\mathrm{rad}^n(M)$, for a non-negative integer n, then an *L-filtration of M (through U)* is a diagram

$$\mathbf{D} = \begin{array}{l|ll} \mathbf{D}_{M/U} & & \}\,(n) \\ & \mathbf{D}_U & \end{array} \tag{1}$$

describing M, where $\mathbf{D}_{M/U}$ and $\mathbf{D}_U$ are diagrams describing M/U and U, respectively, and $\mathbf{D}_U$ is shifted down n rows. Hence the $(n+i)$-th row of $\mathbf{D}$, for $i \geq 1$, is the direct sum of the $(n+i)$-th row of $\mathbf{D}_{M/U}$ and the i-th row of $\mathbf{D}_U$, while the i-th row, for $i \leq n$, is just the i-th row of $\mathbf{D}_{M/U}$.

The question of associativity of L-filtrations arises naturally. It is clear from the definition that

$$M \sim \begin{array}{l|ll} \mathbf{D}_{M/U} & & \}\,(n) \\ & \left(\begin{array}{l|ll} \mathbf{D}_{U/V} & & \}\,(m) \\ & \mathbf{D}_V & \end{array} \right) & \end{array}$$

implies

$$M \sim \begin{array}{l|ll} \left(\begin{array}{l|ll} \mathbf{D}_{M/U} & & \}\,(n) \\ & \mathbf{D}_{U/V} & \end{array} \right) & & \Big\}\,(m+n) \\ & \mathbf{D}_V & \end{array}$$

However, the opposite implication does *not* necessarily hold. For example, take any M with Loewy structure $\begin{smallmatrix} A \\ BC \end{smallmatrix}$. Then

$$M \sim \begin{array}{l|l} (A|B) & \\ & C \end{array} \quad \text{but} \quad M \not\sim \begin{array}{l|l} A & \\ & \left(\begin{array}{l|l} B & \\ & C \end{array} \right) \end{array}.$$

In view of this, if there are no parentheses appearing in an L-filtration, this will mean that all the parentheses are to be considered concentrated to the right.

Part of the motivation for this notation is that if $\mathbf{D}_{M/U}$ is the Loewy structure of M/U in (1), then the first n Loewy layers of M are just the first n rows of $\mathbf{D}$, since $U \subseteq \mathrm{rad}^n(M)$. The following is a sort of converse to this:

Lemma 2 *Suppose $U \subseteq M$ and $L_i(M/rad(U)) \cong L_i(M/U)$ for $i = 1, \ldots, n$. Then $U \subseteq rad^n(M)$.*

Proof The hypothesis implies that for $1 \leq i \leq n$,

$$\frac{M/\mathrm{rad}(U)}{\mathrm{rad}^i(M) + \mathrm{rad}(U)/\mathrm{rad}(U)} \cong \frac{M/U}{\mathrm{rad}^i(M) + U/U}.$$

Hence

$$\mathrm{rad}^i(M) + \mathrm{rad}(U) = \mathrm{rad}^i(M) + U.$$

We prove by induction on i that $U \subseteq \mathrm{rad}^i(M)$ for $i = 0, \ldots, n$. The case $i = 0$ is trivial. Suppose $i < n$ and $U \subseteq \mathrm{rad}^i(M)$. Then $\mathrm{rad}(U) \subseteq \mathrm{rad}^{i+1}(M)$, and since $i + 1 \leq n$,

$$\mathrm{rad}^{i+1}(M) + U = \mathrm{rad}^{i+1}(M) + \mathrm{rad}(U) = \mathrm{rad}^{i+1}(M),$$

which implies $U \subseteq \mathrm{rad}^{i+1}(M)$, and the proof is complete. $\square$

Let H be a subgroup of G, and B a block of kG with block idempotent e. We are interested in the relationship between L-filtrations and induction from H to G under the following hypothesis:

Hypothesis 1 *If S is a simple kG-module lying in B then $S{\downarrow}_H$ is semisimple.*

The following useful observation is due to Benson([1, Lemma 4.1.1]).

Lemma 3 *(Benson's Lemma) Assume Hypothesis 1. If M is a kH-module then*

1. $(rad\ M){\uparrow}.e \subseteq rad(M{\uparrow}).e$
2. $N{\uparrow}/rad^n(N{\uparrow}).e \cong M{\uparrow}/rad^n(M{\uparrow}).e$, *where* $N = M/rad^n(M)$

Corollary 4 *Assume Hypothesis 1. If M is a kH-module, U is a submodule of M, and*

$$M \sim \left. \mathbf{D}_{M/U} \right| \mathbf{D}_U \;\} (n)$$

then

$$M{\uparrow}^G.e \sim \left. \mathbf{D}_{(M/U){\uparrow}^G.e} \right| \mathbf{D}_{U{\uparrow}^G.e} \;\} (n) \quad ,$$

where we are assuming, in each case, that $\mathbf{D}_X$ is a diagram for the module X.

Finally, we will occasionally use a Benson-Carlson module diagram to describe a kG-module M. This is a finite directed graph, with vertices labeled by simple modules, while an edge from a vertex S to a vertex T corresponds to a non-zero element of $\mathrm{Ext}^1_{kG}(S,T)$. The graph must satisfy several additional properties to represent M. For a precise definition, see [4].

3 Simple Modules

For the balance of this paper, we let k be an algebraically closed field of characteristic 3, and A_8 the subgroup of A_9 fixing a point. We also let e_0 denote the principal block idempotent of kA_9 and f_0 the principal block idempotent of kA_8. We will abbreviate the symbols '${\uparrow}_{A_8}^{A_9}$' and '${\downarrow}_{A_8}^{A_9}$' as '$\uparrow$' and '$\downarrow$', respectively.

We label the simple kA_8-modules by their dimensions, with indices when there is more than one of a given dimension. Finally, we denote the projective cover of a module M by P_M.

