# A Computer Simulation for Determining Bone Fragility: the Muscle, Cortical, Cancellous Bone System *

James Buchanan,
U.S. Naval Academy, Annapolis, MD.
buchanan@usna.gov
Robert P. Gilbert, Philippe Guyenne
Department of Mathematical Sciences
University of Delaware, Newark, DE 19716, USA.
gilbert@math.udel.edu, guyenne@math.udel.edu

Miao-jung Ou
UT Joint Institute for Computational Sciences
Oak Ridge National Laboratory
Oak Ridge, TN 37831, USA
miaokey@gmail.com

February 2, 2011

**Abstract**

Based on a model and an algorithm proposed previously by Buchanan, Gilbert, Wirgin and Xu [8], [11] we numerically computed the dependence of the porosity of cancellous bone encased in cortical bone and muscle on the measured ultrasonic wave. The computation shows the bone porosity can be recovered very accurately. The numerical data show that the ultrasonic wave of higher frequency would produce a more accurate bone porosity.

Mathematics Subject Classification (2000): 35A05, 76B40.
Keywords: cancellous and cortical bone, soft tissue, poroelastic medium, Biot's model.

*This work was funded in part by the National Science Foundation Research Grant DMS-0920850

# 1 Introduction

Cancellous bone consists of a trabeculae matrix with an interstitial blood-marrow fluid. Osteoporosis is characterized by a decrease in strength of this bone matrix. Currently, bone mineral density **BMD** is the gold standard for *in vivo* assessment of fracture risk of bones and is measured using x-ray adsorptiometric techniques [9]. However, only 70-80 percent of the variance of bone strength is accounted for by bone density [14]. As the brittleness of bone depends on more factors than bone density, biologists believe that quantitative ultrasound techniques **QUT** can provide an important new diagnostic tool. Moreover, in contrast to x-ray densiometry, ultrasound does not ionize the mineralized tissue, and its implementation is relatively inexpensive. It would be of enormous clinical advantage if an accurate method could be developed using ultrasound interrogation to determine whether one had osteoporosis. The intention of this research to eventually produce an accurate clinical procedure for determining the bone density and other bone parameters describing bone brittleness.
Since the loss of bone density and the destruction of the bone microstructure is most evident in cancellous bone, it is natural to consider the possibility of developing accurate ultrasound models for the isonification of cancellous bone. In this paper we consider the possibility of determining the bone density of cancellous bone *in vivo* by using a simple one-dimensional model of a muscle- cortical bone- cancellous bone sample. If it is possible to accurately determine whether an *in vivo* sample is osteoporotic research in this direction merits further investigation. Moreover, if a higher dimensional model also leads to an accurate prediction of bone density, this would warrant laboratory testing i͡n vivo on biological samples. More specifically, we is determine whether a sonified bone sample is osteoporotic by measuring its refracted acoustic, i.e. the acoustic field is measured when the sample is exposed to impulses from a transducer. From this data the bone parameters can be determined. Such a procedure is referred to as solving an **inverse problem**. We have made several initial investigations of this type using the Biot model of a porous media to represent the cancellous bone [4, 5, 6, 7, 8]. Using computer simulations we have shown that the *in vitro* model determines very well the bone density within an accuracy of 1%; however, other bone parameters such as pore size, shear modulus and bulk modulus are accurate only to within 20% and these bone parameters have a significant effect on bone rigidity. The *in vivo* problem is much more complicated and it may require more sophisticated models than Biot's to determine pore size, shear and bulk modului under these circumstances. The investigation of models different from Biot's is the top[ic of a separate research.

Biot developed a general theory for the ultrasonic propagation in fluid-saturated porous media [1]-[3]. McKelvie and Palmer [15], Williams [16], and Hosokawa and Otani [12] discussed the application of Biot's model for a poroelastic medium to cancellous bones. Recently, Buchanan, Gilbert, Wirgin and Xu [11] proposed a model for the determination of the parameters of cancellous bone using ultrasonic measurements. In the present paper we formulate numerical simulation of a clinical procedure for determining bone fragility ıon vivo. In the present model a transducer is places on one side of the skin of the member to be interrogated and a receiver is place diametrically opposed in contact with the member. The impedance

matching for transducer and receiver with the skin is enhanced by using a sterile jelly on the surface of the skin as is usual in clinical practice.

The muscle is usually modeled as an elastic material; whereas the the corrical and cancellous bone may be modeled by the Biot equations , albeit with different porosities. However, as there is such a disparity of porosities between these two types of bone, in the present model the cortical bone will also be modeled as an elastic material. Our procedure will be to solve this system of equations analytically in the frequency domain and then to fast Fourier transform these solutions to the time domain.

