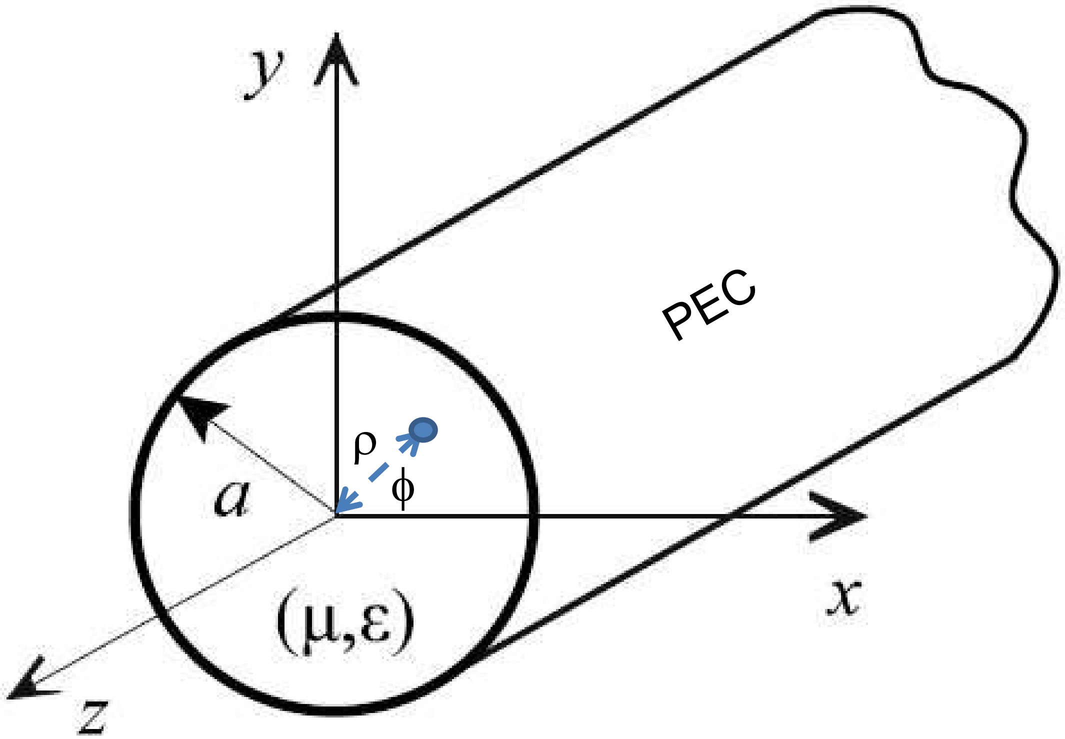
Circular Waveguide



Circular Waveguide

Like any uniform waveguide we can write the field solutions as a transverse term propagating down the waveguide in the z direction.

$$\widetilde{H}_z(\rho,\phi,z) = \widetilde{H}_{zt}(\rho,\phi)e^{-j\beta_z z} \qquad \widetilde{E}_z(\rho,\phi,z) = \widetilde{E}_{zt}(\rho,\phi)e^{-j\beta_z z}$$

Where each transverse term satisfies the wave equation:

$$\nabla^2\widetilde{H}_{zt}(\rho,\phi) + k_c{}^2\widetilde{H}_{zt}(\rho,\phi) = 0$$
$$\nabla^2\widetilde{E}_{zt}(\rho,\phi) + k_c{}^2\widetilde{E}_{zt}(\rho,\phi) = 0$$

$$k_c^2 = \omega^2\mu\varepsilon - \beta_z^2$$

Circular Waveguide

Like any uniform waveguide we can find all the transverse field components once the longitudinal components are determined

$$H_z(\rho,\phi,z) = H_{zt}(\rho,\phi)e^{-j\beta_z z} \qquad E_z(\rho,\phi,z) = E_{zt}(\rho,\phi)e^{-j\beta_z z}$$

$$H_\rho(\rho,\phi,z) = \frac{j}{k_c^2}\left(\frac{\omega\varepsilon}{\rho}\frac{\partial E_z}{\partial\phi} - \beta_z\frac{\partial H_z}{\partial\rho}\right) \qquad E_\rho(\rho,\phi,z) = \frac{-j}{k_c^2}\left(\beta_z\frac{\partial E_z}{\partial\rho} - \frac{\omega\mu}{\rho}\frac{\partial H_z}{\partial\phi}\right)$$

$$H_\phi(\rho,\phi,z) = \frac{-j}{k_c^2}\left(\omega\varepsilon\frac{\partial E_z}{\partial\rho} - \frac{\beta_z}{\rho}\frac{\partial H_z}{\partial\phi}\right) \qquad E_\phi(\rho,\phi,z) = \frac{-j}{k_c^2}\left(\frac{\beta_z}{\rho}\frac{\partial E_z}{\partial\phi} - \omega\mu\frac{\partial H_z}{\partial\rho}\right)$$

Circular Waveguide

We break this into cases as before:

$$H_z = 0 \quad E_z \neq 0$$

$$E_z = 0 \quad H_z \neq 0$$

TM Modes

$$H_\rho(\rho,\phi,z) = \frac{j}{k_c^2}\left(\frac{\omega\varepsilon}{\rho}\frac{\partial E_z}{\partial \phi}\right)$$

$$H_\phi(\rho,\phi,z) = \frac{-j}{k_c^2}\left(\omega\varepsilon\frac{\partial E_z}{\partial \rho}\right)$$

$$E_\rho(\rho,\phi,z) = \frac{-j}{k_c^2}\left(\beta_z\frac{\partial E_z}{\partial \rho}\right)$$

$$E_\phi(\rho,\phi,z) = \frac{-j}{k_c^2}\left(\frac{\beta_z}{\rho}\frac{\partial E_z}{\partial \phi}\right)$$

$$\nabla^2 E_z(\rho,\phi) + k_c{}^2 E_z(\rho,\phi) = 0$$

TE Modes

$$H_\rho(\rho,\phi,z) = \frac{-j}{k_c^2}\left(\beta_z\frac{\partial H_z}{\partial \rho}\right)$$

$$H_\phi(\rho,\phi,z) = \frac{j}{k_c^2}\left(\frac{\beta_z}{\rho}\frac{\partial H_z}{\partial \phi}\right)$$

$$E_\rho(\rho,\phi,z) = \frac{j}{k_c^2}\left(\frac{\omega\mu}{\rho}\frac{\partial H_z}{\partial \phi}\right)$$

$$E_\phi(\rho,\phi,z) = \frac{j}{k_c^2}\left(\omega\mu\frac{\partial H_z}{\partial \rho}\right)$$

$$\nabla^2 H_z(\rho,\phi) + k_c{}^2 H_z(\rho,\phi) = 0$$

Circular Waveguide

We break this into cases as before:

$$H_z = 0 \quad E_z \neq 0$$

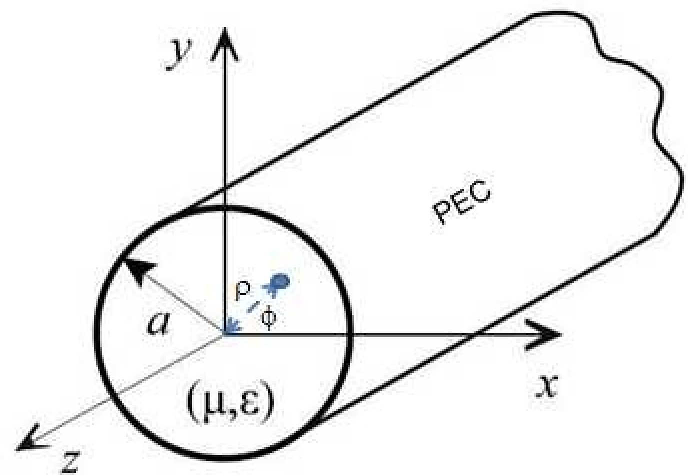
TM Modes

$$H_\rho(\rho,\phi,z) = \frac{j}{k_c^2}\left(\frac{\omega\varepsilon}{\rho}\frac{\partial E_z}{\partial\phi}\right)$$

$$H_\phi(\rho,\phi,z) = \frac{-j}{k_c^2}\left(\omega\varepsilon\frac{\partial E_z}{\partial\rho}\right)$$

$$E_\rho(\rho,\phi,z) = \frac{-j}{k_c^2}\left(\beta_z\frac{\partial E_z}{\partial\rho}\right)$$

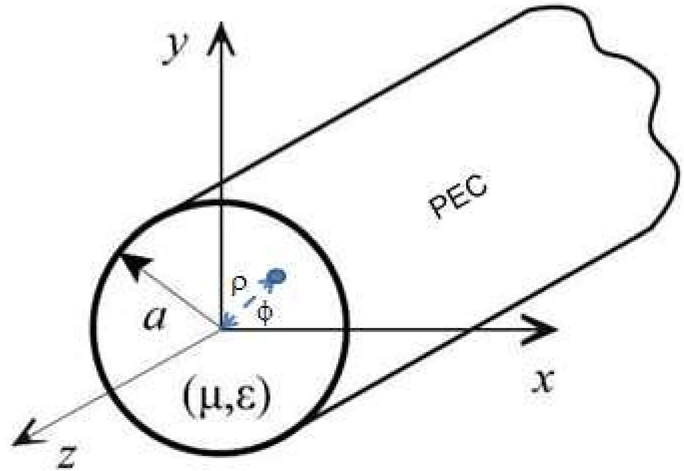
$$E_\phi(\rho,\phi,z) = \frac{-j}{k_c^2}\left(\frac{\beta_z}{\rho}\frac{\partial E_z}{\partial\phi}\right)$$



$$\nabla^2 E_z(\rho,\phi) + {k_c}^2 E_z(\rho,\phi) = 0$$

Circular Waveguide

TM Modes



$$\nabla^2 E_z(\rho,\phi) + k_c{}^2 E_z(\rho,\phi) = 0$$

$$\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + k_c^2\right) E_z(\rho,\phi) = 0$$

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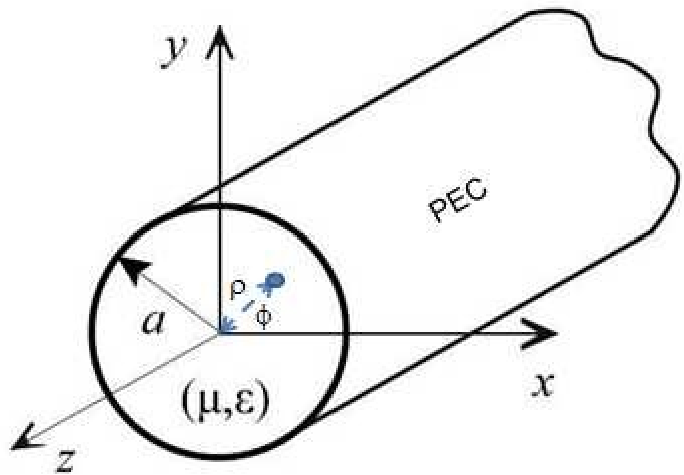
TM Modes



$$\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}+k_c^2\right)E_z(\rho,\phi)=0$$

Separation of Variables

$$E_z(\rho,\phi)=R(\rho)P(\phi)$$

Circular Waveguide

TM Modes

$$\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}+k_c^2\right)E_z(\rho,\phi)=0$$

$$E_z(\rho,\phi)=R(\rho)P(\phi)$$

$$R^{''}P+\frac{1}{\rho}R^{'}P+\frac{1}{\rho^2}RP^{''}+k_c^2RP=0$$

$$\frac{R^{''}P+\frac{1}{\rho}R^{'}P+\frac{1}{\rho^2}RP^{''}+k_c^2RP}{RP}=0$$

$$\frac{R^{''}}{R}+\frac{1}{\rho}\frac{R^{'}}{R}+\frac{1}{\rho^2}\frac{P^{''}}{P}+k_c^2=0$$

Circular Waveguide

TM Modes

$$\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}+k_c^2\right)E_z(\rho,\phi)=0$$

$$E_z(\rho,\phi)=R(\rho)P(\phi)$$

$$\underbrace{\rho^2\frac{R''}{R}+\rho\frac{R'}{R}+\rho^2k_c^2}_{function\ \ of\ \ \rho}+\underbrace{\frac{P''}{P}}_{function\ \ of\ \ \phi}=0$$

Circular Waveguide

TM Modes

$$\underbrace{\rho^2 \frac{R''}{R} + \rho \frac{R'}{R} + \rho^2 k_c^2}_{function\ \ of\ \rho} + \underbrace{\frac{P''}{P}}_{function\ \ of\ \phi} = 0$$

$$\Rightarrow \quad \frac{P''}{P} = -\beta_\phi^2 \qquad (1)$$

$$\Rightarrow \quad \rho^2 \frac{R''}{R} + \rho \frac{R'}{R} + \rho^2 k_c^2 - \beta_\phi^2 = 0 \qquad (2)$$

Circular Waveguide

TM Modes

$$\frac{P''}{P} = -\beta_\phi^2$$

$$P(\phi) = A_o \cos(\beta_\phi \phi) + B_o \sin(\beta_\phi \phi)$$

Circular Waveguide

TM Modes

Bessel's Differential Equation

$$\rho^2 \frac{R''}{R} + \rho \frac{R'}{R} + \rho^2 k_c^2 - \beta_\phi^2 = 0 \qquad (2)$$