Theorem 5 *(Restriction)*

$$A\!\downarrow\, \cong 1 \qquad B\!\downarrow\, \cong 7 \qquad C\!\downarrow\, \sim \begin{matrix}7\\13\\7\end{matrix} \qquad D\!\downarrow\, \cong 21 \qquad E\!\downarrow\, \cong 28 \oplus 13$$

$$F\!\downarrow\, \cong 35 \quad G\!\downarrow\, \cong P_{28} \qquad\qquad H\!\downarrow\, \sim 45_1 \oplus 45_2 \oplus \begin{matrix}35\\1\;28\\35\end{matrix}$$

Proof These follow easily from characters and Frobenius reciprocity. (See the Appendix for induction and restriction of characters.) We also used the fact that $\mathrm{Ext}^1_{kA_8}(28,13) = \{0\}$, and that $\mathrm{Hom}_{kA_9}(C, 7\!\uparrow) \cong k$, as C is a block summand of $7\!\uparrow$.$\square$

Corollary 6 *With $G = A_9$, $H = A_8$, k an algebraically closed field of characteristic three, and $B = B_0$, Hypothesis 1 is satisfied.*

Theorem 7 *(Induction)*

$$1\!\uparrow\, \sim \begin{matrix}A\\B\\A\end{matrix} \qquad\qquad 7\!\uparrow\, \sim C \oplus \begin{matrix}B\\AD\\B\end{matrix} \qquad 13\!\uparrow\, \sim \begin{matrix}E\\F\\E\end{matrix}$$

$$35\!\uparrow\, \sim H \oplus \begin{matrix}F\\DF\\F\end{matrix} \quad 28\!\uparrow\, \sim G \oplus \begin{matrix}E\\AB\\E\end{matrix} \quad 21\!\uparrow\, \sim \begin{matrix}D\\BF\\AED\\BF\\D\end{matrix}$$

After we have proved the theorem, it will follow from Corollary 4 that we have the following L-filtrations of the PIM's of kA_9:

$$P_A \cong P_1\!\uparrow\!.e_0 \sim \left|\begin{matrix}A\\B\\A\end{matrix}\right|\begin{matrix}E\\F\\E\end{matrix} \oplus \begin{matrix}F\\DF\\F\end{matrix}\left|\begin{matrix}A\\B\\A\end{matrix} \oplus \begin{matrix}A\\B\\A\end{matrix} \oplus \begin{matrix}B\\AD\\B\end{matrix} \oplus \begin{matrix}E\\AB\\E\end{matrix}\right|\begin{matrix}E\\F\\E\end{matrix} \oplus \begin{matrix}F\\DF\\F\end{matrix}\left|\begin{matrix}A\\B\\A\end{matrix}\right. \tag{2}$$

$$P_B \cong P_7\!\uparrow\!.e_0 \sim \left|\begin{matrix}B\\AD\\B\end{matrix}\right|\begin{matrix}E\\F\\E\end{matrix} \oplus \begin{matrix}F\\DF\\F\end{matrix}\left|\begin{matrix}B\\AD\\B\end{matrix} \oplus \begin{matrix}B\\AD\\B\end{matrix} \oplus \begin{matrix}A\\B\\A\end{matrix} \oplus \begin{matrix}E\\AB\\E\end{matrix}\right|\begin{matrix}E\\F\\E\end{matrix} \oplus \begin{matrix}F\\DF\\F\end{matrix}\left|\begin{matrix}B\\AD\\B\end{matrix}\right. \tag{3}$$

$$P_D \cong P_{21}{\uparrow}.e_0 \sim \begin{array}{c|c|c} D & & \\ BF & D & \\ AED & BF & D \\ BF & AED & BF \\ D & BF & AED \\ & D & BF \\ & & D \end{array} \tag{4}$$

$$P_E \cong P_{28}{\uparrow}.e_0 \sim \begin{array}{c|c|c|c|c} E & & & & \\ AB & F & & & \\ E & DF & A \quad\; B \quad\;\; E & & \\ & F & B \oplus AD \oplus AB & F & \\ & & A \quad\; B \quad\;\; E & DF & E \\ & & & F & AB \\ & & & & E \end{array} \tag{5}$$

$$P_E \cong P_{13}{\uparrow}.e_0 \sim \begin{array}{c|c|c|c|c} E & & & & \\ F & A \quad\; B & & & \\ E & B \oplus AD & E \quad\; F & & \\ & A \quad\; B & F \oplus DF & A \quad\; B & \\ & & E \quad\; F & B \oplus AD & E \\ & & & A \quad\; B & F \\ & & & & E \end{array} \tag{6}$$

$$P_F \cong P_{35}{\uparrow}.e_0 \sim \begin{array}{c|c|c|c|c} F & & & & \\ DF & A \quad\; B \quad\;\; E & & & \\ F & B \oplus AD \oplus AB & E \quad\; F \quad\;\; F & & \\ & A \quad\; B \quad\;\; E & F \oplus DF \oplus DF & A \quad\; B \quad\;\; E & \\ & & E \quad\; F \quad\;\; F & B \oplus AD \oplus AB & F \\ & & & A \quad\; B \quad\;\; E & DF \\ & & & & F \end{array} \tag{7}$$

Interestingly enough, these also turn out to be the Loewy structures of the projectives, in the sense that the i-th row in each of these diagrams is actually the i-th Loewy layer of the module it describes, though we know of no explanation for this phenomenon.

Proof of Theorem 7 The only difficulty arises in the case of $21{\uparrow}$. We now turn our attention to that case, and will freely use the above L-filtrations, except for that of P_D.

Let $M = B \otimes D$. (See the Appendix for the composition factors of tensor products of simples.) We'll show M has Loewy structure $\begin{array}{c} BF \\ AED \\ BF \end{array}$. Then since $21{\uparrow} \cong (D{\downarrow} \otimes 1){\uparrow} \cong D \otimes 1{\uparrow}$ has a filtration with successive quotients D, M, D, and head and socle isomorphic to D, the Loewy structure of $21{\uparrow}$ follows.

The characters of $B \otimes B$, $B \otimes F$, and $B \otimes E$ show that the head and socle of M each contain one B and one F, while A and E do not occur. If D were in the head, then we would have $M \cong X \oplus D$, where X has a head and socle isomorphic to $B \oplus F$. This would imply the dimension of the endomorphism ring of M is at least 4. However,

$$\begin{aligned} \mathrm{Hom}_{kA_9}(M, M) &\cong \mathrm{Hom}_{kA_9}(B \otimes B, D \otimes D) \\ &\cong \mathrm{Hom}_{kA_9}(A \oplus C \oplus D, D \otimes D) \end{aligned}$$

has dimension less than or equal to three. Hence the head of M is $B \oplus F$, and the Loewy length is $3, 4, 5$, or 6. It's easy to see the length is 3 by eliminating the other cases. For example, if the length is 4, then $\mathrm{rad}^2(M) \hookrightarrow (M/\mathrm{rad}^2(M))^*$, which implies A, E, and D must occur in the second layer, leaving $\mathrm{rad}^2(M)$ uniserial with composition factors B and F, which contradicts the self-duality of M. The other cases are handled similarly.

