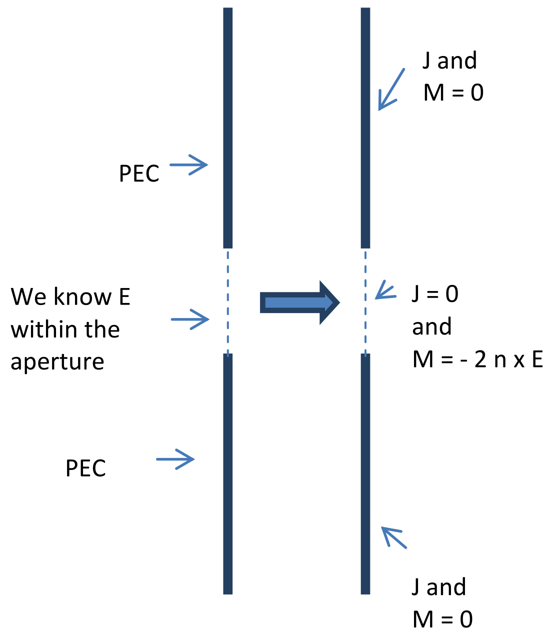
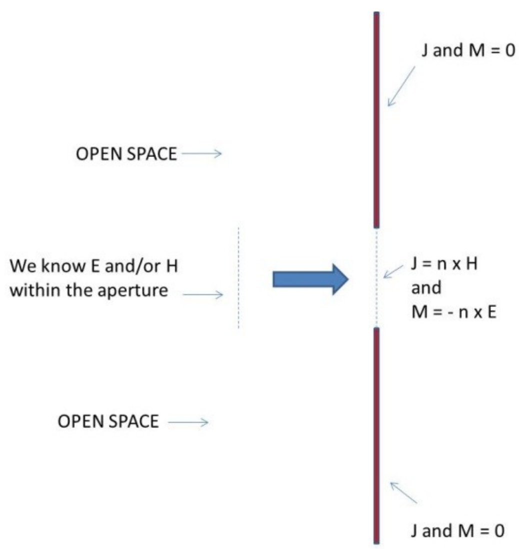
A. Field Equivalence Principle

There are a number of field equivalence principles but let's look at just two of them.



(1) If we know E within an aperture that has a PEC ground plane we can replace the problem with equivalent magnetic current densities.



(2) If we know E and H within an aperture that has no ground plane we can replace the problem with equivalent electric and magnetic current densities

B. Vector potentials

$$R \approx r - r' \cos(\psi)$$
$$A = \frac{\mu}{4\pi} \iint J \frac{e^{-jkR}}{R} ds \approx \frac{\mu}{4\pi} \frac{e^{-jkr}}{r} \iint J\, e^{-jkr' \cos(\psi)} ds$$
$$F = \frac{\varepsilon}{4\pi} \iint M \frac{e^{-jkR}}{R} ds \approx \frac{\varepsilon}{4\pi} \frac{e^{-jkr}}{r} \iint M\; e^{-jkr' \cos(\psi)} ds$$
$$E_\theta \approx -j\omega A_\theta - j\omega\eta F_\phi$$
$$E_\phi \approx -j\omega A_\phi + j\omega\eta F_\theta$$
$$H_\theta \approx -\frac{1}{\eta} E_\phi$$
$$H_\phi \approx \frac{1}{\eta} E_\theta$$

The rectangular to spherical transformation written more explicitly is given as

$$A_\theta \approx \frac{\mu}{4\pi} \frac{e^{-jkr}}{r} \iint \left[J_x \cos(\theta)\cos(\phi) + J_y \cos(\theta)\sin(\phi) - J_z \sin(\theta)\right] e^{-jkr' \cos(\psi)} ds$$
$$A_\phi \approx \frac{\mu}{4\pi} \frac{e^{-jkr}}{r} \iint \left[-J_x \sin(\phi) + J_y \cos(\phi)\right] e^{-jkr' \cos(\psi)} ds$$
$$F_\theta \approx \frac{\varepsilon}{4\pi} \frac{e^{-jkr}}{r} \iint \left[M_x \cos(\theta)\cos(\phi) + M_y \cos(\theta)\sin(\phi) - M_z \sin(\theta)\right] e^{-jkr' \cos(\psi)} ds$$
$$F_\phi \approx \frac{\varepsilon}{4\pi} \frac{e^{-jkr}}{r} \iint \left[-M_x \sin(\phi) + M_y \cos(\phi)\right] e^{-jkr' \cos(\psi)} ds$$

