Design of binary subwavelength diffractive lenses by use of zeroth-order effective-medium theory

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A procedure for designing binary diffractive lenses by use of pulse-width-modulated subwavelength features is discussed. The procedure is based on the combination of two approximate theories, effective-medium theory and scalar diffraction theory, and accounts for limitations on feature size and etch depth imposed by fabrication. We use a closed-form expression based on zeroth-order effective-medium theory to map the desired superwavelength phase to the width of a binary subwavelength feature and to examine the requirements imposed by this technique on fabrication and on analysis. Comparisons are also made to more rigorous approaches. In making these comparisons, we show that a trade-off exists between the exactness of the mapping and the fabrication constraints on the minimum feature. © 1999 Optical Society of America [S0740-3232(99)02805-7]


1. INTRODUCTION

The literature on subwavelength diffractive optics reflects a development similar to that which was true of superwavelength diffractive optics. Analysis tools are developed first; their development is then followed by fabrication technology and, finally, by the development of design tools and methodologies. For subwavelength diffractive optics, rigorous coupled-wave analysis has been used to solve Maxwell’s equations for gratings, and boundary integral techniques and finite-difference time-domain methods have been used to analyze more-general, aperiodic, finite extent elements. Furthermore, the fabrication of subwavelength diffractive optical elements (SWDOE’s) has been spurred by the improved use of electron-beam technology. Design methodologies have been developed for SWDOE’s used as wave plates and antireflection surfaces, which are based on the birefringent nature of subwavelength gratings. However, the design of SWDOE’s for more-general beam-shaping functions, e.g., beam deflection, beam splitting, and focusing, is complicated by the lack of a simple link between subwavelength structure and superwavelength behavior.

Of the papers that have appeared in the literature on subwavelength design, many combine scalar diffraction theory and effective-medium theory (EMT) to provide this link. This combination allows one to encode the desired superwavelength phase transform to use area modulation of binary subwavelength features. The resulting structure is then improved by some means of optimization. In all cases, design of the initial structure includes sampling the superwavelength phase on a lattice whose spacing is of the order of a wavelength, relating the sampled phase to an effective refractive index by means of scalar diffraction theory, and mapping the effective index to a pulse width by means of EMT.

A significant difference among the references cited above is the mapping used to determine pulse width from effective index. Without reference to EMT, Farn suggests a design for a grating deflector that assumes that the mapping is linear. In our research and in that of others, zeroth-order EMT is used to yield a quadratic relationship. In both cases the relationship between index and pulse width is such that the mapping is described by a closed-form expression. In Refs. 9 and 14–18, the mapping between index and fill factor is generated by rigorous determination of the effective index of a binary grating as a function of fill factor. These data are then used to create a look-up table that relates fill factor to phase.

The pulse-width modulation technique for SWDOE design has been applied primarily to grating deflector design. In fact, diffraction efficiencies of the order of 80% have been predicted and experimentally demonstrated for such elements. More recently this technique has been applied to lens design, which prompted us to examine more carefully the application of zeroth-order EMT to lens design.
One obvious advantage to using a closed-form expression over more rigorous techniques is its ability to map directly the desired phase to fill factor. We show here that other advantages to using zeroth-order EMT exist when it is applied to lens design and when fabrication constraints are taken into consideration. These results complement the analyses of grating deflectors that show that zeroth-order EMT produces elements whose diffraction efficiency is less than that for elements designed with a rigorous mapping. Furthermore, if an optimization routine is used to refine the pulse-width-modulated structure, the additional effort to determine the rigorous mapping may not be required. However, we do not address this issue here.

Section 2 is a review of our design procedure and a comparison with other mapping techniques. In Section 3 we discuss the effects that minimum feature constraints have on design and, in Section 4, the effects that they have on computation. In Section 5 the performance of lenses designed by use of linear, zeroth-order EMT (quadratic), and rigorous mappings is compared. Section 6 contains some discussion and our concluding remarks. Although tools for analyzing two-dimensional structures are being developed, we restrict ourselves in this paper to one-dimensional lenses because their analysis is less time consuming than that of two-dimensional lenses.

