Similar Matrices.

Definitions:

1. Two $n \times n$ matrices A, B over a field $\mathbb{F}$ are *similar* if there exists a non-singular matrix $S \in \mathbb{F}$ such that $A = SBS^{-1}$. In that case we write $A \sim B$.

2. A monic polynomial $f(x) = \sum_{i=0}^{d} f_i x^i$ is a *generator* of a matrix sequence $A_0, A_1, \ldots, A_i, \ldots$ if $0 = \sum_{i=0}^{d} f_i A_{i+k}$, for all integer $k \geq 0$. Note that the sequence is completely determined by $f_0, f_1, \ldots, f_d = 1$ and $A_0, A_1, \ldots, A_{d-1}$, since $A_{d+k} = -\sum_{i=0}^{d-1} f_i A_{i+k}$.

3. The *minimal polynomial* — *minpoly* for short — of square matrix A is the monic polynomial $m_A(x) = \sum_{i=0}^{d} m_i x^i$ of minimal degree d such that $0 = m_A(A) = \sum_{i=0}^{d} m_i A^i$. Note that the minimal polynomial of A is the minimal degree generator of the matrix power sequence $A^i, i = 0, 1, 2, \ldots$.

4. The *characteristic polynomial* — *charpoly* for short — of a square matrix A is the determinant of $xI - A$.

5. Let $f(x) \in F[x]$ be a monic polynomial of degree d. The *companion matrix*, C_f, of f is the matrix in $\mathbb{F}^{d \times d}$ with 1's on the first subdiagonal and $-f_i$ in the last column of row i (zero based indexing). For example, for $f(x) = x^3 + 2x^2 - 3x + 1$,

$$C_f = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 3 \\ 0 & 1 & -2 \end{pmatrix}.$$

6. A field is *perfect* if every irreducible polynomial over the field has distinct roots. All fields discussed in this course are perfect: subfields of the complex numbers and finite fields.

Facts (theorems we take for granted, or have easy proof):

1. For a polynomial $f(x)$, if A satisfies $f(x)$, i.e. $f(A) = 0$, then minpoly(A) divides f.

2. A matrix satisfies its charpoly, which is to say, minpoly(A) divides charpoly(A).

3. Let $m_A(x)$ denote the minpoly of $A \in \mathbb{F}^{n \times n}$ and of the matrix sequence $I, A, A^2, \ldots$. Let $m_{A,v}(x)$ and $m_{u,A,v}(x)$, for u, v column vectors in $\mathbb{F}^n$, denote the minpolys of the matrix sequences $v, Av, A^2v, \ldots$ and $u^T v, u^T Av, u^T A^2 v, \ldots$ respectively. The shapes of these sequences are $n \times 1$ and 1×1. Finally let $c_A(x)$ denote the charpoly of A. We have:

$$m_{u,A,v} \mid m_{A,v} \mid m_A \mid c_A.$$

4. For any monic polynomial $f(x)$ and its companion matrix C_f, we have $f = \text{minpoly}(C_f) = \text{charpoly}(C_f)$.

5. λ (possibly in an extension field of the coefficient field) is a root of minpoly(A) if and only if λ is an eigenvalue of A.

6. Similar matrices have the same eigenvalues, the same minpoly, the same charpoly.

7. Call $\hat{f} = f_1(x), f_2(x), f_3(x), \ldots$ a *factor list* if $f_i | f_{i-1}$ for $i > 0$ and each f_i is monic. If some $f_k = 1$ (and thus all succeeding terms are 1 as well), the factor list is said to be *effectively finite*. We are interested only in effectively finite factor lists and are using formally infinite lists only for notational simplicity.

8. Each square matrix A has a unique associated factor list $\hat{f}(A) = f_1(x), f_2(x), \ldots$. The nontrivial (not 1) f_i are the *invariant factors* of A. The minimal polynomial of A is f_1 and the characteristic polynomial of A is $\prod_i f_i$.