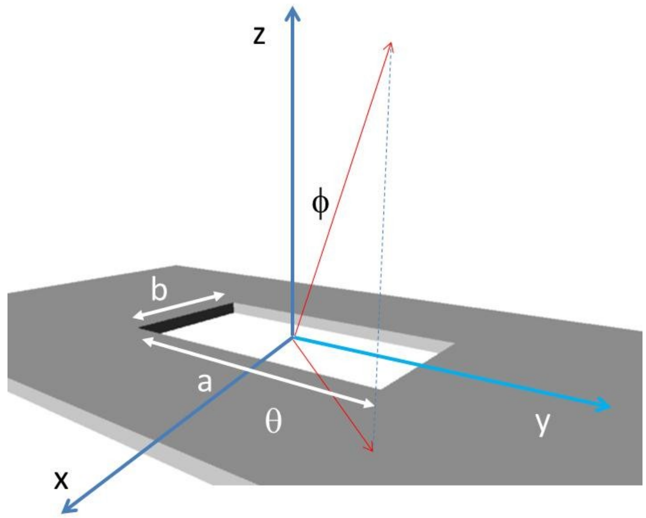
where

$$e^{-jkr' \cos(\psi)} = e^{-jk\left(x' \sin(\theta)\cos(\phi) + y' \sin(\theta)\sin(\phi)\right)}$$

SOLUTIONS STEPS:

(1) Given the E-field in the aperture of a PEC ground plane replace the fields with the equivalent surface currents (J and M)
(2) Use J and M to solve for A and F in the far-field (spherical coordinates)
(3) Use A and F to find E and H
(4) From E and H we can find U or D or anything else.

Example #1: Rectangular Aperture with Uniform E-field



A uniform E-field in the aperture is given by

$$\tilde{E} = a_y E_o$$

Step #1: Find the equivalent J and M in the aperture

$$\tilde{J} = 0$$

$$\tilde{M} = -2\, n\ \times \tilde{E} = -2\, a_z \times a_y E_o = 2E_o a_x$$

Step #2: Find A and F in the far-field

$$A_\theta \approx 0$$
$$A_\phi \approx 0$$

$$F_\theta \approx \frac{\varepsilon}{4\pi}\frac{e^{-jkr}}{r}\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{-\frac{b}{2}}^{\frac{b}{2}}[2E_o\cos(\theta)\cos(\phi)]\, e^{-jk(x'\sin(\theta)\cos(\phi)+y'\sin(\theta)\sin(\phi))}\, dy'dx'$$

$$F_\phi \approx \frac{\varepsilon}{4\pi}\frac{e^{-jkr}}{r}\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{-\frac{b}{2}}^{\frac{b}{2}}[-2E_o\sin(\phi)]\, e^{-jk(x'\sin(\theta)\cos(\phi)+y'\sin(\theta)\sin(\phi))}\, dy'dx'$$

$$F_\theta \approx \frac{\varepsilon}{4\pi}\frac{e^{-jkr}}{r}2abE_o\left[\cos(\theta)\cos(\phi)\frac{\sin(X)}{X}\frac{\sin(Y)}{Y}\right]$$

$$F_\phi \approx -\frac{\varepsilon}{4\pi}\frac{e^{-jkr}}{r}2abE_o\left[\sin(\phi)\frac{\sin(X)}{X}\frac{\sin(Y)}{Y}\right]$$

$$X = \frac{ka}{2}\sin(\theta)\cos(\phi)$$

$$Y = \frac{kb}{2}\sin(\theta)\sin(\phi)$$

$$E_\theta \approx j\frac{abkE_o}{2\pi}\frac{e^{-jkr}}{r}\left[\sin(\phi)\frac{\sin(X)}{X}\frac{\sin(Y)}{Y}\right]$$

$$E_\phi \approx j\frac{abkE_o}{2\pi}\frac{e^{-jkr}}{r}\left[\cos(\theta)\cos(\phi)\frac{\sin(X)}{X}\frac{\sin(Y)}{Y}\right]$$