Solution

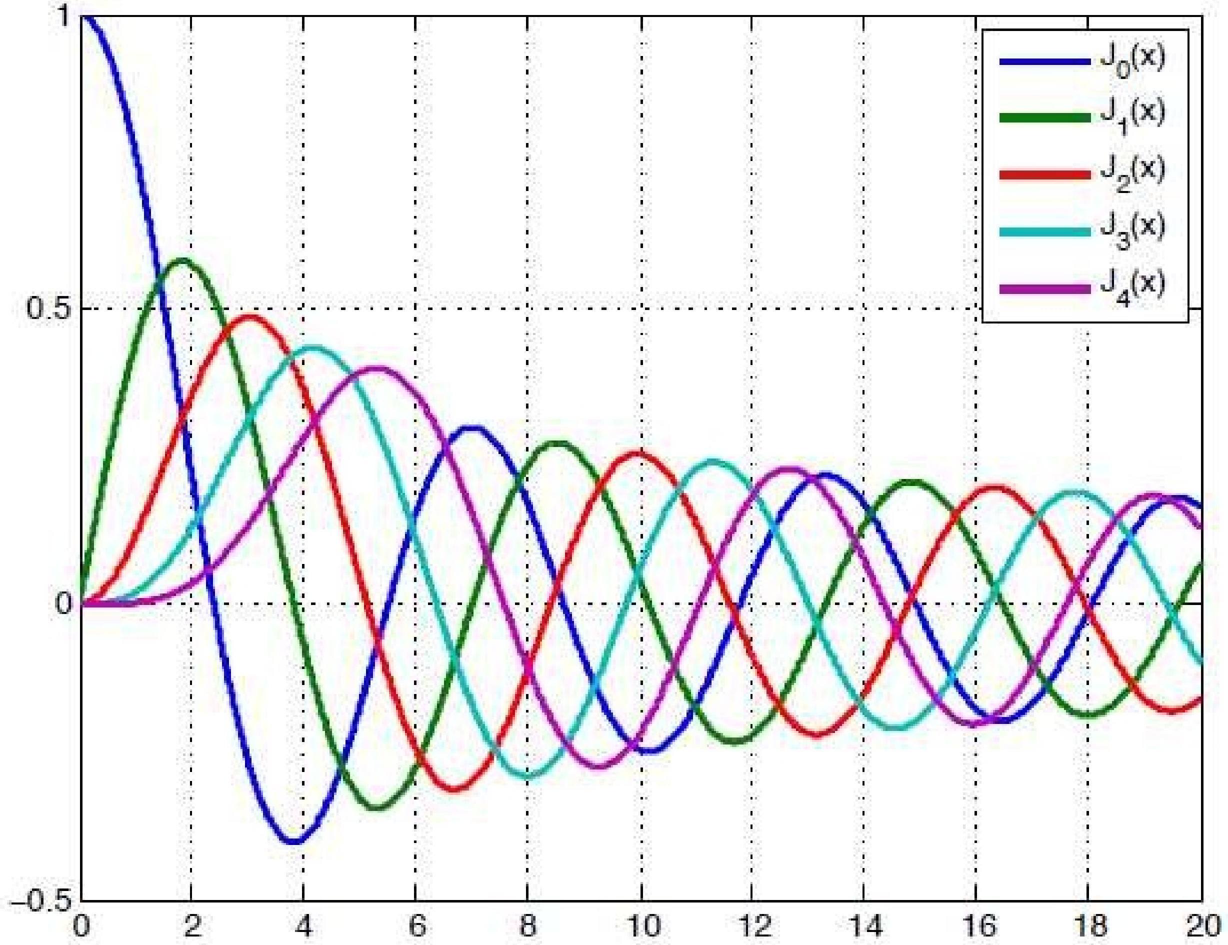
$$R(\rho) = C_o J_{\beta_\phi}(k_c \rho) + D_o Y_{\beta_\phi}(k_c \rho)$$

Bessel function of the 1st kind ($J_{\beta_\phi}(k_c \rho)$)

Bessel function of the 2nd kind or Neumann function ($Y_{\beta_\phi}(k_c \rho)$)