9. Two matrices are similar if and only if they have the same invariant factor list.

10. Each similarity class contains a matrix in *Frobenius Normal Form* (also called Rational Canonical Form), which is a matrix of diagonal blocks, each being the companion matrix of an invariant factor: $\bigoplus_i C_{f_i}$.

11. If $f(x) = g(x)h(x)$ with $\gcd(g, h) = 1$, then $C_f \sim C_g \bigoplus C_h$.

12. $C_{f^2} \not\sim C_f \bigoplus C_f$.

13. Each similarity class contains a matrix in *Generalized Jordan Normal Form* (also called Primary Canonical Form), which is a matrix of diagonal blocks, each being the companion matrix of a power of an irreducible polynomial: $\bigoplus_{i,j} C_{g_i^{e_{i,j}}}$. Specifically, let $f_1, f_2, \ldots, f_k$ be the irreducible factors of a given matrix A. The $g_i, g_2, \ldots, g_l$ are the irreducible factors of f_1, the minimal polynomial of A. The exponent $e_{i,j}$ is the exponent of g_i as a factor of f_j. Note that for each i the exponent list $e_{i,1}, e_{i,2}, \ldots, e_{i,k}$ is nonincreasing and may end in zeroes.

	f_1	f_2	f_3	f_4
g_1	$e_{1,1}$	$e_{1,2}$	$e_{1,3}$	$e_{1,4}$
g_2	$e_{2,1}$	$e_{2,2}$	$e_{2,3}$	$e_{2,4}$
g_3	$e_{3,1}$	$e_{3,2}$	$e_{3,3}$	$e_{3,4}$
g_4	$e_{4,1}$	$e_{4,2}$	$e_{4,3}$	$e_{4,4}$

For example, if the invariant factors are

$$
\begin{aligned}
f_1 &= (x-1)^3(x+1)^2(x-2)^4(x^2+x+1) \\
f_2 &= (x-1)^2(x+1)^2(x-2)^1(x^2+x+1) \\
f_3 &= (x-1)(x-2) \\
f_4 &= 1,
\end{aligned}
$$

Then the exponent table is

	$f_1 = m_A$	f_2	f_3	$c_A = \prod_i f_i$
$x-1$	3	2	1	6
$x+1$	2	2	0	4
$x-2$	4	1	1	6
x^2+x+1	1	1	0	2

Note that the order of the rows is arbitrary, but the order of the columns is not.

The matrix A is similar to

$$
\left[\begin{array}{ccc|cc|c||cc|cc||cccc|c|c||cc|cc}
1 &&&&&&&&&&&&&&&&&&& \\
1 & 1 &&&&&&&&&&&&&&&&&& \\
& 1 & 1 &&&&&&&&&&&&&&&&& \\
\hline
&&& 1 &&&&&&&&&&&&&&&& \\
&&& 1 & 1 &&&&&&&&&&&&&&& \\
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&&&&& 1 &&&&&&&&&&&&&& \\
\hline\hline
&&&&&& -1 &&&&&&&&&&&&& \\
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&&&&&&&&&&&&&& 2 &&&&& \\
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&&&&&&&&&&&&&&& 2 &&&& \\
\hline\hline
&&&&&&&&&&&&&&&&& -1 && \\
&&&&&&&&&&&&&&&& 1 & -1 && \\
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&&&&&&&&&&&&&&&&&&& -1 \\
&&&&&&&&&&&&&&&&&& 1 & -1
\end{array}\right]
$$

and to

$$
\left|\begin{array}{ccccccccccc|ccccccc|cc}
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0 & & & & & & & & & & \ast & & & & & & & & & \\
1 & 0 & & & & & & & & & \ast & & & & & & & & & \\
 & 1 & 0 & & & & & & & & \ast & & & & & & & & & \\
 & & 1 & 0 & & & & & & & \ast & & & & & & & & & \\
 & & & 1 & 0 & & & & & & \ast & & & & & & & & & \\
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