Example #2: Open ended rectangular waveguide antenna within a ground plane

The fundamental mode of a rectangular waveguide for which a>b is given by

$$\tilde{E} = E_o\cos\left(\frac{\pi}{a}x\right)a_y$$

Step #1: Find the equivalent J and M in the aperture

$$\tilde{J} = 0$$

$$\widetilde{M} = -2\,n\times\tilde{E} = -2\,a_z\times a_y\,E_o\cos\left(\frac{\pi}{a}x\right) = -2E_o\cos\left(\frac{\pi}{a}x\right)a_x$$

Step #2: Find A and F in the far-field

$$A_\theta \approx 0$$
$$A_\phi \approx 0$$

$$F_\theta \approx \frac{\varepsilon}{4\pi}\frac{e^{-jkr}}{r}\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{-\frac{b}{2}}^{\frac{b}{2}}\left[2\cos\left(\frac{\pi}{a}x'\right)E_o\cos(\theta)\cos(\phi)\right]e^{-jk(x'\sin(\theta)\cos(\phi)+y'\sin(\theta)\sin(\phi))}\,dy'dx'$$

$$F_\phi \approx \frac{\varepsilon}{4\pi}\frac{e^{-jkr}}{r}\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{-\frac{b}{2}}^{\frac{b}{2}}\left[-2\cos\left(\frac{\pi}{a}x'\right)E_o\sin(\phi)\right]e^{-jk(x'\sin(\theta)\cos(\phi)+y'\sin(\theta)\sin(\phi))}\,dy'dx'$$

$$F_\theta \approx \frac{\varepsilon b}{2\pi}\frac{e^{-jkr}}{r}E_o\cos(\theta)\cos(\phi)\frac{\sin(Y)}{Y}\int_{-\frac{a}{2}}^{\frac{a}{2}}\cos\left(\frac{\pi}{a}x'\right)e^{-jk(x'\sin(\theta)\cos(\phi))}\,dx'$$

$$F_\phi \approx \frac{-\varepsilon b}{2\pi}\frac{e^{-jkr}}{r}E_o\sin(\phi)\frac{\sin(Y)}{Y}\int_{-\frac{a}{2}}^{\frac{a}{2}}\cos\left(\frac{\pi}{a}x'\right)e^{-jk(x'\sin(\theta)\cos(\phi))}\,dx'$$

$$F_\theta \approx -\frac{\varepsilon ab}{4}\frac{e^{-jkr}}{r}E_o\cos(\theta)\cos(\phi)\frac{\sin(Y)}{Y}\frac{\cos(X)}{X^2-\left(\frac{\pi}{2}\right)^2}$$

$$F_\phi \approx \frac{\varepsilon ab}{4}\frac{e^{-jkr}}{r}E_o\sin(\phi)\frac{\sin(Y)}{Y}\frac{\cos(X)}{X^2-\left(\frac{\pi}{2}\right)^2}$$

$$E_\phi \approx -j\frac{kab}{4}\frac{e^{-jkr}}{r}E_o\cos(\theta)\cos(\phi)\frac{\sin(Y)}{Y}\frac{\cos(X)}{X^2-\left(\frac{\pi}{2}\right)^2}$$

$$E_\theta \approx -j\frac{kab}{4}\frac{e^{-jkr}}{r}E_o\sin(\phi)\frac{\sin(Y)}{Y}\frac{\cos(X)}{X^2-\left(\frac{\pi}{2}\right)^2}$$

Rectangular waveguide: Wave impedance of the fundamental mode.

$$Z_w = \frac{Z_o}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = \frac{\sqrt{\frac{\mu}{\varepsilon}}}{\sqrt{1-\frac{\left(\frac{1}{2a\sqrt{\mu\varepsilon}}\right)^2}{f^2}}} = \frac{\sqrt{\frac{\mu}{\varepsilon}}}{\sqrt{1-\left(\frac{\lambda}{2a}\right)^2}}$$

Assume a=0.75 λ

$$Z_w = \frac{\sqrt{\frac{\mu}{\varepsilon}}}{\sqrt{1-\left(\frac{\lambda}{2a}\right)^2}} = 1.34\, Z_o$$

$$R = \frac{(Z_w - Z_o)^2}{(Z_w + Z_o)^2} = \frac{(1.34\, Z_o - Z_o)^2}{(1.34\, Z_o + Z_o)^2} \approx 2.0\ \%$$

