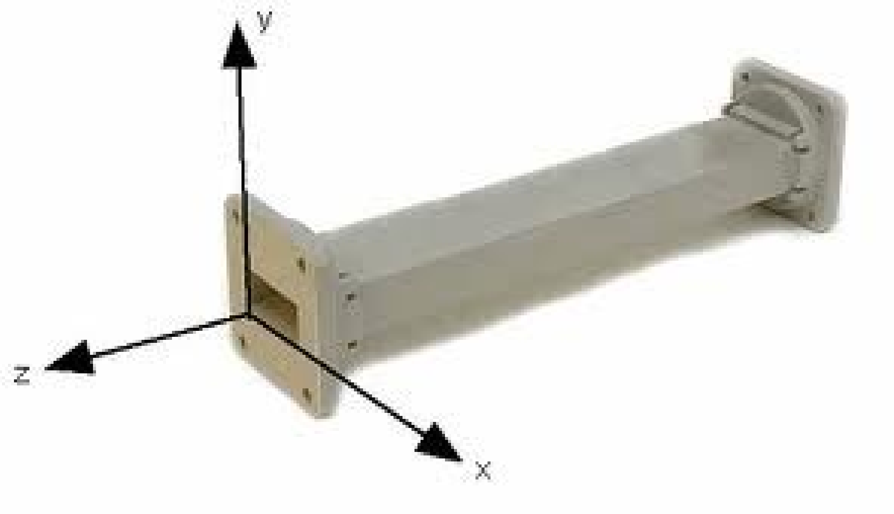
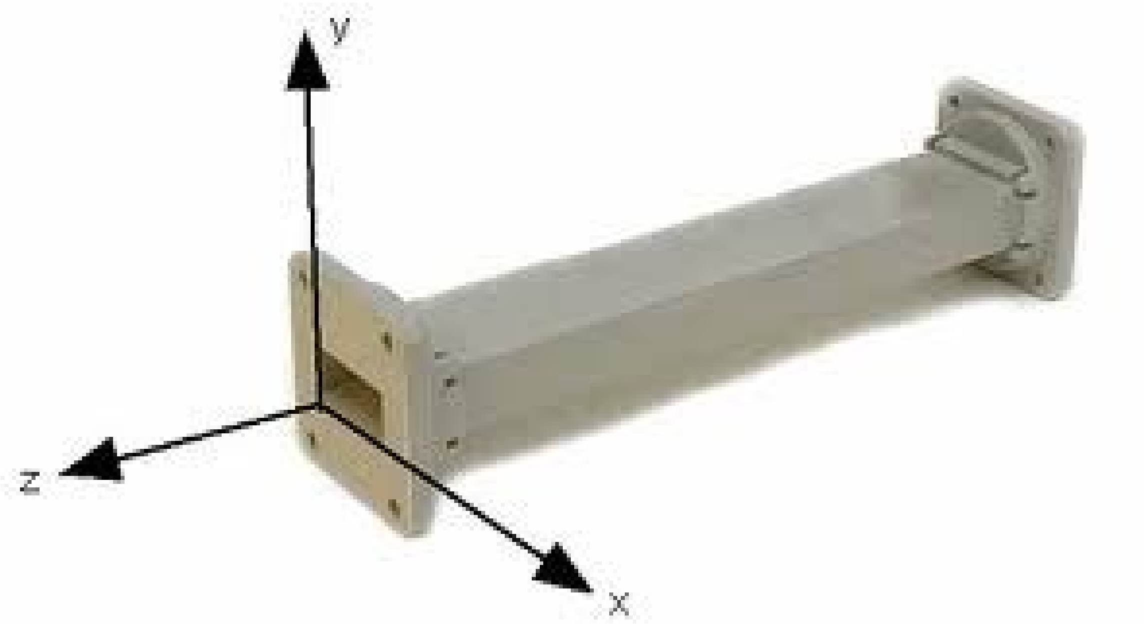
Waveguide Types



Uniform Waveguides



We are interested in finding what electromagnetic field solutions are possible in a uniform infinite waveguide with no sources.