# 2 The Muscle-Cortical and Cancellous Bone Model

In order to facilitate the possibility of obtaining an analytic solution in the frequency domain, we consider a one-dimensional dynamic model. The muscle is considered as an elastic medium. In the muscle ($-x_m < x < 0$ and $L < x < L + x_m$), the elastic displacement $u^m(x)$ satisfies the equation of motion

$$(2\mu^m + \lambda^m)\frac{\partial\, e^m}{\partial\, x} = \rho^m \frac{\partial^2 u^m}{\partial t^2} \tag{2.1}$$

where $\lambda^m, \mu^m$ are Lamé constants and $\rho^m$ is the density for the muscle. We assume that measures have been taken so that the reflection from receiver and transmitter back to the muscle can be ignored. Therefore, we can assume that equation (2.1) is satisfied in $\{x : -x_m < x < 0$ or $L < x < L + x_m\}$. The boundary condition induced by the the transducer corresponds to a stress (traction) given as a function of time

$$(2\mu^m + \lambda^m)\frac{\partial\, u^m}{\partial\, x}(-x_m, t) := F_0(t) \tag{2.2}$$

For describing the bone, we use the respectively the superscripts "h" and "s" as notation to refer to either hard (cortical) bone or soft (cancellous) bone.
In the cortical bone ($0 < x < x_c$ and $L - x_c < x < L$), which we treat as a purely elastic material, the ultrasonic wave satisfies:

$$(2\mu^h + \lambda^h)\frac{\partial^2 u^h}{\partial x^2} = \rho^h \frac{\partial^2 u^h}{\partial t^2} \qquad e^h = \frac{\partial u^h}{\partial x}, \tag{2.3}$$

In the Biot model the motion of the skeletal frame and the interstitial fluid within are tracked by position vectors $u^s$ and $U^s$ respectively; moreover, (Ref. [1]-[3]). However, because of the known frequency dependence of the viscous term in the time harmonic case we propose that instead of the standard Biot equations for the ultrasonic vibrations, be replaced by a

differential-integral equation of the convolution type:

$$\begin{aligned}
\frac{\partial}{\partial x}[\lambda e^s + Q\epsilon^s] &= \frac{\partial^2}{\partial t^2}(\rho_{11}u^s + \rho_{12}U^s) + \int_0^t b(t-\tau)\frac{\partial}{\partial \tau}(u - U^s)(\tau)\, d\tau \\
\frac{\partial}{\partial x}[Qe^s + R\epsilon^s] &= \frac{\partial^2}{\partial t^2}(\rho_{12}u^s + \rho_{22}U^s) - \int_0^t b(t-\tau)\frac{\partial}{\partial \tau}(u^s - U^s)(\tau)\, d\tau \\
e = \frac{\partial u^s}{\partial x}, &\qquad \epsilon = \frac{\partial U^s}{\partial x}
\end{aligned} \tag{2.4}$$

Here, the motion of the frame and fluid within the bone are tracked by the displacements $u^s$ and $U^s$ respectively (Ref. [[1]]-[[3]]). The reason for this alteration comes from the fact that Biot noticed for time harmonic equations the viscous forces acting on the difference of fluid and frame velocities were a function of frequency $\omega$.

## 2.1 The Fourier transformed System

Taking the Fourier transform of (**??**) we obtain

$$(2\mu^m + \lambda^m)\frac{\partial^2 \hat{u}^m}{\partial\, x^2} = -\omega^2 \rho^m \hat{u}^m \tag{2.5}$$

On the surface where the transducer makes contact with the skin, we assume that the stress $\sigma_{xx}(-x_m, t) = C_0 \exp i\omega\, t$. On the other side where the receiver lies we assume that the displacement is fixed and we measure the pressure against the receiver, i.e. $u^m(L + x_m, t)$. We use the modified Biot's equations, i.e. The 1-D modified Biot's equations for cancellous bone are [10]

$$\begin{aligned}
\tilde{\rho}_{11}(\omega)(i\omega)^2\hat{u}^s + \tilde{\rho}_{12}(\omega)(i\omega)^2\hat{U}^s &= P\frac{d^2\hat{u}}{dx^2} + Q\frac{d^2\hat{U}^s}{dx^2} \\
\tilde{\rho}_{12}(\omega)(i\omega)^2\hat{u}^s + \tilde{\rho}_{22}(\omega)(i\omega)^2\hat{U}^s &= Q\frac{d^2\hat{u}^s}{dx^2} + R\frac{d^2\hat{U}^s}{dx^2}
\end{aligned} \tag{2.6}$$

with the superscript $s$ meaning the soft, i.e. cancellous bone, and

$$\begin{aligned}
\tilde{\rho}_{11}(\omega) &:= \rho_{11} + \frac{2\beta\alpha_\infty}{\Lambda}\left(\frac{\rho_f\eta}{i\omega}\right)^{1/2} \\
\tilde{\rho}_{12}(\omega) &:= \rho_{12} - \frac{2\beta\alpha_\infty}{\Lambda}\left(\frac{\rho_f\eta}{i\omega}\right)^{1/2} \\
\tilde{\rho}_{22}(\omega) &:= \rho_{22} + \frac{2\beta\alpha_\infty}{\Lambda}\left(\frac{\rho_f\eta}{i\omega}\right)^{1/2}
\end{aligned} \tag{2.7}$$

and $\rho_{11}$, $\rho_{12}$, $\rho_{22}$ being the mass coupling terms in the Biot's model defined in terms of solid density $\rho_s$, pore fluid density $\rho_f$, $\beta$, $\alpha_\infty$ and $\omega$

$$\rho_{12} := -\beta\rho_f(\alpha_\infty - 1), \quad \rho_{22} := \beta\rho_f\alpha_\infty$$
$$\rho_{11} := (1-\beta)\rho_s + \beta\rho_f(\alpha_\infty - 1) \tag{2.8}$$

The coupling between the fluid part (marrow) and elastic matrix (trabeculer bone) is described by the Johnson-Koplik-Dashen model [13]. In this model, the dynamic tortuosity $\alpha(\omega)$ is expressed as a function of tortuosity $\alpha_\infty$, pore fluid viscosity $\eta$, pore fluid density $\rho_f$, permeability $k$, porosity $\beta$, the angular frequency $\omega$ and the viscous characteristic length $\Lambda$

$$\alpha(\omega) = \alpha_\infty \left(1 + \frac{\eta\beta}{i\omega\alpha_\infty\rho_f k}\sqrt{1 + i\frac{4\alpha_\infty^2 k^2\rho_f\omega}{\eta\Lambda^2\beta^2}}\right), \tag{2.9}$$