This leaves three possibilities for the Loewy structure of $B \otimes D$, which, as above, correspond to the following three possible Loewy structures of $21\!\uparrow$: the one claimed,

$$(i) \quad \begin{array}{c} D \\ BF \\ AEDF \\ B \\ D \end{array} \qquad \text{or} \qquad (ii) \quad \begin{array}{c} D \\ BF \\ AEDB \\ F \\ D \end{array} .$$

Suppose *(i)* is the case. It follows that $21\!\uparrow$ has a quotient with Loewy structure $\begin{array}{c} D \\ BF \\ F \end{array}$ and hence a submodule U with Loewy/socle structure given by one of the following Benson-Carlson module diagrams:

$$(a) \quad \begin{array}{ccccc} & & F & & \\ & / & & \backslash & \\ B & & & & F \\ & \backslash & & / & \\ & & D & & \end{array} \qquad (b) \quad \begin{array}{ccccc} & & F & & \\ & / & & & \\ B & & & & F \\ & \backslash & & / & \\ & & D & & \end{array} \qquad (c) \quad \begin{array}{ccccc} & & F & & \\ & & & \backslash & \\ B & & & & F \\ & \backslash & & / & \\ & & D & & \end{array}$$

In each case, there are two linearly independent homomorphisms from P_F to U, and by considering L-filtration (7), each of these factors to give a homomorphism from $V\!\uparrow$ to U, where $V = \begin{array}{c} 35 \\ 7 \end{array}$. But

$$\begin{array}{rcl} \mathrm{Hom}_{kA_9}(V\!\uparrow, U) & \subseteq & \mathrm{Hom}_{kA_9}(V\!\uparrow, 21\!\uparrow) \\ & \cong & \mathrm{Hom}_{kA_8}(V, 21\!\uparrow\downarrow) \\ & \cong & \mathrm{Hom}_{kA_8}(V, 21 \oplus 21\!\downarrow_{A_7}\uparrow^{A_8}) \\ & \cong & \mathrm{Hom}_{kA_8}(V, 6\!\uparrow_{A_7}^{A_8} \oplus 15\!\uparrow_{A_7}^{A_8}) \end{array}$$

is one-dimensional, a contradiction.

Case *(ii)* is handled similarly, only one ends up with a submodule containing two B's, and obtains a contradiction since $\mathrm{Hom}_{kA_9}(V^*\!\uparrow, 21\!\uparrow) \cong k$, completing the proof of the theorem.$\square$

4 Two-step Modules

In this section we will calculate the Loewy structures of the induced modules of two-step kA_8-modules lying in the principal block. A two-step module is a module with two composition factors and Loewy length two. We will also compute $\dim_k \mathrm{Ext}^1_{kA_9}(S, T)$ for each pair (S, T) of simple kA_9-modules.

Theorem 8 (i) *The second Loewy layers of the PIM's of kA_9 are as stated in Theorem 1.*
(ii) *The following are Loewy and socle structures:*

$$\genfrac{}{}{0pt}{}{13}{1}\uparrow.e_0 \sim \begin{array}{c} E \\ FA \\ EB \\ A \end{array} \qquad \genfrac{}{}{0pt}{}{7}{35}\uparrow.e_0 \sim \begin{array}{c} B \\ ADF \\ BDF \\ F \end{array} \qquad \genfrac{}{}{0pt}{}{7}{13}\uparrow.e_0 \sim \begin{array}{c} B \\ ADE \\ BF \\ E \end{array}$$

$$\genfrac{}{}{0pt}{}{1}{35}\uparrow.e_0 \sim \begin{array}{c} A \\ BF \\ ADF \\ F \end{array} \qquad \genfrac{}{}{0pt}{}{35}{28}\uparrow.e_0 \sim \begin{array}{c} F \\ DFE \\ FAB \\ E \end{array}$$

Proof The L-filtrations of the PIM's of kA_9 give an upper bound on the multiplicities of simples in the second Loewy layers (only simples in the second row of the L-filtration have a possibility of being in the second Loewy layer), while the Loewy structures of the modules in Theorem 7 provide some lower bounds. Putting these together, there remain only three facts to be proved to establish *(i)*:

$$\dim_k \operatorname{Ext}^1_{kA_9}(A, F) \geq 1 \tag{8}$$

$$\dim_k \operatorname{Ext}^1_{kA_9}(B, F) \geq 1 \tag{9}$$

$$\dim_k \operatorname{Ext}^1_{kA_9}(D, D) \geq 1 \tag{10}$$

We get (8) from the long exact sequence in cohomology and the facts that $\operatorname{Ext}^1_{kA_9}(A, 35\uparrow) \cong k$ and that $\operatorname{Ext}^1_{kA_9}(A, D) = \{0\}$. We now turn our attention to the proof of (9).

Let $N = \genfrac{}{}{0pt}{}{13}{1}\uparrow.e_0$. From (6) we have an L-filtration of P_E:

$$P_E \sim \left(\begin{array}{c} E \\ F \\ E \end{array} \middle| \begin{array}{c} A \\ B \\ A \end{array} \right) \Bigg| \left(\begin{array}{c} B \\ AD \\ B \end{array} \middle| \cdots \right) .$$

The quotient represented by this L-filtration is isomorphic to N. Since there is an A in the second Loewy layer of E, A must be in the second Loewy layer of N. The only question is whether B is in the third or fourth layer of N. However, a similar argument shows that E is in the second Loewy layer of N^*. Hence there can not be a uniserial submodule of N of length three with head isomorphic to E, so B must be in the third layer of N, and we have determined the Loewy structures of N and N^*.

