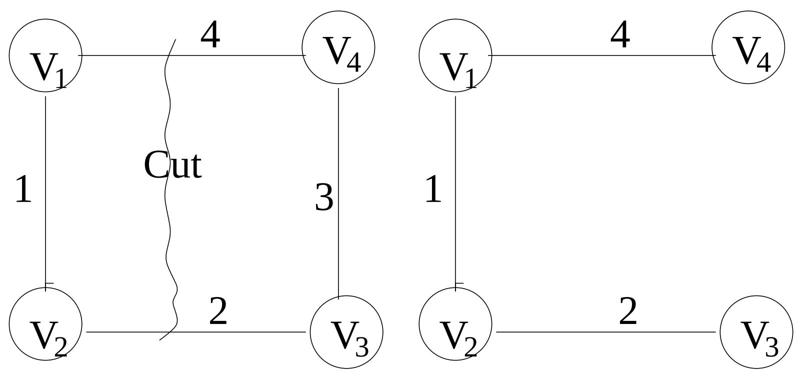
CISC621 Homework Problem On Minimum Spanning Trees

Model Answer

1 Exercise 23.1-2

Fig.1 (a) shows a graph $G = (V, E)$ with cut $(S, V - S)$, where $S = \{V_1, V_2\}$. If we choose $A = \{V_1V_2\}$. Then the cut is respect to A. It is easy to see that the edge V_2V_3 is light, whereas V_1V_4 is not light. However V_1V_4 is safe since $A \cup \{V_1V_4\}$ consists a subset of MST of G (see Fig.1 (b)).



(a) Graph G with a cut (b) MST

Figure 1. Counterexample to the converse of Theorem 23.1.

2 Exercise 23.1-6

2.1 Unique light edge for every cut $\Longrightarrow$ Unique MST

We prove by induction.

When $|V| = 2$. It is straightforward true. Assume when $|V| \leq k$, the statement is true. Now we consider any undirected graph G with $|V| = k + 1$. Arbitralily choose one vertex say V_0. Let $S = \{V_0\}$. Then $(S, V - S)$ is cut of G (see Fig. 2). Denote the subgraph associated with $V - S$ as G'. Since G' has $\leq k$ vertices, it has unique MST, denoted as T'. Then we claim that $T' \cup \{$the light edge of cut$(V, V - S)\}$ is the unique MST of G. Since the light edge is unique for cut $(S, V - S)$, any other edges between V_0 and some vertex in $V - S$ will lead to larger sum of weights of the tree.

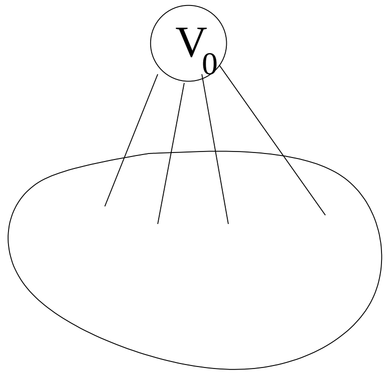
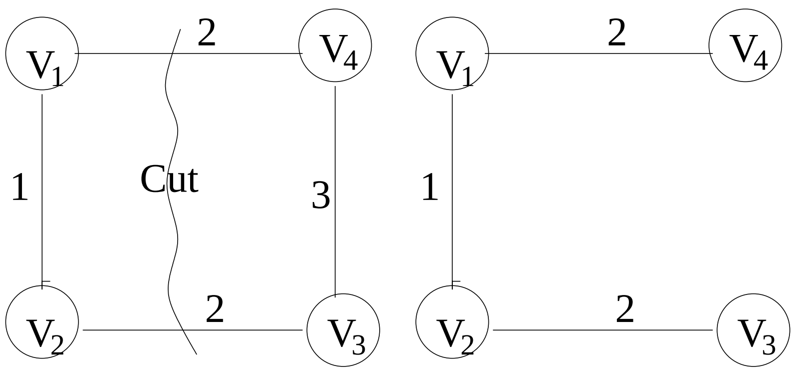


Figure 2. Induction



(a) Graph G (b) MST

Figure 3. Counterexample for Excercise 23.1-6

2.2 Counterexample for converse

Fig. 3 (a) shows a counterexample of the converse of above statement. It is easy to see the cut $(\{V_1, V_2\}, \{V_3, V_4\})$ has no unique light edges. But the graph has unique MST (which is shown in Fig. 3 (b)).