2. BINARY SUBWAVELENGTH DESIGN

In our approach to binary SWDOE’s we link the superwave-length phase \( \theta(x) \) to the binary subwavelength structure \( t(x) \) by means of an intermediate function, the index synthesis function \( g(x) \), which we define below. The binary subwavelength profile \( t(x) \) is given by

\[
t(x) = d f(x) = d \sum_{k=1}^{K} \text{rect} \left[ \frac{x - (z_{k+1} + z_k)/2}{z_{k+1} - z_k} \right],
\]

where \( d \) is the element etch depth; the \( z_k \) values represent nonoverlapping transition points, i.e., for \( k < l \), \( z_k < z_l \); and \( K \) is the total number of transition points, or edges, in the structure. We use the dimensionless (0, 1)-binary amplitude function \( f(x) \) as shorthand to describe the binary nature of \( t(x) \).

The mapping that transforms \( \theta(x) \) into \( g(x) \) accounts for depth and local effects of the binary subwavelength structure. Extension of the zeroth-order approximation of EMT\(^{22,23} \) to arbitrary SWDOE’s yields the relative indices of refraction for \( t(x) \):

\[
\begin{align*}
n_{\text{TE}}^2(x) &= (n_r^2 - 1)g(x) + 1, \quad \text{(2a)} \\
n_{\text{TM}}^2(x) &= (n_r^2 - 1)g(x) + 1, \quad \text{(2b)}
\end{align*}
\]

where \( g(x) \) is the index synthesis function:

\[
g(x) = f(x) + \frac{1}{2} \text{rect} \left( \frac{x}{\Delta} \right). \quad \text{(3)}
\]

The convolution, represented by \(* \), is a heuristic way to account for the averaging that is consistent with the zeroth-order assumption of EMT. We address the value of the subwavelength distance \( \Delta \) below.

From scalar theory, one can express the index of refraction in terms of phase as

\[
n_r(x)(n_r - 1) \frac{\theta(x)}{\theta_0} + 1, \quad \text{(4)}
\]

where the maximum phase \( \theta_0 \) is related to the etch depth:

\[
\theta_0 = \frac{2\pi d n_0}{\lambda}(n_r - 1). \quad \text{(5)}
\]

The constant \( n_r = n_s/n_0 \) is the relative refractive index of the diffractive optical element substrate with refractive index \( n_s \); \( n_0 \) is the index of the surrounding medium. If \( d < \lambda/(n_s - n_0) \), the phase \( \theta(x) \) must be quantized to a range \( \theta_0 \) of phase values, as indicated in Fig. 1. The value of the center phase \( \theta_c \) affects the phase quantization error.

Inversion of Eqs. (2) and combination with Eq. (4) yields

\[
\begin{align*}
g_{\text{TE}}(x) &= \frac{n_{\text{TE}}^2(x) - 1}{n_r^2 - 1}, \\
&= \frac{\{(n_r - 1)[\theta(x)/\theta_0] + 1\}^2 - 1}{n_r^2 - 1}, \quad \text{(6a)} \\
g_{\text{TM}}(x) &= \frac{n_{\text{TM}}^2(x) - 1}{n_r^2 - 1}, \\
&= \frac{\{(n_r - 1)[\theta(x)/\theta_0] + 1\}^2 - 1}{n_r^2 - 1}. \quad \text{(6b)}
\end{align*}
\]

By comparison, Farn’s linear approximation assumes that

\[
g_{\text{TE}}(x) = g_{\text{TM}}(x) = \frac{n_r^2 - 1}{n_r - 1} = \frac{\theta(x)}{\theta_0}. \quad \text{(7)}
\]

Figure 2 is a comparison among Farn’s linear mapping, the quadratic mapping associated with zeroth-order EMT, and a rigorous mapping for a fused-silica substrate (SiO\(_2\), \( n_s = 1.5 \)) with an etch depth of 2 \( \mu \text{m} \), which corresponds to 2\( \pi \) phase. The rigorous mapping was generated for us by Philippe Lalanne, who used the modal technique of Li.\(^{24} \)

Note that independent control of the two polarization states requires two different index synthesis functions, one for each polarization. Independent control of both TE and TM polarizations is not possible by use of only a single binary profile; i.e., one must design the profile for a single polarization state and must be willing to accept whatever response is generated by the orthogonal state. To generate independent polarization functions, the index synthesis functions of Eqs. (6) must be combined in an orthogonal fashion, e.g., by use of a phase shift.\(^{25} \) Unfortunately, the technique proposed in Ref. 25 requires the fab-
rification of a multilevel subwavelength element. In the remainder of this paper we consider only TE designs.