Now consider the L-filtration of P_E arising from (5):

$$P_E \sim \begin{array}{c} E \\ AB \\ E \\ \end{array} \Bigg| \left(\begin{array}{c} F \\ DF \\ F \end{array} \middle| \begin{array}{c} B \\ AD \\ B \end{array} \right) \Bigg| \begin{array}{c} AE \\ BAB \\ AE \end{array} \cdots .$$

If $\mathrm{Ext}^1_{kA_9}(B,F)=\{0\}$, then B must be in the third layer of $\begin{smallmatrix}35\\7\end{smallmatrix}\uparrow.e_0$, and there would be no B in the third layer of P_E, contradicting the Loewy structure of N, and we have established (9).

Now let $M = \begin{smallmatrix}7\\35\end{smallmatrix}\uparrow.e_0$. Examining (3), we see that F must be in the second layer of M. Similarly, B is in the second layer of M^*, so M has no submodule of Loewy length three with head isomorphic to B. Moreover, M has no uniserial submodule with series D,F,F or F,D,F (if so, then the inclusion of this submodule followed by projection onto $7\uparrow.e_0$ is zero, so it would be a submodule of $35\uparrow.e_0$). The reader can now check this leaves only one possibility for the Loewy and socle structure of M.

We now go on to prove (10), completing the proof of the first part of the Theorem. Suppose $\mathrm{Ext}^1_{kA_9}(D,D)=\{0\}$. Examining (4), we see this implies there is no F in the third layer of P_D. By Landrock's Lemma ([7, Lemma 1.9.10]), the multiplicity of F in the third Loewy layer of P_D equals the multiplicity of $D^*\cong D$ in the third Loewy layer of $P_{F^*}\cong P_F$. But the Loewy structure of M shows this is at least 1, a contradicton.

We proceed with the calculation of the Loewy structures of the induced two-step modules from the principal block of kA_8. We see that

$$U \equiv \begin{smallmatrix}13\\7\end{smallmatrix}\uparrow.e_0 \sim \left.\begin{matrix}E\\F\\E\end{matrix}\right|\begin{matrix}B\\AD\\B\end{matrix} \sim \left.\begin{matrix}E\\FB\\E\end{matrix}\right|\begin{matrix}\\AD\\B\end{matrix} \sim \left.\begin{matrix}E\\FB\\ED\end{matrix}\right|\begin{matrix}\\A\\B\end{matrix} \sim \begin{matrix}E\\FB\\EDA\\B\end{matrix}.$$

The second filtration follows from (6) and the fact that $\mathrm{Ext}^1_{kA_9}(B,E)\cong k$, while the third follows because $\mathrm{Ext}^1_{kA_9}(D,E)=\{0\}$. The fourth is obtained by considering the L-filtration of U^*: there is no E or D in the third Loewy layer of U^*, and so the Loewy structures of U, and U^*, follow easily. The others are done in exactly the same way, so we leave them as an exercise. □

Corollary 9 *Suppose M is a kA_8-module of Loewy length two lying in the principal block of kA_8. Then for $i\geq 1$,*

$$L_i(M\uparrow.e_0)\cong L_i((M/rad(M))\uparrow.e_0)\oplus L_{i-1}((rad(M))\uparrow.e_0),$$

where we interpret L_0 to be $\{0\}$.

Proof We first show that $M\uparrow.e_0$ has Loewy length 4; it suffices to show $\mathrm{rad}^4(M\uparrow.e_0)=\{0\}$. We can express M as a sum (not necessarily direct) of submodules, each with a simple head. If U is one such summand, then U^* can be written as a sum of two-step and simple modules, so Theorems 8 and 7 show $\mathrm{rad}^4(U^*\uparrow.e_0)=\{0\}$. It follows that $\mathrm{rad}^4(M\uparrow.e_0)=\{0\}$.

Now let $\mathrm{rad}(M)\cong\oplus_{i=1}^s U_i$ be an expression of $\mathrm{rad}(M)$ as a direct sum of simples. The proof is by induction on s, the case $s=0$ being trivial (the inductive step does take us from M semisimple to $s = 1$). Suppose the Corollary holds for $s-1$. Let $U' = U_1 \oplus \cdots \oplus U_{s-1}$. Our inductive hypothesis says we have the following L-filtration:

$$M\uparrow.e_0 \sim \begin{array}{c|c} L_1(V\uparrow.e_0) & \\ L_2(V\uparrow.e_0) \oplus L_1(U'\uparrow.e_0) & L_1(U_s\uparrow.e_0) \\ L_3(V\uparrow.e_0) \oplus L_2(U'\uparrow.e_0) & L_2(U_s\uparrow.e_0) \\ L_3(U'\uparrow.e_0) & L_3(U_s\uparrow.e_0) \end{array},$$

where $V = M/\mathrm{rad}(M)$. By Theorem 7, the Loewy and socle structure of $U_s\uparrow.e_0$ is of the form

$$\begin{array}{c} X \\ Y \\ X \end{array} \quad \text{or} \quad \begin{array}{c} X \\ YZ \\ X \end{array}$$

for some simples X, Y, and Z where $Y \ncong Z$. In the first case, the result follows immediately from the fact that $\mathrm{rad}^4(M\uparrow.e_0) = \{0\}$. In the second case, this fact still forces X to pass to layer 2. If we then let Y pass to the quotient, the resulting submodule has Loewy structure $\begin{array}{c} Z \\ X \end{array}$, and again the length consideration forces Z to go to the third Loewy layer. By the same reasoning, Y must go to the third layer, and the first three Loewy layers of $M\uparrow.e_0$ are as claimed, and the Corollary is proved. $\square$

5 Projective Indecomposable Modules

In this section, we prove Theorem 1. We will often use Corollary 9 tacitly throughout the proof.

We begin with the Loewy structure of P_E. Let $M = (P_{28}/\mathrm{rad}^3(P_{28}))\uparrow.e_0$. We claim

$$M \sim \begin{array}{c|c} E & \\ AB & F \\ E & DFABE \\ & FBADAB \\ & ABE \end{array} \sim \begin{array}{c|c} E & \\ ABF & \\ EDFABE & \\ & FBADAB \\ & ABE \end{array} \sim \begin{array}{c} E \\ ABF \\ EDFABE \\ FBADAB \\ ABE \end{array}.$$

To see this, consider the modules of Theorem 7. It remains only to show that there are two E's in the third layer, and this follows from the fact that $\mathrm{Ext}^1_{kA_9}(E, E) = \{0\}$. We remark that by Lemma 3, this determines the first three Loewy layers of P_E.