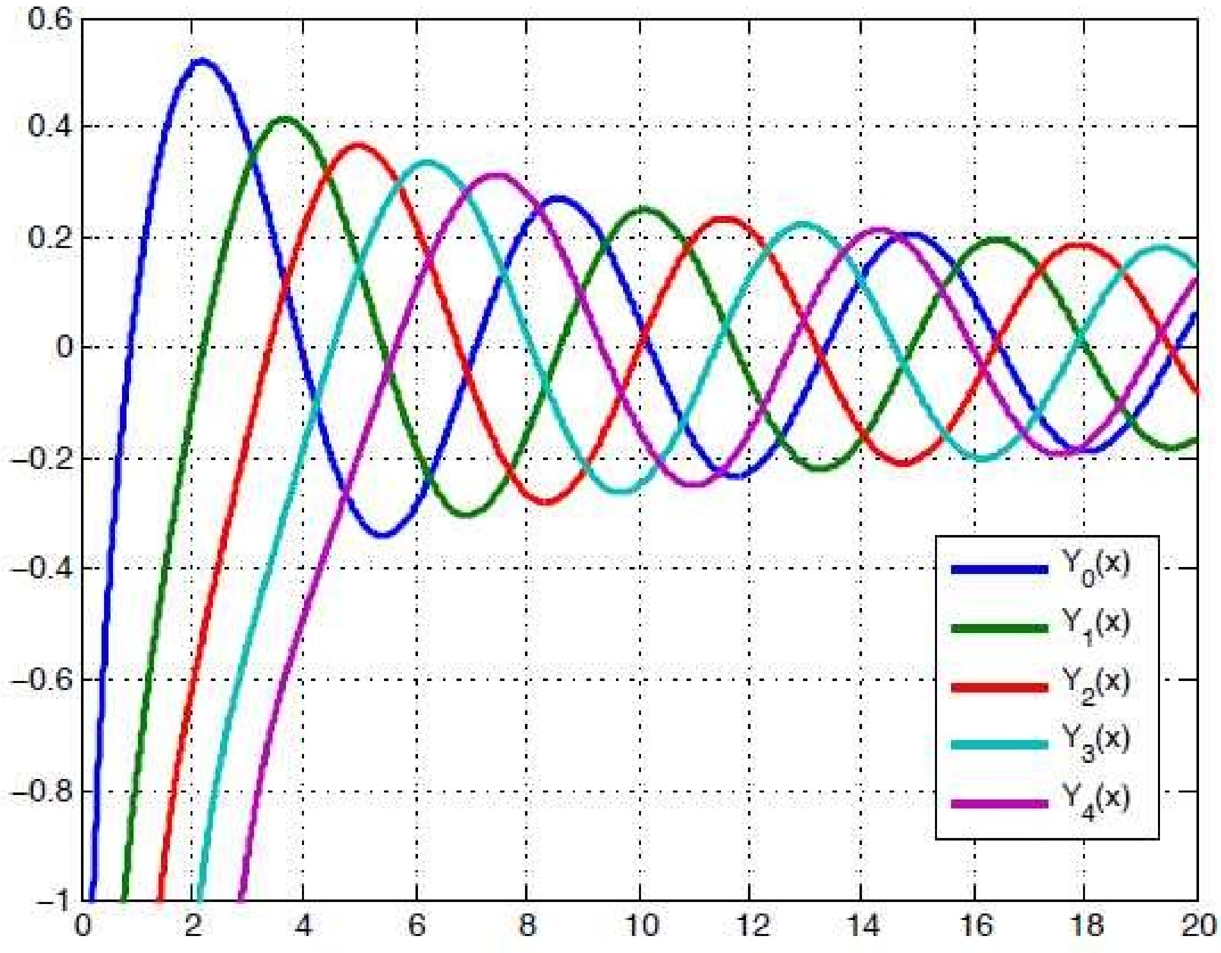
Aside on Bessel Functions

Graphs of $J_n(x)$



Aside on Bessel Functions

Graphs of $Y_n(x)$



Aside on Bessel Functions

A bunch of definitions and identities:

$$J_\alpha(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\alpha+1)} \left(\tfrac{1}{2}x\right)^{2m+\alpha}$$

$$J_n(x) = \frac{1}{\pi} \int_0^\pi \cos(n\tau - x \sin\tau)\,\mathrm{d}\tau.$$

$$J_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-\mathrm{i}\,(n\tau - x\sin\tau)}\,\mathrm{d}\tau.$$

$$Y_\alpha(x) = \frac{J_\alpha(x)\cos(\alpha\pi) - J_{-\alpha}(x)}{\sin(\alpha\pi)}.$$

$$e^{iz\sin\phi} = \sum_{n=-\infty}^{\infty} J_n(z) e^{in\phi},$$

$$J'_\nu(z) = \tfrac{1}{2}(J_{\nu-1}(z) - J_{\nu+1}(z))$$

Aside on Bessel Functions

- The J_n and Y_n are both real functions for real arguments.
- They must therefore represent standing waves (Why?).
- Hankel functions represent traveling waves.

Traveling waves are represented by

Hankel Functions

$$
\begin{aligned}
H_n^{(1)}(x) &= J_n(x) + jY_n(x) \\
H_n^{(2)}(x) &= J_n(x) - jY_n(x)
\end{aligned}
$$

These are called Hankel functions of the first and second kind, respectively.

Aside on Bessel Functions

Small Argument Behavior

- Suppose $\text{Re}(\nu) > 0$.
- Let $\ln \gamma = 0.5772 \Rightarrow \gamma = 1.781$ (i.e. $\ln \gamma$ is "Euler's constant").

Consider the behavior of the Bessel and Neumann functions as $x \to 0$:

$$\begin{aligned}
J_0(x) &\to 1 \\
Y_0(x) &\to \frac{2}{\pi} \ln \frac{\gamma x}{2} \\
J_\nu(x) &\to \frac{1}{\nu!} \left(\frac{x}{2}\right)^\nu \\
Y_\nu(x) &\to -\frac{(\nu-1)!}{\pi} \left(\frac{2}{x}\right)^\nu
\end{aligned}$$

The only Bessel functions finite at the origin are the $J_n(x)$.

Aside on Bessel Functions

Large Argument Behavior

As $x \to \infty$:

$$J_\nu(x) \to \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4} - \frac{\nu\pi}{2}\right)$$

$$Y_\nu(x) \to \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\pi}{4} - \frac{\nu\pi}{2}\right)$$

Given the definition of Hankel functions, we must also have

$$H_\nu^{(1)}(x) \to \sqrt{\frac{2}{j\pi x}} j^{-\nu} e^{jx}$$

$$H_\nu^{(2)}(x) \to \sqrt{\frac{2j}{\pi x}} j^{\nu} e^{-jx}$$

- The $H_\nu^{(2)}$ represent outward traveling waves.
- Why are these all proportional to $x^{-\frac{1}{2}}$?

Aside on Bessel Functions

Imaginary Arguments

- In applications, we get Bessel functions of dimensionless quantities: $B_n(k_\rho \rho)$.
- If k_ρ becomes imaginary, we have evanescence in the ρ direction.