To determine \( t(x) \) from \( g(x) \), one must invert Eq. (3). However, if the intent is to determine as good an initial guess as possible without expending considerable computing resources to do so, it suffices to use simple area-modulation encoding techniques to generate \( f(x) \) from \( g(x) \). In this study we use pulse-width modulation:

\[
    f(x) = \sum_{m=-\infty}^{\infty} \text{rect} \left( \frac{x - m\Delta - g_m\Delta/2}{g_m\Delta} \right),
\]

where \( g_m = g(m\Delta) \). Given this encoding, the index synthesis function is essentially the local fill factor for an area-modulated binary subwavelength structure.

To ensure high-diffraction-efficiency designs, the value for \( \Delta \) should be of the order of the wavelength in the substrate. This behavior has been observed in the design of deflector gratings \(^8\) as well as antireflection gratings.\(^7\)\(^,\)\(^26\)

To determine whether similar behavior was observed in lens design, we calculated the diffraction efficiency of a binary SWDOE designed for operation at a 1-\(\mu\)m wavelength, with a 20-\(\mu\)m focal length, a 12.8-\(\mu\)m diameter, and a 2-\(\mu\)m etch depth in SiO\(_2\). The results, presented in Fig. 3, are consistent with those presented by Farn.\(^8\) To maintain consistency with Farn’s work, in this study we chose as an upper limit on \( \Delta \) half the incident wavelength in the substrate:

\[
    \Delta \leq s_\lambda = \frac{\lambda}{2n_s},
\]

where \( \lambda \) is the wavelength in a vacuum. We refer to \( s_\lambda \) as the subwavelength parameter.

When used as a limit on feature size, the subwavelength parameter imposes a fundamental sampling lattice on the phase and the index synthesis functions. For example, if we consider an application in SiO\(_2\) (\(n_s = 1.5\)) at \( \lambda = 1 \mu\m\), then \( s_\lambda = 1/3 \mu\m\). In gallium arsenide (GaAs), \( \lambda = 1 \mu\m\) and \( n_s = 3 \), \( s_\lambda = 1/6 \mu\m\), i.e., the sampling lattice for GaAs is finer than the one for SiO\(_2\). At longer wavelengths, however (e.g., \( \lambda = 10 \mu\m\) and \( n_s = 3 \)), the sampling lattice is coarser: \( s_\lambda = 5/3 \mu\m\).

As an example of our procedure we present the design of a 20-\(\mu\)m focal-length SiO\(_2\) lens with a 22.72-\(\mu\)m-diameter (\(f/0.88\)) that operates at a wavelength \( \lambda = 1 \mu\m\). The lens functions in free space. We assume that the maximum etch depth is 1.0 \(\mu\m\) and that the minimum feature is 0.1 \(\mu\m\), which corresponds to a 10:1 aspect ratio (depth to width).

Figure 4(a) represents the substrate profile of a continuous-phase diffractive lens that satisfies all the design criteria except the fabrication limitations. (In addition to being continuous phase, the lens has a maximum etch depth of 2.0 \(\mu\m\).) The response of the lens, determined by the boundary element method with TE polarization, is represented in Fig. 5. The diffraction-limited spot size is approximately 2 \(\mu\m\), and its diffraction efficiency (the ratio of energy within this window to the total energy incident upon the SWDOE) is 77%.

Because the 1.0-\(\mu\m\) etch depth yields \( \theta_0 = \pi \), quantization of the superwavelength phase is required.
The expressions for space–bandwidth product and the minimum feature along a single axis for two-dimensional encoding are also indicated in Table 1. Derivation of the expressions for quadratic modulation is presented in Appendix A; similar procedures were used to derive the expressions for the linear mapping. The expression for deflector space–bandwidth product represents the value for only a single period. The element phase and the corresponding index synthesis function for these elements are represented in Figs. 7(a)–9(a) and Figs. 7(b)–9(b), respectively.

Our one-dimensional analysis is appropriate for the design of rotationally symmetric elements, where, to ensure rotational symmetry, coding can occur only along the radial axis. However, elements that function along only one axis will be produced in two dimensions as cylindrical elements. In this case, because binary encoding is in terms of area, use of the second dimension relaxes the requirements on the minimum feature size, at least when the latter is considered along one axis, i.e., $N_{2d} = \sqrt{N_{1d}}$. See Figs. 6–9, which are two-dimensional cylindrical representations of the binary, area-modulated elements. Expressions for the space–bandwidth product and the minimum feature along a single axis for two-dimensional encoding are also indicated in Table 1.