Next, we compute

$$\mathrm{rad}(P_{28})/\mathrm{rad}^4(P_{28})\uparrow.e_0 \sim \begin{array}{c|c} F & \\ DF & ABE \\ F & BADABF \\ & ABEDF \\ & F \end{array} \sim \begin{array}{c} F \\ DFABE \\ FBADABF \\ ABEDF \\ F \end{array}. \tag{11}$$

This follows in the same way; we need only show that there are two F's in the third layer of this module. However, by considering L-filtration (6) and the first three Loewy layers of P_E, we see there are two F's in the fourth layer of P_E, and the Loewy series of this module follows. We conclude that

$$(P_{28}/\mathrm{soc}(P_{28}))\!\uparrow\!.e_0 \sim \begin{array}{c} E \\ AB \\ E \\ \\ \\ \\ \end{array} \left| \begin{array}{c} F \\ DFABE \\ FBADABF \\ ABEDF \\ F \end{array} \right. \sim \begin{array}{c} E \\ ABF \\ EDFABE \\ FBADABF \\ ABEDF \\ F \end{array},$$

and we have determined the first four Loewy layers of P_E.

Next, observe that

$$(\mathrm{rad}^2(P_{28}))\!\uparrow\!.e_0 \sim \begin{array}{c} ABE \\ BADABF \\ ABEDF \\ F \\ \\ \end{array} \left| \begin{array}{c} \\ \\ E \\ AB \\ E \end{array} \right. \sim \begin{array}{c} ABE \\ BADABF \\ ABEDFE \\ FAB \\ E \end{array}.$$

This follows easily from the fact that this module is dual to $(P_{28}/\mathrm{rad}^3(P_{28}))\!\uparrow\!.e_0$. Using this, we see that

$$P_E \cong P_{28}\!\uparrow\!.e_0 \sim \begin{array}{c} E \\ ABF \\ EDF \\ F \\ \\ \\ \\ \end{array} \left| \begin{array}{c} \\ \\ ABE \\ BADABF \\ ABEDFE \\ FAB \\ E \end{array} \right.,$$

and the Loewy structure of P_E follows from our knowledge of its first four Loewy layers.

The Loewy structure of P_B offers a more challenging problem, because the structure of P_7 is much more complicated than that of P_{28}. Hence we begin by analyzing the structure of P_7 a bit further.

Lemma 10 *Let Z be the submodule of $rad^2(P_7)$ containing $rad^3(P_7)$ with composition factors $1, 13, 35, 7$. Then Z has Loewy and socle structure* $\begin{array}{c} 1 \\ 13\ 35 \\ 7 \end{array}$.

To prove the Lemma, it suffices to show there exists a kA_8-module with such a structure–the Lemma then follows from the existence of an injective homommorphism into P_7. To this end, let $U = \left(\begin{array}{c} C \\ H \end{array} \right) \downarrow_{A_8}.f_0$. A quick check with Frobenius reciprocity shows that $\mathrm{soc}(U) \cong 35$ and the head of U is isomorphic to 7, so

$$U \sim \begin{array}{c} 7 \\ 13 \\ 7 \\ \\ \end{array} \left| \begin{array}{c} \\ 35 \\ 1\ 28 \\ 35 \end{array} \right. \sim \begin{array}{c} 7 \\ 13\ 35 \\ 7\ 1\ 28 \\ 35 \end{array}.$$

We claim it suffices to show that the subquotient of U with composition factors $\{13, 1\}$ has Loewy structure $\begin{array}{c}13\\1\end{array}$. (Hence the Benson-Carlson module diagram for U is

$$\begin{array}{ccccc} & & 7 & & \\ & / & & \backslash & \\ 13 & & & & 35 \\ | & \backslash & & / & | \\ 7 & & 1 & & 28 \\ & \backslash & | & / & \\ & & 35 & & \end{array} .)$$

Indeed, if that is the case, just consider the dual of the quotient of U by the submodule $\begin{array}{c}7\;28\\35\end{array}$.

Consider the kA_8 module

$$V = \begin{pmatrix} 10_1 \\ 1 \\ 13 \end{pmatrix} \uparrow_{A_7}^{A_8} \cdot f_0 \sim \;\; {}^{35}\left|\; 1 \oplus 7 \;\right| \; 13 \;\; \oplus \;\; \begin{array}{c}28\\35\\28\end{array} \sim \begin{array}{c}35\\1\;7\;28\\13\;35\\28\end{array} .$$

(The fact that such a uniserial kA_7-module exists is proved by Lemma 3.3.8 of [8].) A Frobenius reciprocity argument between A_8 and A_7 shows that $\begin{array}{c}1\\13\end{array}$ and $\begin{array}{c}7\\13\end{array}$ are isomorphic to submodules of V. (Use $13{\downarrow_{A_7}} \cong 13_{A_7}$, $7{\downarrow_{A_7}} \cong 1_{A_7} \oplus 6_{A_7}$ to show 13 is a submodule of V while 1 and 7 are not. Then show $\begin{array}{c}1\\13\end{array} \downarrow_{A_7} \cong \begin{array}{c}1\\13\end{array}$ and $\begin{array}{c}7\\13\end{array} \downarrow_{A_7} \cong 6 \oplus \begin{array}{c}1\\13\end{array}$.)

If we now let W be the dual of the quotient of V by the submodule $K = \begin{array}{c}28\\35\\28\end{array}$, we see that $\begin{array}{c}1\\13\end{array}$ and $\begin{array}{c}7\\13\end{array}$ are isomorphic to submodules of W^*. This follows from the fact that $\begin{array}{c}1\\13\end{array} \cap K = 0$, so the composition

$$\begin{array}{c}1\\13\end{array} \hookrightarrow V \to V/K \cong W^*$$

is injective, and the same argument works for $\begin{array}{c}7\\13\end{array}$. This shows that W has Loewy and socle structure $\begin{array}{c}13\\1\;7\\35\end{array}$.