$i = \sqrt{-1}$.
The effective elastic constants $P$, $Q$ and $R$ are related to $\beta$, bulk modulus of the pore fluid $K_f$, bulk modulus of the trabeculer bone $K_s$, bulk modulus of the porous skeletal frame $K_b$ and the shear modulus of the composite as well as the skeletal frame $N$:

$$P := \frac{(1-\beta)\left(1-\beta-\frac{K_b}{K_s}\right) + \beta\frac{K_s}{K_f}K_b}{1-\beta-\frac{K_b}{K_s}+\beta\frac{K_s}{K_f}} + \frac{4}{3}N$$
$$Q := \frac{\left(1-\beta-\frac{K_b}{K_s}\right)\beta K_s}{1-\beta-\frac{K_b}{K_s}+\beta\frac{K_s}{K_f}}$$
$$R := \frac{\beta^2 K_s}{1-\beta-\frac{K_b}{K_s}+\beta\frac{K_s}{K_f}} \tag{2.10}$$

It is convenient to write the system of (2.1) and (**??**), we first write it in matrix form

$$\begin{pmatrix} P & Q \\ Q & R \end{pmatrix}\begin{pmatrix} \frac{d^2\hat{u}^s}{dx^2} \\ \frac{d^2\hat{U}^s}{dx^2} \end{pmatrix} = -\omega^2\begin{pmatrix} \tilde{\rho}_{11} & \tilde{\rho}_{12} \\ \tilde{\rho}_{12} & \tilde{\rho}_{22} \end{pmatrix}\begin{pmatrix} \hat{u}^s \\ \hat{U} \end{pmatrix}$$

This implies

$$\begin{pmatrix} \frac{d^2\hat{u}^s}{dx^2} \\ \frac{d^2\hat{U}^s}{dx^2} \end{pmatrix} = \left(\frac{-\omega^2}{PR-Q^2}\right)\begin{pmatrix} R\tilde{\rho}_{11}-Q\tilde{\rho}_{12} & R\tilde{\rho}_{12}-Q\tilde{\rho}_{22} \\ -Q\tilde{\rho}_{11}+P\tilde{\rho}_{12} & -Q\tilde{\rho}_{12}+P\tilde{\rho}_{22} \end{pmatrix}\begin{pmatrix} \hat{u}^s \\ \hat{U}^s \end{pmatrix}$$

Diagonalizing the right hand side using the transformation matrix $\mathbf{P}$ and the diagonal matrix $\mathbf{\Lambda}$

$$\mathbf{P} := \begin{pmatrix} \mathbf{v_1} & \mathbf{v_2} \end{pmatrix}; \quad \mathbf{\Lambda} := \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

we obtain

$$\begin{pmatrix} \frac{d^2\hat{u}^s}{dx^2} \\ \frac{d^2\hat{U}^s}{dx^2} \end{pmatrix} = \mathbf{P}\left(\frac{-\omega^2}{PR-Q^2}\mathbf{\Lambda}\right)\mathbf{P}^{-1}\begin{pmatrix} \hat{u}^s \\ \hat{U}^s \end{pmatrix}$$

For ease of notation, we define intermediate variables $b$ and $c$:

$$\begin{aligned} b &:= -R\tilde{\rho}_{11} + 2Q\tilde{\rho}_{12} - P\tilde{\rho}_{22} \\ c &:= RP(\tilde{\rho}_{11}\tilde{\rho}_{22} - \tilde{\rho}_{12}^2) + Q^2(\tilde{\rho}_{12}^2 - \tilde{\rho}_{22}\tilde{\rho}_{11}) \end{aligned} \tag{2.11}$$

In terms of $b$ and $c$, the eigen values are

$$\lambda_1 = \frac{-b - \sqrt{b^2 - 4c}}{2}; \quad \lambda_2 = \frac{-b + \sqrt{b^2 - 4c}}{2} \tag{2.12}$$

The vectors $\mathbf{v_i}$, $i = 1, 2$ in $\mathbf{P}$ can be expressed as

$$\mathbf{v_i} = \begin{pmatrix} v_{i1} \\ v_{i2} \end{pmatrix} = \frac{1}{\sqrt{1 + \left(\frac{R\tilde{\rho}_{11} - Q\tilde{\rho}_{12} - \lambda_i}{Q\tilde{\rho}_{22} - R\tilde{\rho}_{12}}\right)^2}} \begin{pmatrix} 1 \\ \frac{R\tilde{\rho}_{11} - Q\tilde{\rho}_{12} - \lambda_i}{Q\tilde{\rho}_{22} - R\tilde{\rho}_{12}} \end{pmatrix} \tag{2.13}$$

This leads to the expression for the solution in terms of $v_{ij}$ and $\lambda_j$, $i, j = 1, 2$

$$\begin{aligned} \hat{u}^s(x) &= C_1 v_{11} e^{ik_1 x} + C_2 v_{11} e^{-ik_1 x} + C_3 v_{21} e^{ik_2 x} + C_4 v_{21} e^{-ik_2 x}, \text{ for } 0 < x < L \\ \hat{U}(x)^s &= C_1 v_{12} e^{ik_1 x} + C_2 v_{12} e^{-ik_1 x} + C_3 v_{22} e^{ik_2 x} + C_4 v_{22} e^{-ik_2 x}, \text{ for } 0 < x < L \end{aligned} \tag{2.14}$$

with

$$k_i := \sqrt{\left(\frac{-\omega^2}{PR - Q^2}\right)\lambda_i}, \quad , i = 1, 2 \tag{2.15}$$