For these applications, we define the

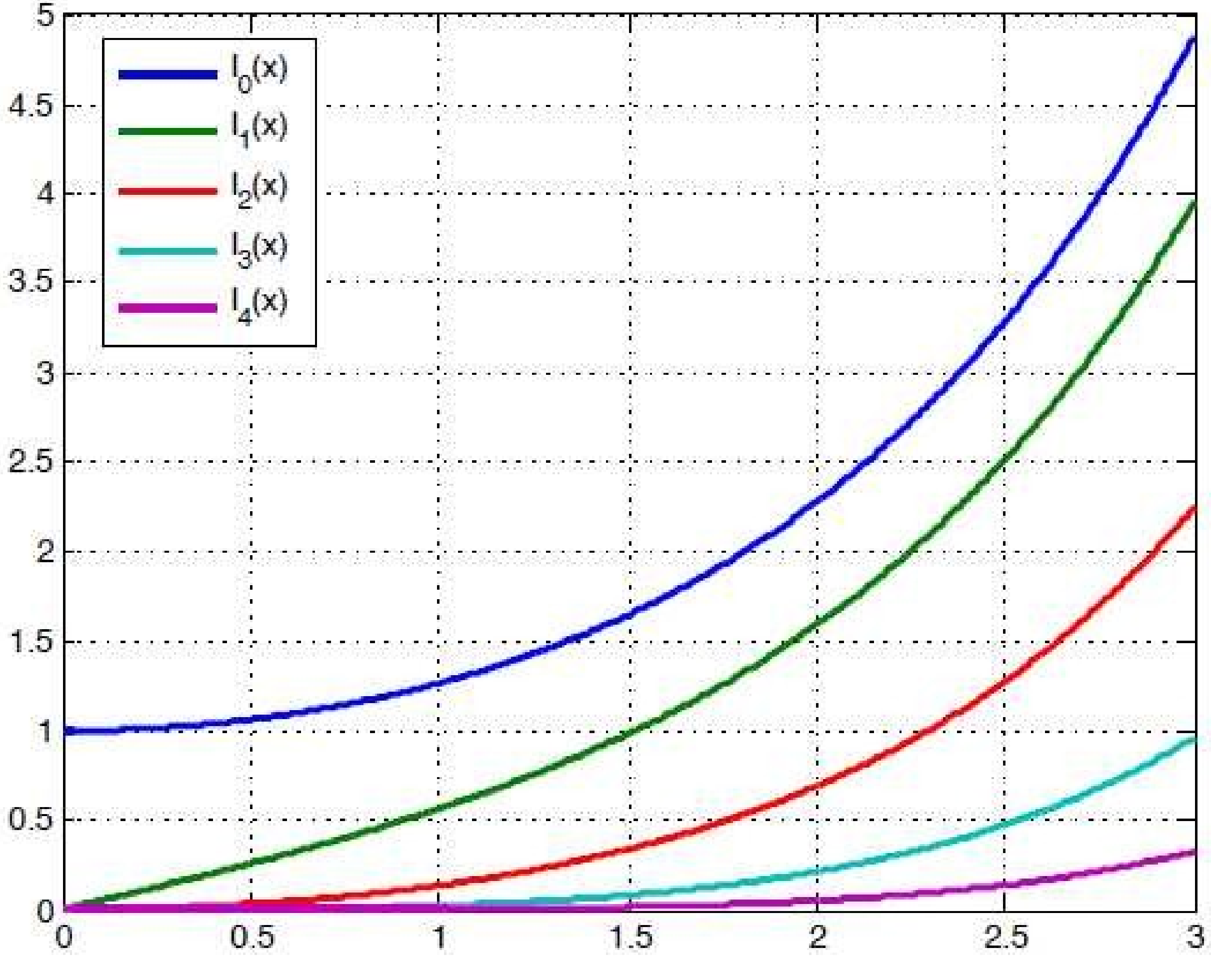
Modified Bessel Functions

$$\begin{aligned} I_n(x) &= j^n J_n(-jx) \\ K_n(x) &= \frac{\pi}{2}(-j)^{n+1} H_n^{(2)}(-jx) \end{aligned}$$

These are real functions of real arguments.

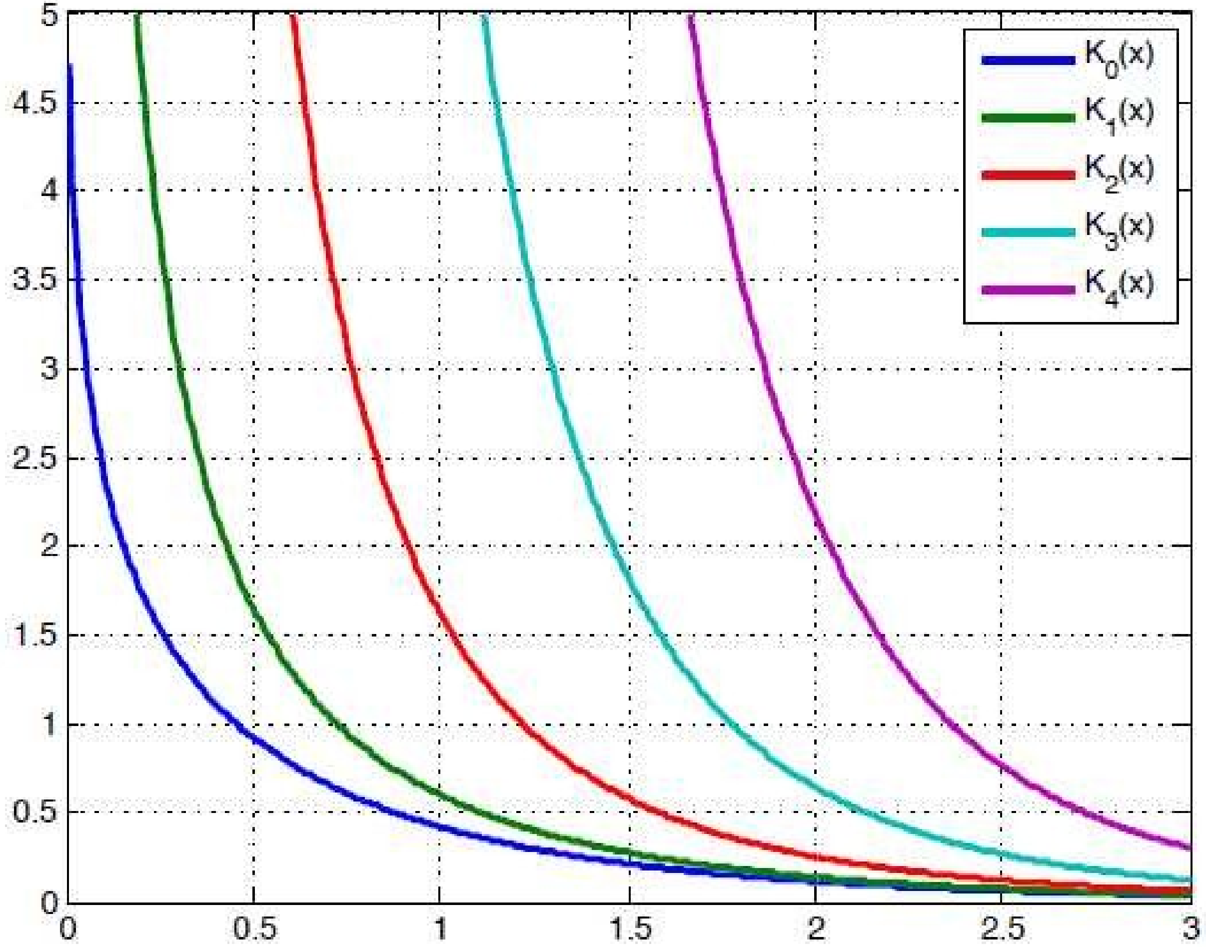
Aside on Bessel Functions

Graphs of $I_n(x)$



Aside on Bessel Functions

Graphs of $K_n(x)$



Aside on Bessel Functions

Summary Points

	Cartesian Coordinates	Cylindrical Coordinates
Standing Waves	$\cos(\beta_x x), \quad \sin(\beta_x x)$	$J_n(\beta_\rho \rho), \quad Y_n(\beta_\rho \rho)$
Traveling Waves	$e^{+j\beta_z z}, \quad e^{-j\beta_z z}$	$H_n^{(1)}(\beta_\rho \rho), \quad H_n^{(2)}(\beta_\rho \rho)$
Evanescent Waves	$e^{+\alpha z}, \quad e^{-\alpha z}$	$I_n(\beta_\rho \rho), \quad K_n(\beta_\rho \rho)$