Example #3: Uniform field in an aperture <u>without</u> a ground plane

The fields within an aperture without a ground plane

$$\tilde{E} = E_o\, a_y$$

$$\tilde{H} = -\frac{E_o}{\eta}\, a_x$$

Step #1: Find the equivalent J and M in the aperture

$$\tilde{J} = n \times \tilde{H} = -\frac{E_o}{\eta} a_z \times a_x = -\frac{E_o}{\eta} a_y$$

$$\tilde{M} = -n \times \tilde{E} = -a_z \times a_y\, E_o = E_o\, a_x$$

$$A_\theta \approx -\frac{\mu}{4\pi}\frac{e^{-jkr}}{r}\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{-\frac{b}{2}}^{\frac{b}{2}}\left[\frac{E_o}{\eta}\cos(\theta)\sin(\phi)\right] e^{-jk\left(x'\sin(\theta)\cos(\phi)+y'\sin(\theta)\sin(\phi)\right)}\, dy'dx'$$

$$A_\phi \approx -\frac{\mu}{4\pi}\frac{e^{-jkr}}{r}\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{-\frac{b}{2}}^{\frac{b}{2}}\left[\frac{E_o}{\eta}\cos(\phi)\right] e^{-jk\left(x'\sin(\theta)\cos(\phi)+y'\sin(\theta)\sin(\phi)\right)}\, dy'dx'$$

$$F_\theta \approx \frac{\varepsilon}{4\pi}\frac{e^{-jkr}}{r}\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{-\frac{b}{2}}^{\frac{b}{2}}\left[E_o\cos(\theta)\cos(\phi)\right] e^{-jk\left(x'\sin(\theta)\cos(\phi)+y'\sin(\theta)\sin(\phi)\right)}\, dy'dx'$$

$$F_\phi \approx \frac{\varepsilon}{4\pi}\frac{e^{-jkr}}{r}\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{-\frac{b}{2}}^{\frac{b}{2}}\left[-E_o\sin(\phi)\right] e^{-jk\left(x'\sin(\theta)\cos(\phi)+y'\sin(\theta)\sin(\phi)\right)}\, dy'dx'$$

$$A_\theta \approx -\frac{\mu}{4\pi}\frac{e^{-jkr}}{r}\frac{E_o}{\eta}\text{ab}\cos(\theta)\sin(\phi)\frac{\sin(X)}{X}\frac{\sin(Y)}{Y}$$

$$A_\phi \approx -\frac{\mu}{4\pi}\frac{e^{-jkr}}{r}\frac{E_o}{\eta}\text{ab}\cos(\phi)\frac{\sin(X)}{X}\frac{\sin(Y)}{Y}$$

$$F_\theta \approx \frac{\varepsilon}{4\pi}\frac{e^{-jkr}}{r}E_o\,\text{ab}\cos(\theta)\cos(\phi)\frac{\sin(X)}{X}\frac{\sin(Y)}{Y}$$

$$F_\phi \approx -\frac{\varepsilon}{4\pi}\frac{e^{-jkr}}{r}E_o\,\text{ab}\sin(\phi)\frac{\sin(X)}{X}\frac{\sin(Y)}{Y}$$

$$E_\theta \approx -j\omega A_\theta - j\omega\eta F_\phi = j\frac{abkE_o}{4\pi}\frac{e^{-jkr}}{r}\sin(\phi)(1+\cos(\theta))\frac{\sin(X)}{X}\frac{\sin(Y)}{Y}$$

$$E_\phi \approx -j\omega A_\phi + j\omega\eta F_\theta = j\frac{abkE_o}{4\pi}\frac{e^{-jkr}}{r}\cos(\phi)(1+\cos(\theta))\frac{\sin(X)}{X}\frac{\sin(Y)}{Y}$$

Example #4. Circular aperture of radius "a" with uniform E-field in the aperture is given by

$$\tilde{E} = a_y E_o$$

Step #1: Find the equivalent J and M in the aperture in cylindrical coordinates

$$\tilde{J} = 0$$

$$\tilde{M} = -2\, n\, \times \tilde{E} = -2\, a_z \times a_y E_o = 2E_o a_x$$

Step #2: Find A and F in the far-field

$$A_\theta \approx 0$$
$$A_\phi \approx 0$$

$$F_\theta \approx \frac{\varepsilon}{4\pi}\frac{e^{-jkr}}{r}\int\int [2E_o\cos(\theta)\cos(\phi)]\, e^{-jk(x'\sin(\theta)\cos(\phi)+y'\sin(\theta)\sin(\phi))}\, dy'dx'$$

$$F_\phi \approx \frac{\varepsilon}{4\pi}\frac{e^{-jkr}}{r}\int\int [-2E_o\sin(\phi)]\, e^{-jk(x'\sin(\theta)\cos(\phi)+y'\sin(\theta)\sin(\phi))}\, dy'dx'$$