On the interface ($x_0 = x_c$ or $x_0 = L - x_c$), the displacement is continuous, and the normal stress and the pore stress in the cortical bone is equal to the normal stress and the pore stress in the cancellous bone. Therefore, the displacements, pressure and stresses satisfy:

$$\begin{aligned} U^h(x_0^+) = \ u^h(x_0^+) &= u^s(x_0^-), \\ \sigma_{xx}^h(x_0^+) \quad &= \sigma_{xx}^s(x_0^-) + \sigma^s(x_0^-). \end{aligned} \tag{2.16}$$

Here $x_0^- = x_c^-$ or $(L - x_c)^+$, and $x_0^+ = x_c^+$ or $(L - x_c)^-$, respectively. In (2.16), we recall that $\sigma_{xx}^a = \lambda^a e^a + Q^a \epsilon^a$, $e^a = \frac{\partial u^a}{\partial x}$, and $\epsilon^a = \frac{\partial U^a}{\partial x}$ There are six equations in (2.16) totally.

# 3 Transient reflection and transmission of waves

Let $\hat{U^a}_0$, $\hat{e^a}$, and $\hat{\epsilon^a}$ be the Fourier transforms of $U_0^a$, $e^a$, and $\epsilon^a$ ($a = m, c, s$), respectively, in the frequency domain $\omega$. Correspondingly, in the muscle ($-x_m < x < 0$) and ($L < x < L + x_m$) and cortical bone ($0 < x < x_c$) and ($L - x_c < x < L$) respectively, we obtain the transformed equations of (2.1) and (2.3) as

$$\frac{\partial^2}{\partial x^2}[(\lambda^m + 2\mu^m)\hat{u^m}] = -\omega^2 \rho^m \hat{u^m}. \tag{3.1}$$

$$\frac{\partial^2}{\partial x^2}[(\lambda^h + 2\mu^h)\hat{u^h}] = -\omega^2 \rho^h \hat{u^h}. \tag{3.2}$$

Whereas in the cancellous bone the equations are, as mentioned before,

$$\begin{aligned} \frac{\partial^2}{\partial x^2}[(\lambda^s + 2\mu^s)\hat{u^s} + Q^s\hat{U^s}] &= -\omega^2(\rho_{11}^s\hat{u^s} + \rho_{12}^s\hat{U^s}) + i\omega b^s(\hat{u^s} - \hat{U^s}), \\ \frac{\partial^2}{\partial x^2}[Q^s\hat{u^s} + R^s\hat{U^s}] &= -\omega^2(\rho_{12}^s\hat{u^s} + \rho_{22}^s\hat{U^s}) - i\omega b^s(\hat{u^s} - \hat{U^s}). \end{aligned} \tag{3.3}$$

On the interface, ($x_0 = x_c$ or $x_0 = L - x_c$), from (2.16), we have the transformed transmission conditions now as

$$\begin{aligned} \hat{U^s}(x_0^-) = \hat{u^h}(x_0^+) &= \hat{u^s}(x_0^-), \\ (\lambda^h + 2\mu^h)\frac{\partial \hat{u^h}}{\partial x}(x_0^+) &= (\lambda^s + 2\mu^s + Q^s)\frac{\partial \hat{u^s}}{\partial x}(x_0^-) + (Q^s + R^s)\frac{\partial \hat{U^s}}{\partial x}(x_0^-), \end{aligned} \tag{3.4}$$

Here, again, $x_0^- = x_c$ or $(L - x_c)^+$, and $x_0^+ = x_c^+$ or $(L - x_c)^-$, respectively.

We represent the elastic wave in the muscle as

$$\begin{aligned} \hat{u}^m(x) =& C_1^m e^{-i\omega k_m x} + C_2^m e^{i\omega k_m x}, && -x_m < x < 0; \\ \hat{u}^m(x) =& C_3^m e^{-i\omega k_m x} + C_4^m e^{i\omega k_m x}, && L < x < L + x_m. \end{aligned} \tag{3.5}$$

where $k_m = \sqrt{\frac{\rho^m}{2\mu^m + \lambda_m}}$, and $C_1^m, C_2^m, C_3^m, C_4^m$ are constants to be determined. However, as the boundary condition at the transducer is $\hat{\sigma}_{xx} = C_0$ and at the opposite side as the receiver is held fixed $\hat{u}^m = 0$ we have the constraints

$$-i(\lambda + 2\mu)\omega k_m \left[C_1^m e^{-i\omega k_m x_m} + C_2^m e^{i\omega k_m x_m}\right] = C_0$$

and

$$C_3^m e^{-i\omega k_m x_{L+x_m}} + C_4^m e^{i\omega k_m x_{L+x_m}} = 0. \tag{3.6}$$

On the interface between the cortical bone and muscle ( $x_0 = 0$ or $x_0 = L$), continuity is required for both skeletal and fluid horizontal displacement, aggregate normal stress. In view of (3.5), on $x = 0$, we have

$$\begin{aligned} \hat{u}^m(0^-) &= \hat{u}^h(0^+), \\ \hat{\sigma}_{xx}^m(0^-) &= \hat{\sigma}_{xx}^h(0^+). \end{aligned} \tag{3.7}$$