Circular Waveguide

TM Modes

$$E_z(\rho,\phi,z) = R(\rho)P(\phi)e^{-j\beta_z z}$$

$$R(\rho) = C_o J_{\beta_\phi}(k_c\rho) + D_o Y_{\beta_\phi}(k_c\rho)$$

$$P(\phi) = A_o \cos(\beta_\phi \phi) + B_o \sin(\beta_\phi \phi)$$

$$E_z(\rho,\phi,z) = \left(C_o J_{\beta_\phi}(k_c\rho) + D_o Y_{\beta_\phi}(k_c\rho)\right)\left(A_o \cos(\beta_\phi \phi) + B_o \sin(\beta_\phi \phi)\right)e^{-j\beta_z z}$$

$$k_c^2 = \omega^2 \mu\varepsilon - \beta_z^2$$

Circular Waveguide

TM Modes

$$E_z(\rho,\phi,z) =$$
$$\left(C_o J_{\beta_\phi}(k_c\rho) + D_o Y_{\beta_\phi}(k_c\rho)\right)\left(A_o\cos(\beta_\phi\phi) + B_o\sin(\beta_\phi\phi)\right)e^{-j\beta_z z}$$

$$k_c^2 = \omega^2\mu\varepsilon - \beta_z^2$$

What we don't know

$$\beta_z, \beta_\phi, A_o, B_o, C_o, D_o$$

Circular Waveguide

TM Modes

$$E_z(\rho,\phi,z) =$$
$$\left(C_o J_{\beta_\phi}(k_c\rho) + D_o Y_{\beta_\phi}(k_c\rho)\right)\left(A_o \cos(\beta_\phi \phi) + B_o \sin(\beta_\phi \phi)\right) e^{-j\beta_z z}$$

$$k_c^2 = \omega^2 \mu\varepsilon - \beta_z^2$$

What we don't know

$$\beta_z, \beta_\phi, A_o, B_o, C_o, D_o$$

How do we find them?

Circular Waveguide

TM Modes

$$E_z(\rho,\phi,z) = \left(C_o J_{\beta_\phi}(k_c\rho) + D_o Y_{\beta_\phi}(k_c\rho)\right)\left(A_o \cos(\beta_\phi \phi) + B_o \sin(\beta_\phi \phi)\right) e^{-j\beta_z z}$$

Boundary conditions:

$$E_z(a,\phi,z) = 0$$

What else?

Circular Waveguide

TM Modes

$$E_z(\rho,\phi,z) =$$
$$\left(C_o J_{\beta_\phi}(k_c\rho) + D_o Y_{\beta_\phi}(k_c\rho)\right)\left(A_o \cos(\beta_\phi\phi) + B_o \sin(\beta_\phi\phi)\right)e^{-j\beta_z z}$$

Boundary conditions:

$$E_z(a,\phi,z) = 0$$

$$E_z(0,\phi,z) = finite$$

$$E_z(\rho,\phi,z) = E_z(\rho,\phi+2\pi,z)$$

Circular Waveguide

TM Modes

$$E_z(\rho,\phi,z) =$$
$$\left(C_o J_{\beta_\phi}(k_c\rho) + D_o Y_{\beta_\phi}(k_c\rho)\right)\left(A_o\cos(\beta_\phi\phi) + B_o\sin(\beta_\phi\phi)\right)e^{-j\beta_z z}$$

Boundary conditions:

$$E_z(a,\phi,z) = 0$$

$$E_z(0,\phi,z) = finite \Rightarrow D_o = 0$$

$$E_z(\rho,\phi,z) = E_z(\rho,\phi+2\pi,z) \Rightarrow \beta_\phi = m$$

m=0,1,2,3,….

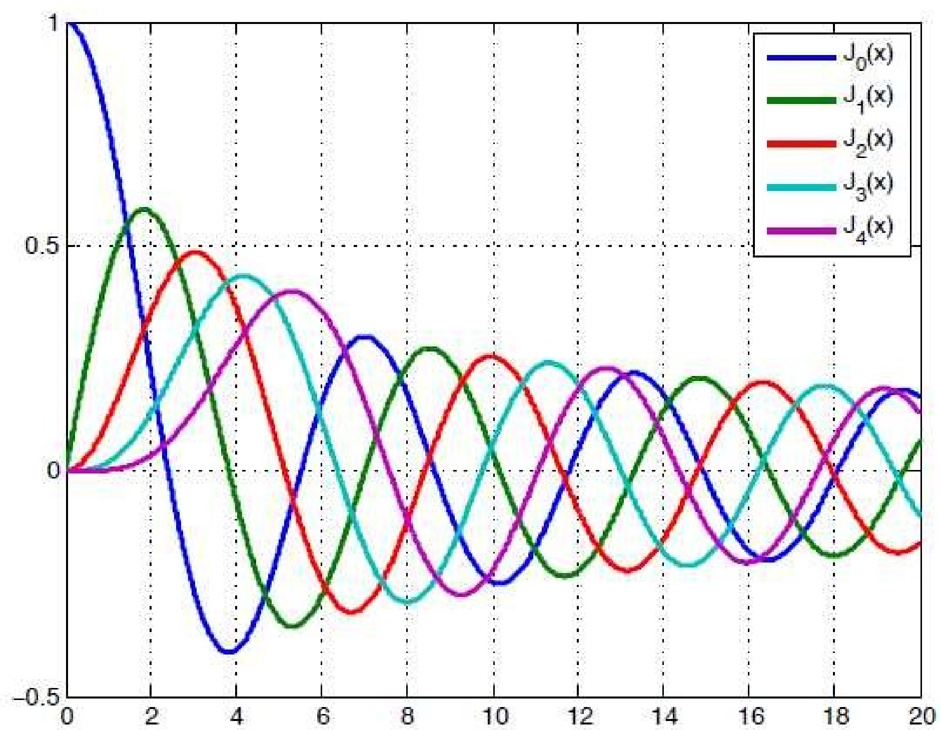
Circular Waveguide

TM Modes

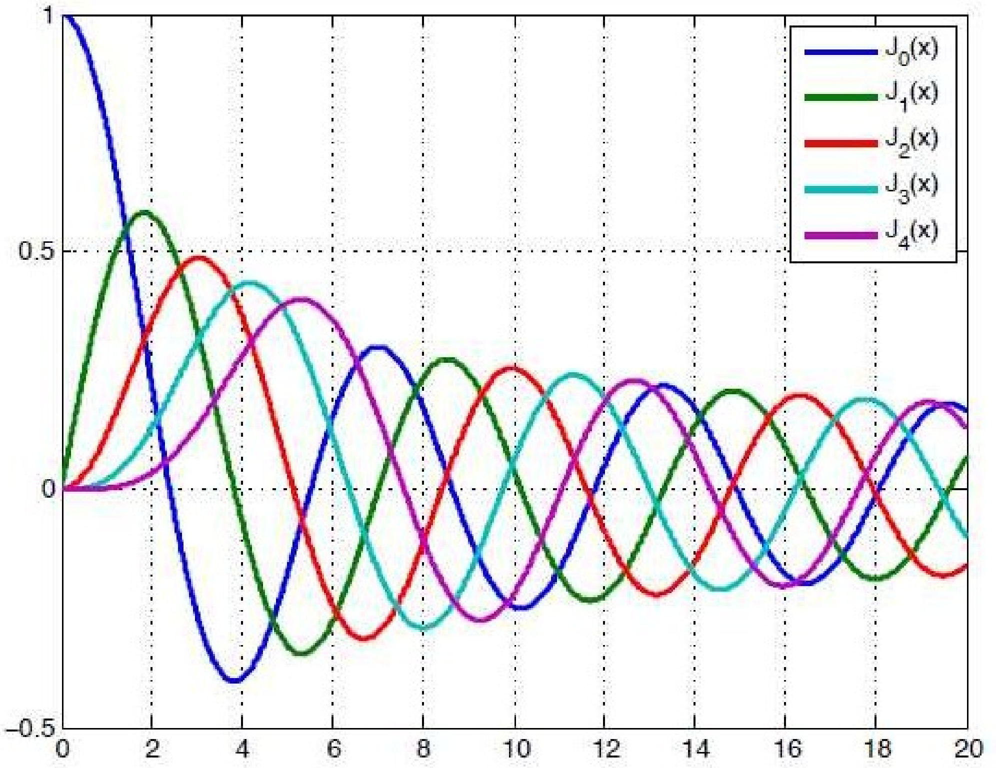
$$E_z(\rho,\phi,z) = \left(A_o\cos(m\phi) + B_o\sin(m\phi)\right)J_m(k_c\rho)e^{-j\beta_z z}$$