We can always find those solutions by solving:

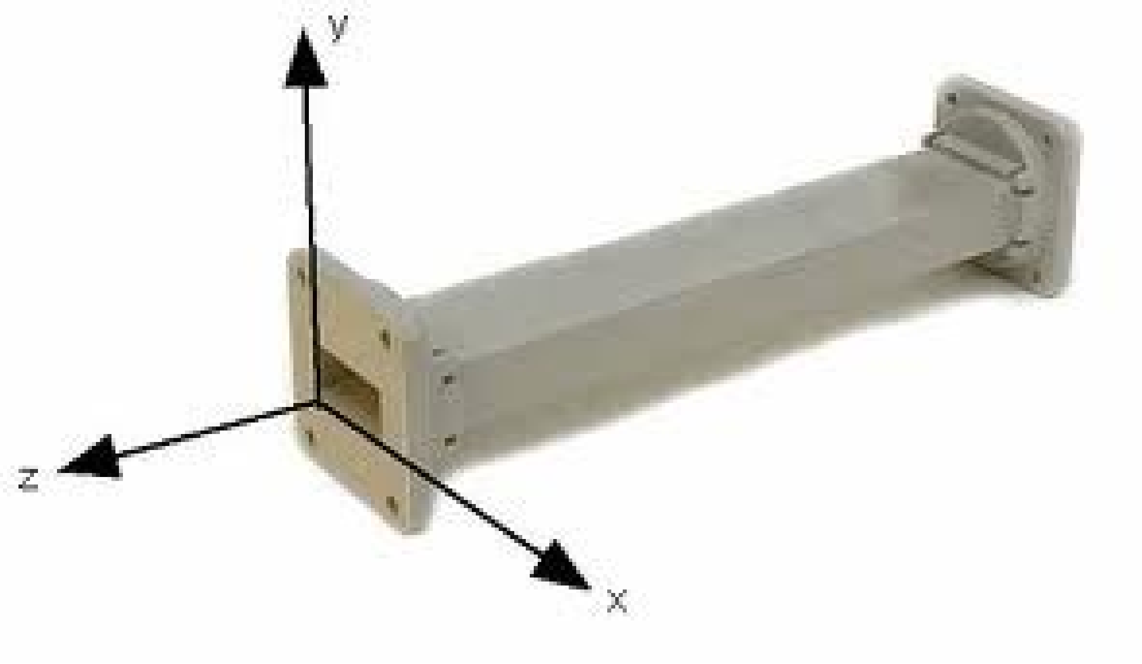
$$\nabla^2 \tilde{E}(x, y, z) + k^2 \tilde{E}(x, y, z) = 0$$

or

$$\nabla^2 \tilde{H}(x, y, z) + k^2 \tilde{H}(x, y, z) = 0$$

Subject to boundary conditions.

Uniform Waveguides



Because the cross section does not change in the z direction

$$\tilde{E}(x,y,z) = \tilde{E}_t(x,y)e^{-j\beta_z z}$$

$$\tilde{H}(x,y,z) = \tilde{H}_t(x,y)e^{-j\beta_z z}$$

Uniform Waveguides

$$E_x = \frac{-j}{(k_c{}^2)}\left[\omega\mu\frac{\partial H_z}{\partial y} + \beta_z\frac{\partial E_z}{\partial x}\right]$$

$$H_x = \frac{j}{(k_c{}^2)}\left[\omega\varepsilon\frac{\partial E_z}{\partial y} - \beta_z\frac{\partial H_z}{\partial x}\right]$$

$$E_y = \frac{j}{(k_c{}^2)}\left[\omega\mu\frac{\partial H_z}{\partial x} - \beta_z\frac{\partial E_z}{\partial y}\right]$$

$$H_y = \frac{-j}{(k_c{}^2)}\left[\omega\varepsilon\frac{\partial E_z}{\partial x} + \beta_z\frac{\partial H_z}{\partial y}\right]$$

Three Cases:

(1) $E_z=0$, $H_z \neq 0$ TE

$$E_x = \frac{-j}{(k_c{}^2)}\left[\omega\mu\frac{\partial H_z}{\partial y}\right]$$

$$E_y = \frac{j}{(k_c{}^2)}\left[\omega\mu\frac{\partial H_z}{\partial x}\right]$$

$$H_x = \frac{-j}{(k_c{}^2)}\left[\beta_z\frac{\partial H_z}{\partial x}\right]$$

$$H_y = \frac{-j}{(k_c{}^2)}\left[\beta_z\frac{\partial H_z}{\partial y}\right]$$

$$\nabla_t{}^2 H_z(x,y) + k_c{}^2 H_z(x,y) = 0$$

(2) $E_z \neq 0$, $H_z = 0$ TM

$$E_x = \frac{-j}{(k_c{}^2)}\left[\beta_z\frac{\partial E_z}{\partial x}\right]$$

$$E_y = \frac{-j}{(k_c{}^2)}\left[\beta_z\frac{\partial E_z}{\partial y}\right]$$

$$H_x = \frac{j}{(k_c{}^2)}\left[\omega\varepsilon\frac{\partial E_z}{\partial y}\right]$$