The expressions for space–bandwidth product indicate that, for deflectors and linear phase lenses, the feature sizes required for a linear mapping are larger (albeit not substantially so) than those required for a quadratic mapping. For quadratic phase lenses, the linear mapping ac-

\[
N_{\min} = \text{ceil} \left( \frac{1}{dg_{\min}} \right),
\]

where
\[
dg_{\min} = \min_{m} \left| g_{m} - g_{m-1} \right|, \quad m = [1, M - 1].
\]

The ceiling function produces the next-largest integer for a given input.
Fortunately, two-dimensional analysis of cylindrical elements is more difficult than the one-dimensional analysis of rotationally symmetric ones.)

The reduced tolerances on linear phase lenses as compared with quadratic phase lenses are also evident. However, the range of values for minimum feature is still small, and fabrication technology limits the space–bandwidth product that can actually be realized. Current technology is capable of realizing minimum features of the order of 0.2–0.3 μm over an area approximately 0.5 × 0.5 mm², which corresponds to $S \approx 2500 \times 2500$.

Fabrication of an element whose required space–bandwidth product is larger than this requires spatial quantization of the binary pattern $f(x)$, as represented in Fig. 12.27.

Returning now to the design example of Section 2, we find that spatial quantization of the SWDOE represented

<table>
<thead>
<tr>
<th>Table 1. Expressions for Space–Bandwidth Product and Minimum Feature</th>
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<tbody>
<tr>
<td>Element</td>
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<td>---------</td>
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<td></td>
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<tr>
<td>Linear mapping</td>
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<tr>
<td>Deflector</td>
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<tr>
<td>Linear phase lens</td>
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</tbody>
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Note that the required minimum feature size is related to the smallest phase change, the accurate representation of slowly varying phase functions is more difficult than it is for rapidly varying ones. This does not mean that such phase functions cannot be fabricated; rather, they are more sensitive to fabrication constraints on minimum feature size. It is true for any design that, if the required minimum feature size $\delta$ is less than the minimum feature size that can be fabricated $\delta_{\text{min}}$, spatial quantization of the profile is required. Thus binary profiles corresponding to slowly varying phase functions (e.g., deflectors with small deflection angles and lenses with large $f$-numbers) are more likely to be spatially quantized and to suffer a reduction in performance than those with rapidly varying phases. This is reflected in Figs. 10 and 11, which represent the minimum feature and the space–bandwidth product for one- and two-dimensional deflectors and lenses designed with the quadratic mapping associated with zeroth-order EMT. The lens diameter was 100 μm. The minimum lens $f$-number occurs when the width of the last zone is one wavelength. The maximum value occurs when the lens consists of only a single zone. From these curves it appears that fabrication of high-index lenses for long wavelengths appears to be within the reach of current technology.

Note that the required minimum feature becomes smaller when the refractive index increases for a given wavelength. For a fixed refractive index, the minimum feature becomes larger as the wavelength increases. Most evident in the figures is the reduction in tolerances when two- and one-dimensional encoding are compared. Thus the fabrication of cylindrical elements is easier than the fabrication of rotationally symmetric elements. (Unfortunately, two-dimensional analysis of cylindrical elements is more difficult than the one-dimensional analysis of rotationally symmetric ones.)

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in Fig. 4(d) to a 0.1-μm minimum feature yields the structure shown in Fig. 4(e). The loss in diffraction efficiency for this example is minimal—only 1.6%; however, we investigate this loss further in Section 5. The response of the lens is also represented in Fig. 5.

The sampling distance $D$ has an effect not only on design but also on the computational resources required for analysis. In Section 4 we discuss some of the practical considerations that one needs to keep in mind when performing one-dimensional boundary element method analysis.

4. COMPUTATION CONSIDERATIONS

Computational resources for the analysis of a binary SW-DOE are related to the total number of nodes $J$, or samples, used to represent the diffractive surface, as shown in Fig. 13. For one-dimensional boundary element method analysis, matrix size and number of multiplications are both proportional to $J^3$.  