On $x = L$,

$$\begin{aligned} \hat{u}^m(L^+) &= \hat{u}^h(L^-), \\ \hat{\sigma}_{xx}^m(L^+) &= \hat{\sigma}_{xx}^h(L^-), \end{aligned} \tag{3.8}$$

(3.7) and (3.8) are equivalent to:

$$\begin{aligned} \hat{u}^m(0^-) &= \beta^h \hat{U}^h(0^+) + (1 - \beta^h)\hat{u}^h(0^+), \\ \lambda^m \frac{\partial \hat{u}^m}{\partial x}(0^-) &= (\lambda^h + 2\mu^h + Q^h)\frac{\partial \hat{u}^h}{\partial x}(0^+) + (R^h + Q^h)\frac{\partial \hat{U}^h}{\partial x}(0^+), \\ \lambda^m \frac{\partial \hat{u}^m}{\partial x}(0^-) &= \frac{1}{\beta^h}\left[Q^h \frac{\partial \hat{u}^h}{\partial x}(0^+) + R^h \frac{\partial \hat{U}^h}{\partial x}(0^+)\right], \end{aligned} \tag{3.9}$$

$$
\begin{aligned}
\hat{u}^m(L^+) &= \beta^h \hat{U}^h(L^-) + (1-\beta^h)\hat{u}^h(L^-), \\
\lambda^m \frac{\partial \hat{u}^m}{\partial x}(L^+) &= (\lambda^h + 2\mu^h + Q^h)\frac{\partial \hat{u}^h}{\partial x}(L^-) + (R^h + Q^h)\frac{\partial \hat{U}^h}{\partial x}(L^-), \\
\lambda^m \frac{\partial \hat{u}^m}{\partial x}(L^+) &= \frac{1}{\beta^h}\left[ Q^h \frac{\partial \hat{u}^h}{\partial x}(L^-) + R^h \frac{\partial \hat{U}^h}{\partial x}(L^-) \right].
\end{aligned} \tag{3.10}
$$

# 4 Symbolic Calculations in the Frequency Domain

# 5 Numerical tests

In the paper [4], it is shown that from the measured data by the receivers (see Figure 1), the porosities of the cortical and cancellous bones are uniquely determined locally. In this section, we will compute the dependence of the porosity $\beta^a$, where $a = h, s$ on the measured transmitted wave $c_4$ at the receivers. This is done by computing the solution (6.19) forwardly.

Theoretically, the linear system (6.19) has a unique solution. But computationally, its condition number is about $10^{20}$ for practical data. To produce accurate solutions, we first rescale the matrix $\mathbf{M}$. Suppose $m_j$ is the largest element of the $jth$ column, $j = 1, 2, \cdots, 20$, then dividing the $jth$ column by $m_j$, we get a new matrix $\mathbf{M}_{new} = \mathbf{MQ}$, where $\mathbf{Q}$ is a diagonal matrix with $\mathbf{Q}(j,j) = \frac{1}{m_j}$, $j = 1, 2, \cdots, 20$. Let $\mathbf{X} = \mathbf{Q}^{-1}\mathbf{C}$, then we have $\mathbf{M}_{new}\mathbf{QX} = \mathbf{B}$. Now we want to recondition the matrix $\mathbf{M}_{new}$ and rewrite the problem as

$$\mathbf{P}^{-1}\mathbf{M}_{new}\mathbf{X} = \mathbf{P}^{-1}\mathbf{B}$$

where $\mathbf{P}^{-1}$ is the preconditioner of the matrix $\mathbf{M}_{new}$.
We need $\mathbf{P}$ be a matrix which is similar to $\mathbf{M}_{new}$ and $\mathbf{P}^{-1}\mathbf{M}_{new}$ has a much smaller condition number than that of $\mathbf{M}_{new}$. In our computation, we choose $\mathbf{P}$ to be a diagonal matrix with

$$\mathbf{P}(i,i) = \mathbf{M}_{new}(i,i)$$

in (6.19). Here $i = 1, 2, \cdots, 20$.

In the numerical computation, in addition to the constants listed in Table 1 and Table 2, we used also the following data, $A = 1$, $L = 0.05$, $x_c = 0.01$ $x_s = -0.1$, $s = 0.02$, $\hat{f} = c_0^2$. We experimented with different waves $\omega$. In Figure 2, we plotted the real part of constant $c4$. (the imaginary part of $c_4$ is neglected as it is about one-tenth of the real part) in the transmitted wave against the porosity of cancellous bones $\beta^a (a = h, s)$ for $\omega = 10^5$.