We now prove that W is isomorphic to a submodule of U. Indeed, an easy calculation shows that the B_1 component of $W\!\uparrow$ is isomorphic to $\begin{pmatrix} C \\ H \end{pmatrix}$, so Frobenius reciprocity shows there exists a non-zero kA_8-homomorphism $\theta : W \to U$. On the other hand, θ must have zero kernel since 35 is the unique simple submodule of U. This shows that the subquotient with composition factors 13,1 of U actually is a non-zero extension, and completes the proof of the Lemma.$\square$

Lemma 10 implies we have the following L-filtration:

$$P_7 \sim \begin{array}{c|c|c|c} 7 & & & \\ 13\ 35 & & & \\ & 7\ 7\ 28 & 1 & \\ & & 13\ 35 & \\ & & & 7 \end{array} .$$

Inducing, projecting onto B_0, and applying Corollary 4, we get

$$P_B \sim \begin{array}{c|c|c|c} B & & & \\ ADEF & & & \\ BFDF & BBE & A & \\ EF & ADADAB & BEF & \\ & BBE & AFDF & B \\ & & EF & AD \\ & & & B \end{array} \tag{12}$$

At this point, we know that the third Loewy layer of P_B contains at least one B, one D, and two F's, and by Landrock's Lemma, precisely one E. Using this, and the fact that $\mathrm{Ext}^1_{kA_9}(B,B)=\{0\}$, we evaluate the following L-filtration:

$$\begin{aligned}
(P_7/\mathrm{rad}^3 P_7)\!\uparrow\!.e_0 \quad &\sim \quad \begin{array}{c|c|c} B & & \\ AD & EF & \\ B & FDFBBE & A \\ & EFADADAB & B \\ & BBE & A \end{array} \\
&\sim \quad \begin{array}{c|c|c|c|c} B & & & & \\ ADEF & & & & \\ B & BB & FDFE & & A \\ & & & EFADADAB & B \\ & & & BBE & A \end{array} \\
&\sim \quad \begin{array}{c|c} B & \\ ADEF & \\ BFDFBBE & A \\ EFADADAB & B \\ BBE & A \end{array} .
\end{aligned} \tag{13}$$

So there are three B's in the third layer of P_B, and (12) becomes

$$P_B \sim \begin{array}{c|c|c} B & & \\ ADEF & & \\ BFDFBBE & A & \\ EFADADAB & BEF & \\ BBE & AFDF & B \\ & EF & AD \\ & & B \end{array} . \tag{14}$$

By Landrock's Lemma, there are two E's in the fourth layer of P_B, so by Lemma 2, the A must go to the third layer in (14), and we have shown there is an A in the third layer of P_B. This completes the calculation of the third layer, and shows that the Loewy structure of $(P_7/\mathrm{rad}^3 P_7)\!\uparrow\!.e_0$ is obtained by removing the vertical line from (13).

Lemma 11 *If U is the kA_8-module in the proof of Lemma 10, then $U{\uparrow}.e_0$ has Loewy structure*
$$\begin{array}{c} B \\ ADEF \\ BFDFBAE \\ EFADBABF \\ BAEDF \\ F \end{array}.$$

Proof We first observe that $\mathrm{rad}(U)$ is the sum of the two submodules W (from the proof of Lemma 10) and $W' = H{\downarrow}.f_0$, and that

$$W'{\uparrow}.e_0 \sim \begin{array}{c} F \\ DFAE \\ FBABF \\ AEDF \\ F \end{array} \quad \text{and} \quad W{\uparrow}.e_0 \sim \begin{array}{c} E \\ FBA \\ EADBF \\ BADF \\ F \end{array}.$$

The structure of W is an easy exercise, using the structure of P_E. The only difficulty arising in the analysis of W' is to show there are two F's in the third layer. To see this, examine (7) to observe there does not exist a uniserial kA_9-module with series D, F, F, and evaluate the L-filtration

$$W'{\uparrow}.e_0 \sim \left(\begin{array}{c} 35 \\ 1\ 28 \\ 35 \end{array}\right){\uparrow}.e_0 \sim \begin{array}{c|c} F & \\ DF & AE \\ F & BABF \\ & AEDF \\ & F \end{array} \sim \begin{array}{c|c} F & \\ DFAE & \\ FF & BAB \\ & AEDF \\ & F \end{array}.$$

It follows that $(\mathrm{rad}(U)){\uparrow}.e_0$ has Loewy length 5, from which we easily derive its Loewy structure, and we get an L-filtration of $U{\uparrow}.e_0$ through $(\mathrm{rad}(U)){\uparrow}.e_0$:

$$U{\uparrow}.e_0 \sim \begin{array}{c|c} B & \\ AD & EF \\ B & FDFBAE \\ & EFADBABF \\ & BAEDF \\ & F \end{array}. \tag{15}$$

Now $P_7 = P_C{\downarrow}.f_0$ and $P_B = P_7{\uparrow}.e_0$, so P_B has a submodule $\left(\begin{array}{c} 7 \\ 13 \\ 7 \end{array}\right){\uparrow}.e_0$ with quotient isomorphic to $U{\uparrow}.e_0$. This, together with our knowledge of the first three layers of P_B, implies the Loewy tructure of U is obtained by removing the line from the above L-filtration.□

Now Lemma 11, (14), and Landrock's Lemma show that the fourth Loewy layer of P_B is as claimed. We continue by calculating the Loewy structure of

$$(\mathrm{rad}^2\ P_7){\uparrow}.e_0 \sim \begin{array}{c|c} BBAE & \\ ADADBABEF & \\ BBAEFDF & B \\ EF & AD \\ & B \end{array}.$$

By (13) and duality, this module has Loewy length 5. From this it is easy to see the Loewy structure is obtained by removing the line. Finally, this gives us a filtration of P_B:

$$P_B \sim \begin{array}{c|c} B & \\ ADEF & \\ BFDF & BBAE \\ EF & ADADBABEF \\ & BBAEFDFB \\ & EFAD \\ & B \end{array} .$$

Since the third row of this L-filtration is $L_3(P_B)$ and the fourth row is $L_4(P_B)$, we can remove the line to obtain the Loewy structure of P_B.

We now proceed with the remaining three projectives. With the momentum generated by Landrock's Lemma, the work is relatively easy.