To determine an order-of-magnitude estimate for $J$, we need to determine the surface length $L$. The surface is divided into $M$ regions that either contain a binary feature or are constant; thus

$$L = M_n l_{nt} + M_{rt} = M + 2M_d,$$  \hspace{1cm} (13)  

where $M_n$ and $M_t$ are the number of subperiods that have no phase transition and the number of subperiods that have a phase transition, respectively. To arrive at our final expression we used the fact that $M = M_n + M_t$, and the surface length $l_{nt}$ of a subperiod that has constant value is $l_{nt} = \Delta$. The surface length $l_t$ of a subperiod that contains a binary feature is $l_t = 2d + \Delta$. Note that the length is independent of the width $\delta$ of the binary feature. Therefore the surface length is a function of the number of subperiods $M$ and of the depth $d$ and is not a function of the space–bandwidth product $S$ of the diffractive optical element.

If the boundary is sampled at equally spaced intervals that are a fraction of the wavelength in the substrate, the total number of nodes $J$ is

$$J = \frac{L}{\frac{\lambda}{\alpha n_s}} = \frac{L}{\frac{2}{s_s}},$$  \hspace{1cm} (14)  

where $\alpha > 1$ is the sampling rate, i.e., the number of samples per wavelength. Because $L < M(2d + \Delta)$, we can place an upper bound on $J$:

$$J < \frac{\alpha M (2d + \Delta)}{2} = \frac{aD}{2s_s} \left(1 + \frac{2d}{s_s} \right) = J_{ub}. \hspace{1cm} (15)$$  

Fig. 8. Same as in Fig. 7, except for linear phase lens design.

Fig. 9. Same as in Fig. 7, except for quadratic phase lens design.

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We have used Eq. (11) and $s_\lambda$ to arrive at our final expression for $J_{ub}$. Note that, although the upper bound is a function of illumination wavelength, element width, etch depth, material properties, and sampling rate, it is independent of the specific phase function encoded. This is because the boundary length within a subperiod is independent of the feature width $\delta$.

The quantity $\sigma$,

$$\sigma = \frac{J_{ub}}{aD} = \frac{1}{2s_\lambda} \left( 1 + \frac{2d}{s_\lambda} \right),$$

is represented in Fig. 14 as a function of wavelength and index of refraction for $d = 1 \mu m$. Because $J/\alpha$ is the surface length represented as the number of substrate wavelengths, the quantity $J/\alpha D$ is the number of substrate wavelength sections per linear distance, i.e., wavelength density. Note that, for $\lambda = 1 \mu m$ and $n = 1.5$, $\sigma = 21$, whereas, for $n = 3.0$, $\sigma = 78$. Since computational resources are proportional to $J^3$, analysis of a GaAs structure requires 51 times more resources than the same structure realized in SiO$_2$. At long wavelengths the increase is smaller but still of the order of 10–20 times greater than in SiO$_2$. Unfortunately, this limits the nature of the SWDOE's that one can analyze readily. For example, analysis of a symmetric structure with 2000 nodes over half its extent requires five hours of CPU time and 309 MB of RAM with four 250-MHz processors in a Sun Enterprise 450.

5. LENS DESIGN

In this section we examine the effect on lens performance as a function of the index mapping. The improvement in the performance of deflector gratings when using a rigorous, as opposed to a linear or a quadratic, mapping has been noted previously. However, as we show here, it is important to consider not only the accuracy of the mapping but also its effect on minimum feature. The comparison in Ref. 9 did not consider the minimum feature and how minimum feature limitations affect element performance.

Our space–bandwidth product analysis indicates that the fabrication requirements are less stringent for a linear phase lens than for a quadratic phase lens. In fact, the proportional increase in minimum feature size for a linear lens as opposed to a quadratic lens is

$$\frac{\delta_{\text{lin}}}{\delta_{\text{quad}}} = 2\sqrt{\frac{2n_0 f}{\lambda}}.$$  

However, before one resigns oneself to the fabrication of linear phase lenses, it is important to characterize their performance, especially if both linear and quadratic phase lenses still require spatial quantization.

We designed several 20-\mu m focal-length lenses for use in SiO$_2$ at $\lambda = 1 \mu m$. To remove influences due to etch depth limitations, the etch depth was 2.0\mu m, or $2_{\text{phase}}$. Because the analysis of structures that have deep etches...
requires considerable computational resources, the lens diameter was limited to 12.80 μm (f/0.64); i.e., the lens consisted only of the central zone. We considered both linear and quadratic phase lenses. For each, binary profiles were generated with linear, quadratic, and rigorous mappings, which were then spatially quantized assuming a 0.1-μm minimum feature. The diffraction efficiency of each lens into a diffraction-limited 3.81-μm spot is listed in Table 2. As a point of reference, the diffraction efficiency of the continuous 2π-phase linear and quadratic lenses was also determined.