# 6 Appendix

Introducing the new unknowns $\hat{u}^a_x = \frac{\partial \hat{u}^a}{\partial x}$ and $\hat{U}^a_x = \frac{\partial \hat{U}^a}{\partial x}$, the systems (3.2) , (3.3) can be decoupled into the following matrix form

$$\frac{\partial}{\partial x}\begin{pmatrix}\hat{u}^a\\ \hat{U}^a\\ \hat{u}^a_x\\ \hat{U}^a_x\end{pmatrix} = \begin{pmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ l^a_{11} & l^a_{12} & 0 & 0\\ l^a_{21} & l^a_{22} & 0 & 0\end{pmatrix}\begin{pmatrix}\hat{u}^a\\ \hat{U}^a\\ \hat{u}^a_x\\ \hat{U}^a_x\end{pmatrix}, \tag{6.1}$$

where $a = s$ or $a = h$,

$l^a_{11} = -\omega^2 a^a_{11} + i\omega b^a l^a_1, = 6\times 10^7 - \mathbf{i}4\times 10^7 \quad l^a_{12} = -\omega^2 a^a_{12} - i\omega b^a l^a_1, = -5\times 10^7 + \mathbf{i}4\times 10^7$
$l^a_{21} = -\omega^2 a^a_{21} - i\omega b^a l^a_2, = -1\times 10^7 + \mathbf{i}7\times 10^6 \quad l^a_{21} = -\omega^2 a^a_{22} + i\omega b^a l^a_2, = 9\times 10^6 - \mathbf{i}7\times 10^6$

and

$$l^a_1 = \frac{Q^a + R^a}{x^a_c}, = -1.3\times 10^{-7} + \mathbf{i}4\times 10^{-9} \qquad l^a_2 = \frac{Q^a + \lambda^a + 2\mu^a}{x^a_c},$$
$$= -2\times 10^{-8} + \mathbf{i}7\times 10^{-10}$$
$$a^a_{11} = \frac{R^a\rho^a_{11} - Q^a\rho^a_{12}}{x^a_c}, = -5\times 10^{-5} + \mathbf{i}1\times 10^{-6} \qquad a^a_{12} = \frac{R^a\rho^a_{12} - Q^a\rho^a_{22}}{x^a_c}, = 2\times 10^-$$
$$a^a_{21} = \frac{(\lambda^a + 2\mu^a)\rho^a_{12} - Q^a\rho^a_{11}}{x^a_c}, = 1\times 10^{-5} - \mathbf{i}6\times 10^{-7} \quad a^a_{22} = \frac{(\lambda^a + 2\mu^a)\rho^a_{22} - Q^a\rho^a_{12}}{x^a_c}, =$$
$$x^a_c = (\lambda^a + 2\mu^a)R^a - (Q^a)^2 = 5\times 10^8 + \mathbf{i}4\times 10^8.$$

Here $a = h$ represents hard (cortical) bone and $a = s$ represents soft (cancellous) bone. For $a = s$, the system (6.1) is for $x_c < x < L - x_c$. For For $a = h$, the system (6.1) may be for $0 < x < x_c$ or for $L - x_c < x < L$. Therefore, the system (6.1) consists of twelve equations in total.

The homogeneous systems (6.1) can be easily found to have solutions of the following form: For $0 < x < x_c$,

$$\begin{pmatrix}\hat{u}^h\\ \hat{U}^h\\ \hat{u}^h_x\\ \hat{U}^h_x\end{pmatrix} = \mathbf{E}^h(x)\mathbf{C}^h = \mathbf{E}^h(x)\begin{pmatrix}C^h_1\\ C^h_2\\ C^h_3\\ C^h_4\end{pmatrix}. \tag{6.2}$$

For $x_c < x < L - x_c$,

$$\begin{pmatrix}\hat{u}^s\\ \hat{U}^s\\ \hat{u}^s_x\\ \hat{U}^s_x\end{pmatrix} = \mathbf{E}^s(x)\mathbf{C}^s = \mathbf{E}^s(x)\begin{pmatrix}C^s_1\\ C^s_2\\ C^s_3\\ C^s_4\end{pmatrix}. \tag{6.3}$$

For $L - x_c < x < L$,

$$\begin{pmatrix} \hat{u}^h \\ \hat{U}^h \\ \hat{u}^h_x \\ \hat{U}^h_x \end{pmatrix} = \mathbf{E}^h(x)\mathbf{D}^h = \mathbf{E}^h(x) \begin{pmatrix} D^h_1 \\ D^h_2 \\ D^h_3 \\ D^h_4 \end{pmatrix}. \tag{6.4}$$

Here $\mathbf{E}^a(x) = \mathbf{E}^a_0 e^{\mathbf{\Lambda}^a_0 x}$, and

$$\mathbf{\Lambda}^a_0 = \begin{pmatrix} \gamma^a_1 & 0 & 0 & 0 \\ 0 & \gamma 2^a & 0 & 0 \\ 0 & 0 & \gamma^a_3 & 0 \\ 0 & 0 & 0 & \gamma^a_4 \end{pmatrix}, \tag{6.5}$$

$$\mathbf{E}^a_0 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ ((\gamma^a_1)^2 - l^a_{11})/l^a_{12} & ((\gamma^a_2)^2 - l^a_{11})/l^a_{12} & ((\gamma^a_3)^2 - l^a_{11})/l^a_{12} & ((\gamma^a_4)^2 - l^a_{11})/l^a_{12} \\ \gamma^a_1 & \gamma^a_2 & \gamma^a_3 & \gamma^a_4 \\ \frac{\gamma^a_1((\gamma^a_1)^2 - l^a_{11})}{l^a_{12}} & \frac{\gamma^a_2((\gamma^a_2)^2 - l^a_{11})}{l^a_{12}} & \frac{\gamma^a_3((\gamma^a_3)^2 - l^a_{11})}{l^a_{12}} & \frac{\gamma^a_4((\gamma^a_4)^2 - l^a_{11})}{l^a_{12}} \end{pmatrix}, \tag{6.6}$$

where $\mathbf{E}^a_0$ and $\mathbf{\Lambda}^a_0$ are the eigenvector and eigenvalue matrices of the $4 \times 4$ matrix in (6.1), and

