Cut Property

Let an undirected graph $G = (V, E)$ with edge weights be given. A tree in $G$ is a subgraph $T = (V', E')$ which is connected and contains no cycles.

A spanning tree is one reaching all the vertices: $V' = V$.

In the rest of this discussion we will equate tree $T$ with its set of edges $E'$. Note that $E'$ determines $T$ since it is connected, i.e. $V' = \{u \in V : (u, v) \in E' \text{ for some } v \in V\}$.

The weight of a tree (or of any set of edges) is the sum of its edge weights.

A minimal spanning tree (MST) is a spanning tree whose weight is not greater than the weight of any other spanning tree of $G$.

The cut defined by a set of vertices $S$ is the set of all edges that cross from $S$ to $V - S$: $\text{cut}(S) = \{(u, v) \in E : u \in S, v \in V - S\}$.

A light (or lightest) edge in a set of edges is one whose weight is no greater than that of any other edge of the set.

If $X$ is a set of edges, a set of vertices $S$ is said to respect $X$ if $\text{cut}(S) \cap X = \emptyset$. In other words, no edge of $X$ crosses from $S$ to $V - S$.

Cut Property. Let $X$ be a set edges that is a subset of some MST $T$. Let $S$ be a set of vertices whose cut respects $X$ and let $(u, v)$ be a light edge of cut$(S)$. Then there is a MST containing $X \cup \{(u, v)\}$.

In other words, a light edge of cut$(S)$ can be added to $X$ and it will still be a subset of some MST.

Proof. If $T$ contains $(u, v)$ we are done. If not, adjoin $(u, v)$ to $T$ forming a cycle within $T \cup \{(u, v)\}$. This cycle must contain at least one other edge $(w, z)$ of cut$(S)$. Then $T' = T \cup \{u, v\} \cap \{(w, z)\}$ is a spanning tree of weight no greater than that of $T$, so $T'$ is a MST. qed.