The data indicate that the quadratic lenses outperform the linear lenses and that, even when spatially quantized, the quadratic-mapped quadratic phase lens outperforms the others. Although the data are derived from only a single example, we have observed the same general trends in other designs, and, even if the data are not definitive, we consider the data representative.

With regard to minimum feature, as expected, the minimum feature for the linear phase lenses is larger than for the quadratic phase lenses. The case for fabrication of a linear phase lens can be made when one compares the performance demonstrated by the lenses prior to spatial quantization. Although the diffraction efficiency of the quadratic-mapped quadratic lens is higher than the linear-mapped linear lens, the former lens requires a minimum feature that is one tenth the size required by the latter. Were it possible to realize elements with features of the order of 20 nm, then fabrication of the spatially unquantized linear-mapped linear phase lens would be justified. However, if such features could be realized, it is likely that the performance of the quadratic-mapped quadratic lens would approach 80% diffraction efficiency with spatial quantization to 20 nm.

The behavior of the minimum feature for the different lenses and the different mappings can be explained with reference to the lens index synthesis functions represented in Fig. 15. Of the linear phase lenses, the rigorous-mapped lens has the shallowest slope of the three mappings at its edge and, therefore, requires the smallest minimum feature. Of the quadratic phase lenses, the linear-mapped lens exhibits the shallowest slope at its center. The minimum feature sizes predicted by the formulas in Table 1 agree well with the values that we obtained from our designs.

It is interesting that prior to spatial quantization the variations in lens performance are slight for the different mappings of the quadratic phase lens. However, examination of Fig. 15 reveals that the value of the index synthesis functions at the lens edge is small for all the lenses except the linear- and the quadratic-mapped quadratic lenses. Thus this region contains many small features that cannot be fabricated, which explains the 20% drop in diffraction efficiency of these lenses after spatial quantization. The linear- and the quadratic-mapped quadratic lenses are also affected by quantization, but these lenses lose only approximately 10% in diffraction efficiency. Also, the similarity between the rigorous-mapped quadratic lens and the linear-mapped linear lens explains the high diffraction efficiency of the latter lens.

Our analysis indicates that, when fabrication constraints are considered, lens performance should not be based solely on the exactness of the mapping. In fact, for the example that we considered, zeroth-order EMT was capable of balancing electromagnetic theory against fabrication reality exceptionally well. Furthermore, we were able to produce such high-quality results with a minimal amount of computational effort.

### 6. DISCUSSION AND CONCLUDING REMARKS

Although pulse-width modulation design of diffractive gratings and comparisons between mapping techniques have appeared in the literature, it is important to note
that in this paper we have concentrated on lens, not grating, design and have also considered fabrication constraints on etch depth and minimum feature. Consideration of fabrication constraints is reflected in the two quantization steps in our design procedure: phase quantization, and spatial quantization. Because of the non-linear nature of the associated phase functions, the issue of phase quantization is more critical for lenses and beam splitters than it is for grating deflectors. This point has not previously been addressed in the literature and should be addressed more fully. Furthermore, although spatial quantization is recognized as a necessary part of the fabrication process, its effect on performance has not been previously considered during design. Finally, our examination of the sensitivity of a particular mapping to constraints on minimum feature was heretofore an unexplored issue and is one that we believe also needs to be more fully characterized. Note that the distinction between a closed-form mapping and a rigorous one is the ability to invert the closed-form relationship between fill factor and index, which allows a designer to account a priori for the effects of subwavelength area-modulation encoding.

We investigated the relationship among material properties, design, and computational resources necessary for analysis. We conclude that element design and fabrication, but unfortunately not analysis, is simpler for high-refractive-index materials and long wavelengths. We also derived expressions for the space–bandwidth product and for the minimum feature of one- and two-dimensional encoded deflectors, linear lenses, and quadratic lenses, assuming both a linear and a quadratic phase to pulse-width mapping. The expressions for minimum feature were validated by comparison with minimum feature values taken from designs.