$$\gamma^a_1 = \frac{1}{\sqrt{2}}\sqrt{l^a_{22} + l^a_{11} - \sqrt{(l^a_{11} - l^a_{22})^2 + 4l^a_{12}l^a_{21}}},$$
$$\gamma^a_2 = -\frac{1}{\sqrt{2}}\sqrt{l^a_{22} + l^a_{11} - \sqrt{(l^a_{11} - l^a_{22})^2 + 4l^a_{12}l^a_{21}}},$$
$$\gamma^a_3 = \frac{1}{\sqrt{2}}\sqrt{l^a_{22} + l^a_{11} + \sqrt{(l^a_{11} - l^a_{22})^2 + 4l^a_{12}l^a_{21}}},$$
$$\gamma^a_4 = -\frac{1}{\sqrt{2}}\sqrt{l^a_{22} + l^a_{11} + \sqrt{(l^a_{11} - l^a_{22})^2 + 4l^a_{12}l^a_{21}}}.$$

The coefficients $C^h_i, C^s_i, D^h_i,\ i = 1, 2, 3, 4$ are to to be determined.

Now we discuss the determination of the constant vector

$$\mathbf{C} = (C^m_1 \quad C^m_2 \quad C^h_1 \quad C^h_2 \quad C^h_3 \quad C^h_4 \quad C^s_1 \quad C^s_2 \quad C^s_3 \quad C^s_4 \quad D^h_1 \quad D^h_2 \quad D^h_3 \quad D^h_4 \quad C^m_3 \quad C^m_4)^T$$

by the boundary conditions and the interface coupling.

Let

$$\mathbf{M}_1 = \begin{pmatrix} 1 & 1 \\ -\lambda^m i\omega k_m & \lambda^m i\omega k_m \\ -\lambda^m i\omega k_m & \lambda^m i\omega k_m \end{pmatrix}, \tag{6.7}$$

$$\mathbf{M}_2 = \begin{pmatrix} 1-\beta^h & \beta^h & 0 & 0 \\ 0 & 0 & \lambda^h + 2\mu^h + Q^h & R^h + Q^h \\ 0 & 0 & Q^h/\beta^h & R^h/\beta^h \end{pmatrix}, \tag{6.8}$$

$$\mathbf{M}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda^h + 2\mu^h + Q^h & R^h + Q^h \\ 0 & 0 & Q^h/\beta^h & R^h/\beta^h \end{pmatrix}, \tag{6.9}$$

$$\mathbf{M}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda^s + 2\mu^s + Q^s & R^s + Q^s \\ 0 & 0 & Q^s/\beta^s & R^s/\beta^s \end{pmatrix}, \tag{6.10}$$

$$\mathbf{M}_5 = \begin{pmatrix} e^{-i\omega k_m L} & e^{i\omega k_m L} & 0 \\ -i\omega\lambda^m k_m e^{-i\omega k_m L} & i\omega\lambda^m k_m e^{i\omega k_m L} & 0 \\ -i\omega\lambda^m k_m e^{-i\omega k_m L} & i\omega\lambda^m k_m e^{i\omega k_m L} & 0 \end{pmatrix}, \tag{6.11}$$

$$\mathbf{M}_6 = \begin{pmatrix} 1 & -e^{2ik_0 x_s} & -1 \\ 1 & e^{2ik_0 x_s} & -1 \\ 0 & e^{-ik_0 s} & e^{ik_0 s} \\ 0 & \frac{ik_0}{\rho_0\omega^2} e^{-ik_0 s} & -\frac{ik_0}{\rho_0\omega^2} e^{ik_0 s} \end{pmatrix}, \tag{6.12}$$

$$\mathbf{M}_7 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \lambda^m i\omega k_m e^{i\omega k_m s} & -\lambda^m i\omega k_m e^{-i\omega k_m s} \\ e^{i\omega k_m s} & e^{-i\omega k_m s} \end{pmatrix}, \tag{6.13}$$

$$\mathbf{M}_8 = \begin{pmatrix} \lambda^m i\omega k_m e^{-i\omega k_m (L+s)} & -\lambda^m i\omega k_m e^{i\omega k_m (L+s)} & -e^{ik_0(L+s)} \\ e^{-i\omega k_m (L+s)} & e^{i\omega k_m (L+s)} & \frac{ik_0}{\rho_0\omega^2} e^{ik_0(L+s)} \end{pmatrix}, \tag{6.14}$$

In matrix notations, the condition on $x = 0$ is

$$\mathbf{M}_1(\mathbf{C_1^m} \quad \mathbf{C_2^m})^T = \mathbf{M}_2\mathbf{E}^h(0)\mathbf{C}^h = \mathbf{M}_2\mathbf{E}_0^h\mathbf{C}^h, \tag{6.15}$$

The condition on $x =_c$ is

$$\mathbf{M}_3\mathbf{E}_0^h e^{\mathbf{\Lambda}_0^h x_c}\mathbf{C}^h = \mathbf{M}_4\mathbf{E}_0^s e^{\mathbf{\Lambda}_0^s x_c}\mathbf{C}^s. \tag{6.16}$$

The condition on $x = L - x_c$ is

$$\mathbf{M}_3\mathbf{E}_0^h e^{\mathbf{\Lambda}_0^h (L-x_c)}\mathbf{D}^h = \mathbf{M}_4\mathbf{E}_0^s e^{\mathbf{\Lambda}_0^s (L-x_c)}\mathbf{C}^s. \tag{6.17}$$

