Names for certain domains.
$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ are the *natural numbers*

$\mathbb{Z} = \mathbb{N} \cup -\mathbb{N}$ are the *integers*

$\mathbb{Q} = \{a/b : a \in \mathbb{Z}, 0 < b \in \mathbb{Z}, \text{ and } \gcd a, b = 1\}$ are the *rational numbers*

$\mathbb{R}$ are the *real numbers.*

$\mathbb{C}$ are the *complex numbers.*

For any ring $\mathrm{R}, \mathrm{R}[x]$ is the ring of polynomials with coefficients in R.

Let S be a set and consider the following operators on S,

$+ : S \times S \to S$ (binary sum)

$0 :\to S$ (nullary zero element)

$- : S \to S$ (unary negation)

$\times : S \times S \to S$ (binary product). Often we write ab for $a \times b$)

$1 :\to S$ (nullary identity element)

$^{-1} : S^* \to S^*$, where $S^* = S - \{0\}$. (unary inverse)

The following properties are often encountered. These assertions are for all $a, b, c \in S$,

$P1 : (a + b) + c = a + (b + c),$ additive associativity
$P2 : a + 0 = a = 0 + a,$ (2-sided) additive identity element
$P3 : a + (-a) = 0 = (-a) + a,$ (2-sided) additive inverses
$P4 : a + b = b + a,$ additive commutativity
$P5 : (a \times b) \times c = a \times (b \times c),$ multiplicative associativity
$P6 : a \times 0 = 0 = 0 \times a,$ (2-sided) absorbing element ("zero" element)
$P7 : a \times 1 = a = 1 \times a,$ (2-sided) multiplicative identity element
$P8 : a \times b = b \times a,$ multiplicative commutativity
$P9 : a \times b = 0$ if and only if $a = 0$ or $b = 0$ (no zero divisors)
$P10 :$ Ascending chain condition, Noetherian domain
$P11 :$ Unique Factorization property
$P12 :$ Every ideal is principal
$P13 :$ Let S^* denote the set of nonzero elements in S. $\exists d : S^* \to \mathbb{R}^+$ such that
$P13a : d(a) >= 0 \ \forall a \in S^*$.
$P13b : d(ab) >= d(a), \ \forall a, b \in S^*$.
$P13c : \forall a, b > 0, \ \exists q, r \in S$ such that $a = qb + r$ and $(r = 0$ or $d(r) < d(b))$.
$P14 :$ If $a \neq 0, \exists b \in S$ such that $a \times b = 1 = b \times a,$ (2-sided) inverses.
The inverse is normally denoted as a^{-1}

$D = (S, +)$, such that P1 is a *semi-group*.
$D = (S, +, 0)$, such that P1,P2 is a *monoid*.
$D = (S, +, 0, -)$, such that P1,P2,P3 is a *group*.
$D = (S, +, 0, -)$, such that P1,P2,P3,P4 is an *abelian* (or commutative) group.
$D = (S, +, 0, -, \times)$, such that P1,P2,P3,P4,P5,P6 is a *ring*.
(Remark: A ring is a group additively and a monoid multiplicatively.)
$D = (S, +, 0, -, \times, 1)$, such that P1,P2,P3,P4,P5,P6,P7,P8 is a *commutative ring with 1*, or CR1, for short.
$D = (S, +, 0, -, \times, 1)$, such that P1,P2,P3,P4,P5,P6,P7,P8,P9 is an *integral domain*, ID.
$D = (S, +, 0, -, \times, 1)$, such that P1,P2,P3,P4,P5,P6,P7,P8,P9,P10,P11 is a *unique factorization domain*, UFD.
$D = (S, +, 0, -, \times, 1)$, such that P1,P2,P3,P4,P5,P6,P7,P8,P12 is a *principal ideal ring*, PIR.
$D = (S, +, 0, -, \times, 1)$, such that P1,P2,P3,P4,P5,P6,P7,P8,P9,P12 is a *principal ideal domain*, PID.
$D = (S, +, 0, -, \times, 1)$, such that P1,P2,P3,P4,P5,P6,P7,P8,P9,P13 is an *Euclidean domain*, ED.
$D = (S, +, 0, -, \times, 1, \ ^{-1})$, such that P1,P2,P3,P4,P5,P6,P7,P8,P9,P14 is a *field*.

Theorem: $field \subset ED \subset PID \subset UFD \subset ID \subset CR1 \subset ring$,
and $PIR \subset CR1$.

Further definitions:

An element a in a CR1 D is a *unit* if $\exists b \in D$ such that $ab = 1$. Such a b is called the *inverse* of a and is normally written a^{-1}. Lemma: If such b exists it is unique.

Elements a and b in an ID, D, are *associates* if $\exists$ unit $u \in D$ such that $au = b$.

With respect to elements $a, b \in D$, a CR1:
An a is said to *divide* b if $\exists c \in D : a \times c = b$, and we write $a|b$ in this case.
An element a is said to be a *zero divisor* if if $\exists c \neq 0 : a \times c = 0$.
Be careful: $a|0$ is true of all a (let $c = 0$), but more often than not, a is not a zero divisor!

Remarks and examples
Fields are our bread and butter. Examples are $\mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{Z}_p$, for prime p.

ED's are important because the quotient/remainder is a basis for an extended greatest common divisor (EGCD) algorithm. Examples are $\mathbb{Z}, F[x]$, for field F.

PID's are PIR's with no zero divisors. No important examples that aren't EDs.

PIR's are important because this is the most general class of rings in which EGCD is defined: For every a, b there exists d, s, t such that $d = \gcd(a, b) = sa + tb$. Example is $\mathbb{Z}_n$, for composite n.

UFD's are important because factorization is important. Example: $F(x, y)$, multivariate polynomials over a field. Note, $\gcd(a, b)$ exists (is well defined) in a UFD, but in general, extended gcd is not.

ID's are important because lack of zero divisors implies a cancellation law: $ab = ac$ and $a \neq 0 \Rightarrow b = c$.

CR1's are important because some basic definition (eg. matrix determinant) and algorithms (eg. most matrix multiplication schemes) are valid at this generality.