Using Landrock's Lemma and (11), to determine the third Loewy layer of P_F we need only show it contains three F's. This is accomplished as follows. Let $R = \mathrm{rad}^2(P_{35})/\mathrm{rad}^4(P_{35})$, and let M be the submodule of R with quotient 13. Then M has Loewy and socle structure $\begin{smallmatrix} 35\ 35 \\ 1\ 28\ 7 \end{smallmatrix}$. Indeed, this is clearly the socle structure of M. To see that it is the Loewy structure, let $W = \mathrm{rad}(P_{28})/\mathrm{soc}(P_{28})$, which has Loewy and socle structure $\begin{smallmatrix} 35 \\ 1\ 7\ 28 \\ 35 \end{smallmatrix}$. An inclusion from W into its injective hull, P_{35}, induces an inclusion from $W/\mathrm{soc}(W)$ to M, which shows M has L-filtration

$$M \sim \begin{array}{c|c} 35 & 35 \\ & 1\ 7\ 28 \end{array}$$

and therefore the Loewy structure of M is as claimed.

Let N be the quotient of M by 7. If 35 is a submodule of N then M has L-filtration

$$M \sim \begin{array}{c|c} 35 & 35 \\ 1\ 28 & 7 \end{array} .$$

This suffices to prove $L_3(P_F)$ contains three F's, since M occurs in an L-filtration of P_{35}, which we can then induce to get an L-filtration of P_F. Our knowledge of the multiplicities of B and E in the fourth layer of P_F then forces *both* copies of F in this L-filtration to pass to layer 3.

If 35 is not a submodule of N then $N \sim \begin{smallmatrix} 35 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 35 \\ 28 \end{smallmatrix}$. A similar argument yields the desired result in this case.

Now we have determined the first three Loewy layers of P_F, and by our usual argument L-filtration (7) becomes

$$P_F \sim \begin{array}{c|c|c} F & & \\ DFABE & & \\ FBADABEFF & & \\ ABEFDFDF & \begin{array}{c} A \\ B \\ A \end{array} \oplus \begin{array}{c} B \\ AD \\ B \end{array} \oplus \begin{array}{c} E \\ AB \\ E \end{array} & \\ EFF & & \begin{array}{c} F \\ DF \\ F \end{array} \end{array}$$

Since there are two B's in the fifth layer of P_F, the A must go to the fourth layer, and we have determined $L_4(P_F)$. Finally, consider the L-filtration of P_F through $(\mathrm{rad}^2 P_{35})\!\uparrow\!.e_0$. Since the third and fourth rows of this L-filtration are precisely $L_3(P_F)$ and $L_4(P_F)$, the stucture of P_F follows.

We now look at P_A: to get the third layer we need only show there are two A's, but this follows from the fact that there are two B's in the fourth layer. This allows us to remove the first two lines from L-filtration (2). It follows that there are two A's and one D in the fourth layer. This, combined with Landrock's Lemma, shows $L_4(P_A)$ is as claimed. We now use the fact that $(\mathrm{rad}^2 P_1)\!\uparrow\!.e_0 \cong ((P_1/\mathrm{rad}^3 P_1)\!\uparrow\!.e_0)^*$ has Loewy length 5 to get an L-filtration

$$P_A \sim \begin{array}{c|c} A & \\ BEF & \\ AFDF & EAAB \\ EF & ABBBADFE \\ & EAABDFFA \\ & FEB \\ & A \end{array}.$$

Of course, the third and fourth rows are just $L_3(P_A)$ and $L_4(P_A)$, so we can remove the line.

We now know the multiplicity of each simple module in each Loewy layer of P_D, except for the simple module D itself, so we are almost home. Let M be the submodule of $\mathrm{rad}^2(21\!\uparrow)$ with simple head D. Since the Loewy series of $21\!\uparrow$ is also its socle series, we see M must have Loewy length 3, so M has Loewy series D, BF, D; D, B, D; or D, F, D. Say, for example, the first case holds. Then, since there are a B and an F in the fifth layer of P_D, Lemma 2 forces the D to go to the fourth layer in the following L-filtration:

$$\left(\begin{smallmatrix} 21 \\ 21 \end{smallmatrix}\right)\!\uparrow \sim \begin{array}{c|c} D & \\ BF & D \\ AED & BF \\ BF & AED \\ D & BF \\ & D \end{array} \sim \begin{array}{c|c} D & \\ BFD & \\ AEDBF & \\ BFAE & D \\ D & BF \\ & D \end{array}.$$

Of course, the same argument works for the other two cases. This shows that the Loewy structure of $\left(\begin{smallmatrix} 21 \\ 21 \end{smallmatrix}\right)\!\uparrow$ is obtained by removing the line. Since there are a B and an F in the sixth layer of P_D, exactly the same argument works to show there is a D in the fifth layer. Hence the Loewy structure of P_D is as claimed, and Theorem 1 is, at long last, proved.

Appendix

Irreducible Brauer Characters of A_7 mod 3

	$1A$	$2A$	$4A$	$5A$	$7A$	$7B$
1	1	1	1	1	1	1
10_1	10	-2	0	0	$b7$	$**$
10_2	10	-2	0	0	$**$	$b7$
13	13	1	-1	-2	-1	-1
6	6	2	0	1	-1	-1
15	15	-1	-1	0	1	1

$$b7 = \tfrac{1}{2}(-1 + i\sqrt{7})$$

$$** = \tfrac{1}{2}(-1 - i\sqrt{7})$$

Loewy Structure of PIM's of A_7 mod 3

1	10_1	10_2	13		
10_1 10_2 13 13	1	1	1 1	6	15
1 1 1 1 1	10_2 13	10_1 13	10_1 10_2 13	15	6
10_1 10_2 13 13	1	1	1 1	6	15
1	10_1	10_2	13		

Decomposition Matrix of A_8 mod 3

	1	7	13	28	35	21	45_1	45_2
χ_1	1							
χ_2		1						
χ_3	1		1					
χ_4		1	1					
χ_8				1				
χ_9					1			
χ_{12}	1	1	1		1			
χ_{13}	1			1	1			
χ_{14}		1		1	1			
χ_5						1		
χ_6						1		
χ_7						1		
χ_{10}							1	
χ_{11}								1

Decomposition Matrix of A_9 mod 3

	A	B	D	E	F	C	H	G
1	1							
8	1	1						
21_1			1					
21_2			1					
28		1	1					
35_1					1			
35_2					1			
42	1			1				
48		1		1				
56			1		1			
84	1	1		1	1			
105	1	1	1	1	1			
120	2	1		1	2			
168	1	2	2	1	2			
27						1		
189							1	
216						1	1	
162								1

