Solution to Problem 2

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Problem Statement 2.a

In any group of $n \leq 2$ people, there are two people with the same number of friends in the group.Assume that friendship is symmetric but not reflexive.

Theorem 2.a

In any undirected graph with vertices $n \geq 2$, there should be atleast two vertices which have the same degree.

Proof 2.a

Since we assume the friendship to be symmetric and not reflexive,therefore no self-loops are allowed and there can be an egde only between two distinct vertices. The prrof is given by contradiction. Let us assume that in an undirected graph with $n \geq 2$ vertices no two vertices have the same degree. So the degrees of all the n vertices should be $0, 1, 2, ..., n-2, n-1$. But a vertex with degree $n-1$ should have an edge to every other vertex in the undirected graph and so there should be no vertex with degree 0. This contradicts our assumption.

Hence there should be atleast two vertices which have the same degree.

Problem statement 2.b

Every group of six people contains either three mutual friends or three mutual strangers.

Theorem 2.b

In any undirected graph with six vertices, there exist atleast three mutually connected vertices or three mutually unconnected vertices.

Proof 2.b

Picking out any arbitrary vertex and using the pigeon-hole principle,there are atleast three vertices that are connected or unconnected to this vertex.

Consider the first case.

Let B, C, D be the vertices that are connected to vertex A. If there is an edge between any two of B, C, D,then A together with the other two vertices that are connected form a mutually connected trio.If there is no such edge between B, C, D, then B, C, D form a mutually unconnected trio.

Consider the second case.

Let B, C, D be the vertices that are unconnected to vertex A. If there are edges between B, C and C, D and D, B then B, C, D form the mutually connected trio.If any two vertices in B, C, D are unconnected, then A together with this unconnected pair form the mutually unconnected trio.

Grading Policy

Points

$0 - 1$ Each theoram

$0 - 4$ Each proof