Our design results reveal that a trade-off exists between the exactness of a mapping and the fabrication constraints. In our example the zeroth-order mapping performed marginally better than the rigorous mapping prior to the imposition of fabrication constraints. However, after spatial quantization it provided 10% more in diffraction efficiency because it contained many large features that the fabrication could resolve. This was not true of the results generated by the rigorous mapping. The degradation in performance as a function of minimum feature size still needs to be determined.

The results of our lens design are encouraging in that they reveal that the losses in diffraction efficiency from an expected upper bound are minimal. The performance of the quadratic-mapped, spatially quantized quadratic lens is sufficiently high that optimization of the structure to regain the 8% in diffraction efficiency lost as part of the design might only waste valuable computer resources.

To address this issue we need to determine how much efficiency can be regained through optimization, the computational costs of such an optimization, and the sensitivity of the optimization to starting point. In the same manner in which diffraction efficiency upper bounds have been derived for superwavelength diffractive lenses, it would be helpful to the design process to determine diffraction efficiency upper bounds for subwavelength lenses.

**APPENDIX A**

Here we derive approximate expressions for the space–bandwidth product and for the minimum feature for two binary subwavelength elements, assuming a quadratic index mapping: one that functions as a deflector, and the other that provides focusing. A similar procedure was used to derive the expressions for a linear index mapping.

1. **Grating Deflector**

A single-period representation of the phase for a linear blazed grating is

\[ \theta(x) = \frac{2\pi x}{W}, \quad x = [0, W], \tag{A1} \]

where, for a given deflection angle \( \phi \), the grating period \( W \) is

\[ W = \frac{\lambda}{n_s \sin \phi}. \tag{A2} \]

Sampling and quadratic mapping of the phase yields

\[ g_m = \frac{2}{n_r + 1} \left( \frac{1}{M} \right)^2 + \frac{n_r - 1}{n_r + 1} \left( \frac{2m - 1}{M^2} \right), \quad m = [0, M - 1]. \tag{A3} \]

We use the difference between samples to determine the minimum feature necessary to resolve the smallest change in \( g_m \):

\[ dg_m = \frac{N^{-1}}{n_r + 1} \left( \frac{1}{M} \right)^2 + \frac{n_r - 1}{n_r + 1} \left( \frac{1}{M^2} \right), \quad m = [1, M - 1], \tag{A4} \]

whose minimum absolute value is obtained when \( m = 1 \):

\[ dg_{\min} = N^{-1} = \frac{2}{n_r + 1} \left( \frac{1}{M} \right)^2 + \frac{n_r - 1}{n_r + 1} \left( \frac{1}{M^2} \right). \tag{A5} \]

The space–bandwidth product \( S \) required for a single period of the grating is

\[ S = MN = \left( \frac{n_r + 1}{2} \right) \left( \frac{1}{2 + (n_r - 1)/M} \right) M^2 \approx \left( \frac{n_r + 1}{2} \right) M^2, \tag{A6} \]

and the minimum feature \( \delta \) is

\[ \delta = \frac{W}{S} = \frac{2\lambda}{(n_s + n_0) M^2 \sin \phi}. \tag{A7} \]

Use of the minimum number of samples \( M_{\min} \),

\[ M_{\min} = \frac{W}{S_{\min}} = \frac{2n_s}{\sin \phi}, \tag{A8} \]

in relations (A6) and (A7) yields

\[ S = \left( \frac{n_r + 1}{2} \right) \left( \frac{2n_s}{\sin \phi} \right)^2, \tag{A9} \]

\[ \delta = \frac{S_{\min}}{n_s (n_s + n_0)}. \tag{A10} \]
Note that the minimum feature is directly proportional to the wavelength and angle and is inversely proportional to the cube of the refractive index.

2. Lenses

To design a focusing element we start with the expression for a quadratic lens phase that has a focal length \( f \) and diameter \( D \):

\[
\theta(x) = \frac{2\pi L}{\lambda} + \frac{2\pi n_0}{\lambda} \times \left[ 1 - \sqrt{1 + \left( \frac{x}{f} \right)^2} \right] \text{mod} 2\pi,
\]

\[x = [-D/2, D/2],\]

where \( L \) is the number of zones:

\[
L = \text{ceil} \left\{ \frac{n_0}{\lambda} \sqrt{1 + \left( \frac{D}{2f} \right)^2} - 1 \right\}. \tag{A12}
\]