Irreducible Brauer Characters of A_8 mod 3

	$1A$	$2A$	$2B$	$4A$	$4B$	$5A$	$7A$	$7B$
1	1	1	1	1	1	1	1	1
7	7	-1	3	-1	1	2	0	0
13	13	5	1	1	-1	-2	-1	-1
21	21	-3	1	1	-1	1	0	0
28	28	-4	4	0	0	-2	0	0
35	35	3	-5	-1	-1	0	0	0
45_1	45	-3	-3	1	1	0	$b7$	$**$
45_2	45	-3	-3	1	1	0	$**$	$b7$

Loewy Structure of PIM's of A_8 mod 3

1	7	13	28	35			
13 35	13 35	1 7	35	1 28 7	21		
1 1 7 28	1 7 7 28	13 35	1 7 28	13 35 35	21	45_1	45_2
13 35	13 35	1 7	35	1 28 7	21		
1	7	13	28	35			

Irreducible Brauer Characters of A_9 mod 3

	$1A$	$2A$	$2B$	$4A$	$4B$	$5A$	$7A$	$10A$
A	1	1	1	1	1	1	1	1
B	7	3	-1	1	-1	2	0	-2
C	27	7	3	1	-1	2	-1	2
D	21	1	-3	-1	1	1	0	1
E	41	5	1	-1	1	-4	-1	0
F	35	-5	3	-1	-1	0	0	0
G	162	6	-6	0	-2	-3	1	1
H	189	-11	-3	1	1	-1	0	-1

Induction and Restriction

Here we list how the simple modules induce and restrict between kA_7 and kA_8 (as modules, from [8]), and kA_8 and kA_9 (only as characters).

$$\begin{aligned}
&1_{A_8}\downarrow_{A_7} \cong 1 \\
&7_{A_8}\downarrow_{A_7} \cong 1 \oplus 6 \\
&13_{A_8}\downarrow_{A_7} \cong 13 \\
&21_{A_8}\downarrow_{A_7} \cong 6 \oplus 15 \\
&28_{A_8}\downarrow_{A_7} \sim \begin{array}{c} 13 \\ 1\ 1 \\ 13 \end{array} \\
&35_{A_8}\downarrow_{A_7} \cong 10_1 \oplus 10_2 \oplus 15 \\
&45_{1,A_8}\downarrow_{A_7} \cong P_{10_1} \\
&45_{2,A_8}\downarrow_{A_7} \cong P_{10_2}
\end{aligned}$$

$$\begin{aligned}
&1_{A_7}\uparrow^{A_8} \cong 1 \oplus 7 \\
&6_{A_7}\uparrow^{A_8} \sim \begin{array}{c} 7 \\ 13 \\ 7 \end{array} \oplus 21 \\
&10_{1,A_7}\uparrow^{A_8} \cong 35 \oplus 45_1 \\
&10_{2,A_7}\uparrow^{A_8} \cong 35 \oplus 45_2 \\
&13_{A_7}\uparrow^{A_8} \sim \begin{array}{c} 28 \\ 35 \\ 28 \end{array} \oplus 13 \\
&15_{A_7}\uparrow^{A_8} \sim \begin{array}{c} 35 \\ 1\ 28 \\ 35 \end{array} \oplus 21
\end{aligned}$$

$$\begin{aligned}
A\downarrow_{A_8} &= 1 \\
B\downarrow_{A_8} &= 7 \\
C\downarrow_{A_8} &= 7 + 7 + 13 \\
D\downarrow_{A_8} &= 21 \\
E\downarrow_{A_8} &= 13 + 28 \\
F\downarrow_{A_8} &= 35 \\
G\downarrow_{A_8} &= P_{28} = 3(28) + 2(35) + 1 + 7 \\
H\downarrow_{A_8} &= 45_1 + 45_2 + 35 + 35 + 1 + 28
\end{aligned}$$

$$\begin{aligned}
1_{A_8}\uparrow^{A_9} &= 2A + B \\
7_{A_8}\uparrow^{A_9} &= A + 2B + C + D \\
13_{A_8}\uparrow^{A_9} &= 2E + F \\
21_{A_8}\uparrow^{A_9} &= A + 2B + 3D + E + 2F \\
28_{A_8}\uparrow^{A_9} &= A + B + 2E + G \\
35_{A_8}\uparrow^{A_9} &= D + 3F + H \\
45_{i,A_8}\uparrow^{A_9} &= C + 2H
\end{aligned}$$

Tensor Products

The following is a list of the decomposition of the tensor products of some of the irreducible Brauer characters of kA_9. For the tensor squares, we have written the decomposition in two parts–the first part is the character of the symmetric square, the second part is the character of the alternating square.

$$\begin{aligned}
B \otimes B &= (A + C) + D \\
C \otimes C &= (A + 2B + 3D + 2F + E + G) + (6A + 4B + 2C + D + 3F + 4E) \\
D \otimes D &= (D + H) + (3A + 2B + C + 3F + 2E) \\
E \otimes E &= (A + 3B + C + 6D + 3F + G + H) \\
&\quad +(8A + 8B + 2C + 3D + 6E + 7F + H) \\
F \otimes F &= (B + C + D + G + 2H) + (7A + 6B + 3D + C + 7F + 6E)
\end{aligned}$$

$$
\begin{aligned}
B \otimes C &= 2A + 4B + 2D + 2E + F \\
B \otimes D &= A + 2B + D + E + 2F \\
B \otimes E &= A + B + 2E + F + G \\
B \otimes F &= D + F + H
\end{aligned}
$$

$$
\begin{aligned}
C \otimes D &= A + 2B + C + 3D + E + 2F + G + H \\
C \otimes E &= 6A + 6B + C + 3D + 6E + 6F + 2G + H \\
C \otimes F &= 3A + 2B + C + 2D + 2E + 6F + 3H
\end{aligned}
$$

$$
\begin{aligned}
D \otimes E &= 3A + 3B + C + 2D + 3E + 3F + G + 2H \\
D \otimes F &= A + 2B + C + 2D + E + 2F + G + 2H
\end{aligned}
$$

$$
E \otimes F = 4A + 4B + 2C + 3D + 3E + 7F + G + 4H
$$

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