$$E_z(a,\phi,z) = 0 \quad \Rightarrow \quad J_m(k_c a) = 0$$

Graphs of $J_n(x)$



Circular Waveguide



	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
$p=1$	2.405	3.832	5.136	6.380	7.588	8.771
$p=2$	5.520	7.016	8.417	9.761	11.065	12.339
$p=3$	8.654	10.173	11.620	13.015	14.732	
$p=4$	11.792	13.324	14.796			

Circular Waveguide

TM Modes

$$E_z(\rho,\phi,z) = \left(A_o\cos(m\phi) + B_o\sin(m\phi)\right)J_m(k_c\rho)e^{-j\beta_z z}$$

$$E_z(a,\phi,z) = 0 \Rightarrow J_m(k_c a) = 0$$

$$\Rightarrow k_c a = \chi_{m,p}$$

$$\Rightarrow k_c = \frac{\chi_{m,p}}{a}$$

$\chi_{m,p}$ denotes the p^th^ root of $J_m(x)$

Circular Waveguide

TM Modes

$$E_z^{m,p}(\rho,\phi,z) = \left(A_o \cos(m\phi) + B_o \sin(m\phi)\right) J_m(k_c\rho) e^{-j\beta_z z}$$

$$k_c = \frac{\chi_{m,p}}{a}$$

$$\beta_z = \sqrt{\omega^2 \mu\varepsilon - k_c^2}$$

$$\beta_z^{m,p} = \sqrt{\omega^2 \mu\varepsilon - \left(\frac{\chi_{m,p}}{a}\right)^2}$$

$\chi_{m,p}$ denotes the p[th] root of $J_m(x)$

Circular Waveguide

TM Modes

$$E_z^{m,p}(\rho,\phi,z) = A_{m,p}\sin(m\phi-\phi_o)J_m\left(\frac{\chi_{m,p}}{a}\rho\right)e^{-j\beta_z z}$$

$$k_c = \frac{\chi_{m,p}}{a}$$

$$\beta_z = \sqrt{\omega^2\mu\varepsilon - k_c^2}$$

$$\beta_z^{m,p} = \sqrt{\omega^2\mu\varepsilon - \left(\frac{\chi_{m,p}}{a}\right)^2}$$

$$H_\rho(\rho,\phi,z) = \frac{j}{k_c^2}\left(\frac{\omega\varepsilon}{\rho}\frac{\partial E_z}{\partial\phi}\right)$$

$$H_\phi(\rho,\phi,z) = \frac{-j}{k_c^2}\left(\omega\varepsilon\frac{\partial E_z}{\partial\rho}\right)$$

$$E_\rho(\rho,\phi,z) = \frac{-j}{k_c^2}\left(\beta_z\frac{\partial E_z}{\partial\rho}\right)$$

$$E_\phi(\rho,\phi,z) = \frac{-j}{k_c^2}\left(\frac{\beta_z}{\rho}\frac{\partial E_z}{\partial\phi}\right)$$

m=0,1,2,3, …

$\chi_{m,p}$ denotes the p[th] root of $J_m(x)$

Circular Waveguide

TM Modes Cutoff Frequencies

$$E_z^{m,p}(\rho,\phi,z) = A_{m,p} \sin(m\phi - \phi_o) J_m\left(\frac{\chi_{m,p}}{a}\rho\right) e^{-j\beta_z z}$$

$$\beta_z^{m,p} = \sqrt{\omega^2 \mu\varepsilon - \left(\frac{\chi_{m,p}}{a}\right)^2}$$

$$f^{m,p} = \frac{1}{2\pi\sqrt{\mu\varepsilon}} \frac{\chi_{m,p}}{a}$$

m=0,1,2,3, …

$\chi_{m,p}$ denotes the p[th] root of $J_m(x)$

Circular Waveguide

We break this into cases as before:

$$E_z = 0 \quad H_z \neq 0$$

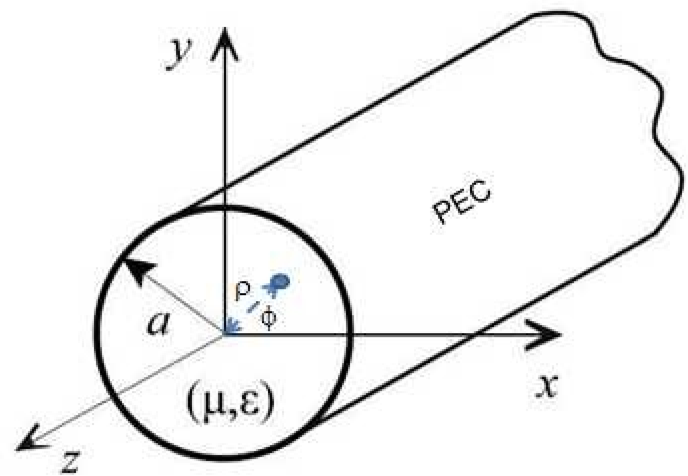
TE Modes

$$H_\rho(\rho,\phi,z) = \frac{-j}{k_c^2}\left(\beta_z \frac{\partial H_z}{\partial \rho}\right)$$

$$H_\phi(\rho,\phi,z) = \frac{j}{k_c^2}\left(\frac{\beta_z}{\rho}\frac{\partial H_z}{\partial \phi}\right)$$

$$E_\rho(\rho,\phi,z) = \frac{j}{k_c^2}\left(\frac{\omega\mu}{\rho}\frac{\partial H_z}{\partial \phi}\right)$$

$$E_\phi(\rho,\phi,z) = \frac{j}{k_c^2}\left(\omega\mu \frac{\partial H_z}{\partial \rho}\right)$$



$$\nabla^2 H_z(\rho,\phi) + k_c{}^2 H_z(\rho,\phi) = 0$$

Circular Waveguide

TE Modes

$$H_z(\rho,\phi,z) =$$
$$\left(C_o J_{\beta_\phi}(k_c\rho) + D_o Y_{\beta_\phi}(k_c\rho)\right)\left(A_o \cos(\beta_\phi \phi) + B_o \sin(\beta_\phi \phi)\right)e^{-j\beta_z z}$$