The sampled representation of the index synthesis function is

\[
g_m = \frac{2}{n_s + 1} \left\{ L + \frac{n_0}{\lambda} \times \left[ 1 - \sqrt{1 + \left( \frac{m\Delta}{f} \right)^2} \right] \text{mod} 1 \right\} + \frac{n_r - 1}{n_s + 1} \left\{ L + \frac{n_0}{\lambda} \times \left[ 1 - \sqrt{1 + \left( \frac{\Delta}{f} \right)^2} \right] \text{mod} 1 \right\}^2,
\]

\[m = [-M/2, M/2],\]

where we assume the minimum number of samples for \( M \):

\[
M = \frac{D}{s_h}. \tag{A14}
\]

Since the phase function has zero slope at the origin, \( dg_{\min} \) is obtained between \( m = 0 \) and \( m = 1 \):

\[
dg_{\min} = -\frac{2}{n_s + 1} \left\{ \frac{n_0}{\lambda} \left[ 1 - \sqrt{1 + \left( \frac{\Delta}{f} \right)^2} \right] \right\} + \frac{n_r - 1}{n_s + 1} \left\{ \frac{n_0}{\lambda} \left[ 1 - \sqrt{1 + \left( \frac{\Delta}{f} \right)^2} \right]^2 \right\}
\]

\[= \frac{2n_0}{\lambda} \left[ \sqrt{1 + \left( \frac{\Delta}{f} \right)^2} - 1 \right] \frac{n_r}{n_s + 1} + \frac{n_r - 1}{n_s + 1} \left[ \frac{n_0}{\lambda} \left[ 1 - \sqrt{1 + \left( \frac{\Delta}{f} \right)^2} \right]^2 \right]
\]

\[\approx \left( \frac{n_r}{n_s + 1} \right) \frac{n_0s_h^2}{f^2} = \frac{s_h}{(n_r + 1)2f}. \tag{A15}\]

We assumed that \( \Delta f \ll 1 \) and \( \Delta \approx s_h \) to arrive at our final expression. Thus

\[
N = \frac{(n_r + 1)}{s_h} 2f, \tag{A16}
\]

\[
S = MN = \frac{(n_r + 1)D}{s_h^2} 2f, \tag{A17}
\]

\[
\delta = \frac{D}{S} = \frac{s_h^2}{(n_r + 1)2f}. \tag{A18}
\]

As with the deflector, the minimum feature is sensitive to the substrate refractive index. It is inversely proportional to the focal length; thus short focal lengths are preferred. The minimum feature size is also directly proportional to the square of the wavelength, which makes lens design even more advantageous for long wavelengths than deflector design.

To reduce space-bandwidth product requirements, we also consider a linear phase lens, whose zone spacing is the same as a quadratic phase lens but whose phase is linear within each zone:

\[
\theta_l(x) = 2\pi \sum_{i=1}^{L} \left( 1 - \frac{|x - x_{i-1}|}{W_i} \right) \text{rect} \left( \frac{x - x_{i-1}}{W_i} \right). \tag{A19}
\]

The locations of the zone edges \( x_l, l = [0, L], \) are

\[x_l = \left[ 2f(\lambda l/n_0) + (\lambda l/n_0)^2 \right]^{1/2}, l = [0, L], \tag{A20}\]

and the zone widths \( W_l \) are

\[W_l = x_l - x_{l-1}, l = [1, L]. \tag{A21}\]

Within each zone, the phase representation is similar to that of the grating deflector. Because of its width, the slowest change in phase occurs within the first zone; thus

\[
dg_{\min} = \frac{2}{n_r + 1} \frac{\Delta}{W_l} + \frac{n_r - 1}{n_r + 1} \left[ \frac{\Delta}{W_l} \right]^2, \tag{A22}\]

\[
N \approx \frac{n_r + 1}{2} W_l, \tag{A23}\]

where

\[W_l = \left[ 2f(\lambda l/n_0) + (\lambda l/n_0)^2 \right]^{1/2}. \tag{A24}\]

Finally,

\[
S = MN \approx \frac{(n_r + 1)D}{s_h^2} \sqrt{\frac{f\lambda}{2n_0}}, \tag{A25}\]

\[
\delta \approx \frac{s_h^2}{(n_r + 1)} \sqrt{\frac{2n_0}{f\lambda}}. \tag{A26}\]

It is interesting that, although the sensitivity to wavelength and focal length has been reduced, the minimum feature remains inversely proportional to the cube of the substrate refractive index.

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