Due to rotational symmetry the observed fields are independent of ϕ. So without any loss of generality we can set ϕ=0. This implies that

$$F_\theta \approx \frac{\varepsilon}{4\pi}\frac{e^{-jkr}}{r}[2E_o\cos(\theta)]\int_0^a\int_0^{2\pi} e^{-jk(\cos(\phi')\rho'\sin(\theta))}\, \rho' d\rho' d\phi'$$

$$F_\phi \approx 0$$

where $x' = \cos(\phi')\rho'$

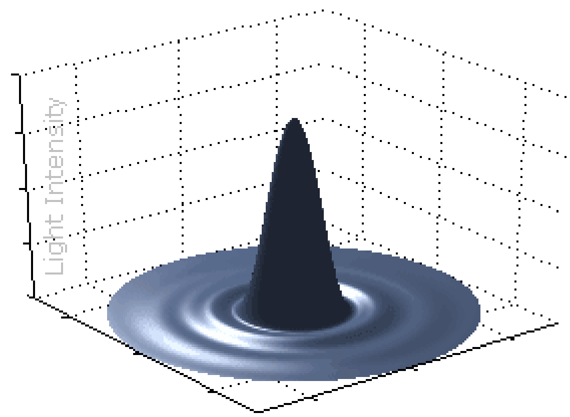
The above integrals above can be solved analytically with the help of a couple of special integrals:

$$J_o(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{jx\cos(\phi')}d\phi'$$

$$\int_0^1 J_o(x\rho)\rho d\rho = \frac{J_1(x)}{x}$$

$$E_\phi \approx jka^2\frac{e^{-jkr}}{r}E_o\cos(\theta)\cos(\phi)\,\frac{J_1(ka\,sin(\theta))}{ka\,sin(\theta)}$$

$$E_\theta \approx jka^2\frac{e^{-jkr}}{r}E_o\sin(\phi)\,\frac{J_1(ka\,sin(\theta))}{ka\,sin(\theta)}$$



It is this effect that limits the resolution of imaging systems. For a given diameter lens at a given wavelength the width of the main lobe of the airy disk is the smallest spot size possible as shown below.

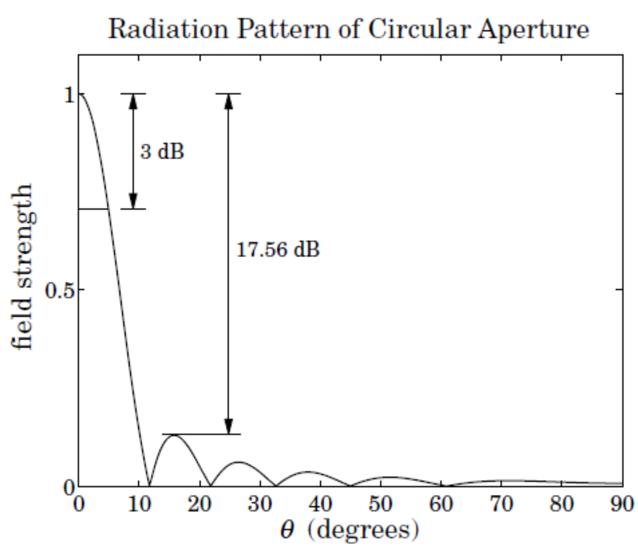


Fig. 17.9.2 Radiation pattern of circular aperture ($a = 3\lambda$).

The first null angle is called the Rayleigh diffraction limit. It is also referred to as the diffraction limited spot size and plays a big role in specifying the fundamental resolution limit of imaging systems. It depends only on the wavelength of the light and the size of the aperture. In terms of the diameter D=2a of the optical aperture the Rayleigh diffraction limit is given as:

$$\Delta\theta = 1.22\frac{\lambda}{D}$$