Boundary conditions:

$$E_\phi(a,\phi,z) = 0$$

$$H_z(0,\phi,z) = finite$$

$$H_z(\rho,\phi,z) = H_z(\rho,\phi + 2\pi, z)$$

Circular Waveguide

TE Modes

$$H_z(\rho,\phi,z) = A_o \sin(m(\phi-\phi_o))J_m(k_c\rho)e^{-j\beta_z z}$$

Boundary conditions:

$$E_\phi(a,\phi,z) = 0$$

$$H_z(0,\phi,z) = finite$$

$$H_z(\rho,\phi,z) = H_z(\rho,\phi+2\pi,z)$$

$$H_\rho(\rho,\phi,z) = \frac{-j}{k_c^2}\left(\beta_z \frac{\partial H_z}{\partial \rho}\right)$$

$$H_\phi(\rho,\phi,z) = \frac{j}{k_c^2}\left(\frac{\beta_z}{\rho}\frac{\partial H_z}{\partial \phi}\right)$$

$$E_\rho(\rho,\phi,z) = \frac{j}{k_c^2}\left(\frac{\omega\mu}{\rho}\frac{\partial H_z}{\partial \phi}\right)$$

$$E_\phi(\rho,\phi,z) = \frac{j}{k_c^2}\left(\omega\mu \frac{\partial H_z}{\partial \rho}\right)$$

Circular Waveguide

TE Modes

$$H_z(\rho,\phi,z) = A_o \sin(m\phi - \phi_o) J_m(k_c\rho) e^{-j\beta_z z}$$

$$E_\phi(\rho,\phi,z) = A_o \frac{j\omega\mu}{k_c} \sin(m\phi - \phi_o) J_m^{'}(k_c\rho) e^{-j\beta_z z}$$

$$E_\rho(\rho,\phi,z) = \frac{j}{k_c^2}\left(\frac{\omega\mu}{\rho}\frac{\partial H_z}{\partial\phi}\right)$$

$$E_\phi(\rho,\phi,z) = \frac{j}{k_c^2}\left(\omega\mu\frac{\partial H_z}{\partial\rho}\right)$$

Circular Waveguide

TE Modes

$$H_z(\rho,\phi,z) = A_o \sin(m\phi - \phi_o) J_m(k_c\rho) e^{-j\beta_z z}$$

$$E_\phi(\rho,\phi,z) = A_o \frac{j\omega\mu}{k_c} \sin(m\phi - \phi_o) J_m^{'}(k_c\rho) e^{-j\beta_z z}$$

$$E_\phi(a,\phi,z) = 0 \quad \Rightarrow \quad J_m^{'}(k_c a) = 0$$

$$\Rightarrow \quad k_c^{m,p} = \frac{\chi_{m,p}^{'}}{a}$$

$\chi_{m,p}^{'}$ denotes the p^th root of $J_m^{'}(x)$

Circular Waveguide

TE Modes

$$H_z(\rho,\phi,z) = A_o \sin(m\phi - \phi_o) J_m(k_c\rho) e^{-j\beta_z z}$$

$$E_\phi(\rho,\phi,z) = A_o \frac{j\omega\mu}{k_c} \sin(m\phi - \phi_o) J_m^{'}(k_c\rho) e^{-j\beta_z z}$$

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
$p=1$	3.832	1.841	3.054	4.201	5.317	6.416
$p=2$	7.016	5.331	6.706	87.015	9.282	10.520
$p=3$	10.173	8.536	9.969	11.346	12.682	13.987
$p=4$	13.324	11.706	13.170			

$\chi_{m,p}^{'}$ denotes the p[th] root of $J_m^{'}(x)$

Circular Waveguide

TE Modes

$$H_z^{m,p}(\rho,\phi,z) = A_o \sin(m\phi - \phi_o) J_m(\frac{\chi'_{m,p}}{a}\rho) e^{-j\beta_z z}$$

$$E_\phi^{m,p}(\rho,\phi,z) = A_o \frac{j\omega\mu}{k_c} \sin(m\phi - \phi_o) J'_m(\frac{\chi'_{m,p}}{a}\rho) e^{-j\beta_z z}$$

$\chi'_{m,p}$ denotes the pth root of $J'_m(x)$

$$\beta_z^{m,p} = \sqrt{\omega^2 \mu\varepsilon - \left(\frac{\chi'_{m,p}}{a}\right)^2}$$

$$f^{m,p} = \frac{1}{2\pi\sqrt{\mu\varepsilon}} \frac{\chi'_{m,p}}{a}$$

Circular Waveguide

TE Modes

$$H_z^{m,p}(\rho,\phi,z) = A_o \sin(m\phi - \phi_o) J_m\left(\frac{\chi'_{m,p}}{a}\rho\right) e^{-j\beta_z z}$$

$$f^{m,p} = \frac{1}{2\pi\sqrt{\mu\varepsilon}} \frac{\chi'_{m,p}}{a}$$

TM Modes

$$E_z^{m,p}(\rho,\phi,z) = A_{m,p} \sin(m\phi - \phi_o) J_m\left(\frac{\chi_{m,p}}{a}\rho\right) e^{-j\beta_z z}$$

$$f^{m,p} = \frac{1}{2\pi\sqrt{\mu\varepsilon}} \frac{\chi_{m,p}}{a}$$

What is the dominant mode?

Circular Waveguide

TE Modes

$$f^{m,p} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\frac{\chi'_{m,p}}{a}$$

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
$p=1$	3.832	1.841	3.054	4.201	5.317	6.416
$p=2$	7.016	5.331	6.706	87.015	9.282	10.520
$p=3$	10.173	8.536	9.969	11.346	12.682	13.987
$p=4$	13.324	11.706	13.170			

TM Modes

$$f^{m,p} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\frac{\chi_{m,p}}{a}$$

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
$p=1$	2.405	3.832	5.136	6.380	7.588	8.771
$p=2$	5.520	7.016	8.417	9.761	11.065	12.339
$p=3$	8.654	10.173	11.620	13.015	14.732	
$p=4$	11.792	13.324	14.796			

Circular Waveguide

TE Modes

$$f^{m,p} = \frac{1}{2\pi\sqrt{\mu\varepsilon}} \frac{\chi'_{m,p}}{a}$$

	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
$p = 1$	3.832		3.054	4.201	5.317	6.416
$p = 2$	7.016	5.331	6.706	87.015	9.282	10.520
$p = 3$	10.173	8.536	9.969	11.346	12.682	13.987
$p = 4$	13.324	11.706	13.170			

TM Modes

$$f^{m,p} = \frac{1}{2\pi\sqrt{\mu\varepsilon}} \frac{\chi_{m,p}}{a}$$

	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
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