The condition on $x = L$ is

$$\mathbf{M}_5(\mathbf{C_3^m} \quad \mathbf{C_4^m} \quad \mathbf{c4})^T = \mathbf{M}_2\mathbf{E}^h(L)\mathbf{D}^h = \mathbf{M}_2\mathbf{E}_0^h e^{\mathbf{\Lambda}_0^h L}\mathbf{D}^h. \tag{6.18}$$

Collecting (6.12)-(6.18) we obtain a $20 \times 20$ matrix equation

$$\mathbf{MC} = \mathbf{B}, \tag{6.19}$$

where

$$\mathbf{M} = \begin{pmatrix} \mathbf{M}_6 & \mathbf{M}_7 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{M}_1 & -\mathbf{M}_2\mathbf{E}_0^h & 0 & 0 & 0 \\ 0 & 0 & \mathbf{M}_3\mathbf{E}_0^h e^{\mathbf{\Lambda}_0^h x_c} & -\mathbf{M}_4\mathbf{E}_0^s e^{\mathbf{\Lambda}_0^s x_c} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{M}_4\mathbf{E}_0^s e^{\mathbf{\Lambda}_0^s (L-x_c)} & -\mathbf{M}_3\mathbf{E}_0^h e^{\mathbf{\Lambda}_0^h (L-x_c)} & 0 \\ 0 & 0 & 0 & 0 & \mathbf{M}_2\mathbf{E}_0^h e^{\mathbf{\Lambda}_0^h L} & -\mathbf{M}_5 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{M}_8 \end{pmatrix}, \tag{6.20}$$

$$\mathbf{B} = [\mathbf{0} \quad -\tfrac{\hat{\mathbf{f}}\mathbf{e}^{\mathbf{ik_0 x_s}}}{\mathbf{ik_0 c_0^2}} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0}].$$

# References


[1] Biot, M. A., "Theory of propagation of elastic waves in a fluid-saturated porous solid. I. Low-frequency range," J. Acoust Soc. Am. 28, 168–178 (1956).

[2] Biot, M. A., "Theory of propagation of elastic waves in a fluid-saturated porous solid. I. Higher-frequency range," J. Acoust Soc. Am. 28, 179–191 (1956).

[3] Biot, M. A., "General theory of acoustic propagation in porous dissipative media," J. Acoust Soc. Am. 34, 1254–1264 (1962).

[4] J. L. Buchanan and R. P. Gilbert: *Measuring Osteoporosis Using Ultrasound*, in ADVANCES IN SCATTERING AND BIOMEDICAL ENGINEERING eds. D. I. Fotiadis, C. V. Massalas, World Scientific, (2004) 484-494.

[5] Buchanan, J. L., Gilbert, R. P. and Khashanah, K.: *Determination of the parameters of cancellous bone using low frequency acoustic measurements*, Computational Acoustics (2004), 99-126.

[6] J. Buchanan and R.P. Gilbert: *Determination of the parameters of cancellous bone using high frequency acoustic measurements* Mathematical and Computer Model. (2007) **45** 281-308.

[7] J. Buchanan and R.P. Gilbert: *Determination of the parameters of cancellous bone using high frequency acoustic measurements II: Inverse problems*, J. Computational Acoustics 15 (2) (2007) pp.199-220

[8] J. L. Buchanan, R. P. Gilbert, A. Wirgin and Y. Xu: *Transient reflection and transmission of ultrasonic waves in cancellous bones*, Appl. Math. Comp. 142, 561–573, (2003).

[9] Chaffai,S., Padilla, F., Berger, G. and Languier, P.: *In vitro measurement of the frequency dependent attenuation in cancellous bone between 0.2 and 2 MHz*, J. Acoust, Soc. Amer.,**108**, 1281-1289, (2000).

[10] Z.E.A. Fellah, J.Y. Chapelon, S. Berger, W. Lauriks and C. Depollier: *Ultrasonic wave propagation in human cancellous bone: Application of Biot theory*, J. Acoust. Soc. Am., **116**, 61-73, (2004).

[11] R. P. Gilbert, Y. Xu and Shangyou Zhang: "Computing Porosity of Cancellous Bone Using Ultrasonic Waves"

[12] Hosokawa, A. and T. Otani, "Ultrasonic wave propagation in bovine cancellous bone," Acoust Soc. Am. 101, 558–562 (1997).

[13] D.L. Johnson, J. Koplik and R. Dashen: *Theory of dynamic permeability and tortuousity in fluid-saturated porous media*, J. Fluid. Mech. **176** ,379-402, (1987).

[14] C. M. Langton, T. J. Haire, P. S. Ganney, C. A. Dobson, M. J. Fagan, G. Sisias, and R. Phillips: *Stochastically simulated assessment of anabolic treatment following varying degrees of cancellous bone resorption*, Bone, **27**), 111-118 (2000.

[15] McKelvie, T.J. and S.B. Palmer, "The interaction of ultrasound with cancellous bone," Phys. Med. Biol. 36, 1331–1340 (1991).

[16] Williams, J. L. "Ultrasonic wave propagation in cancellous and cortical bone: prediction of some experimental results by Biot's theory," Acoust Soc. Am. 91, 1106–1112 (1992).

[17] Y. Sun, L. Kong, B. Liu, L. Song, S. Yang, T. Wei, "Comparative study of single-thread, double-thread, and triple-thread dental implant: a three-dimensional finite element analysis," ISSN 1 746-7233, England, UK, "World Journal of Modelling and Simulation", Vol. 3 (2007) No.4, pp.310-314.