$$H_y = \frac{-j}{(k_c{}^2)}\left[\omega\varepsilon\frac{\partial E_z}{\partial x}\right]$$

$$\nabla_t{}^2 E_z(x,y) + k_c{}^2 E_z(x,y) = 0$$

Uniform Waveguides

$$E_x = \frac{-j}{(k_c{}^2)}\left[\omega\mu\frac{\partial H_z}{\partial y} + \beta_z\frac{\partial E_z}{\partial x}\right]$$

$$H_x = \frac{j}{(k_c{}^2)}\left[\omega\varepsilon\frac{\partial E_z}{\partial y} - \beta_z\frac{\partial H_z}{\partial x}\right]$$

$$E_y = \frac{j}{(k_c{}^2)}\left[\omega\mu\frac{\partial H_z}{\partial x} - \beta_z\frac{\partial E_z}{\partial y}\right]$$

$$H_y = \frac{-j}{(k_c{}^2)}\left[\omega\varepsilon\frac{\partial E_z}{\partial x} + \beta_z\frac{\partial H_z}{\partial y}\right]$$

Three Cases:

(3) $E_z=0$, $H_z = 0$ TEM

$$k_c = \sqrt{k^2 - \beta_z{}^2} = 0$$

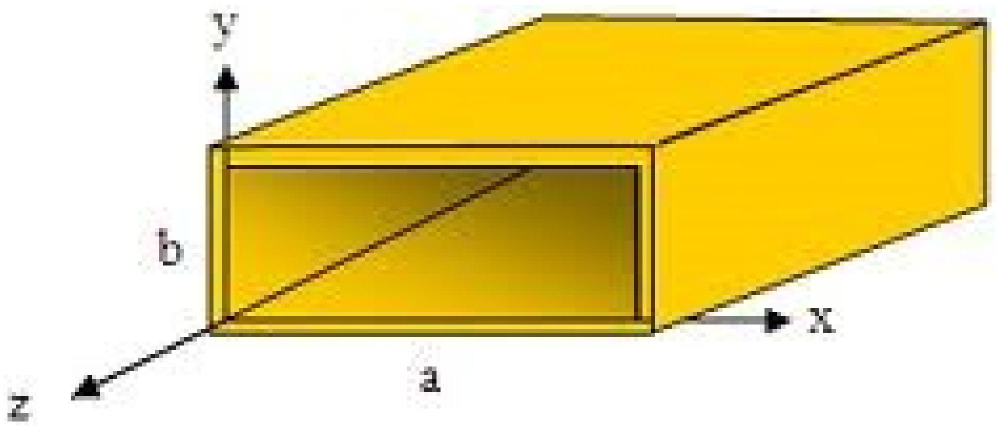
$$\Rightarrow \quad \beta_z = k = \omega\sqrt{\varepsilon\mu}$$

$$\nabla_t{}^2 E_x(x,y) + \overset{=0}{k_c{}^2} E_x(x,y) = 0$$

$$\Rightarrow \quad \nabla_t{}^2 E_x(x,y) = 0$$

$$\nabla_t{}^2 E_y(x,y) = 0$$

Rectangular Waveguides



TM Modes

$$E_x = \frac{-j}{(k_c{}^2)}\left[\beta_z \frac{\partial E_z}{\partial x}\right]$$

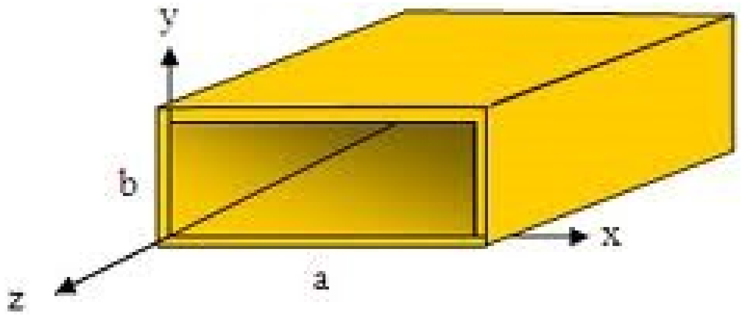
$$E_y = \frac{-j}{(k_c{}^2)}\left[\beta_z \frac{\partial E_z}{\partial y}\right]$$

$$\nabla_t{}^2 E_z(x,y) + k_c{}^2 E_z(x,y) = 0$$

$$H_x = \frac{j}{(k_c{}^2)}\left[\omega\varepsilon \frac{\partial E_z}{\partial y}\right]$$

$$H_y = \frac{-j}{(k_c{}^2)}\left[\omega\varepsilon \frac{\partial E_z}{\partial x}\right]$$

Rectangular Waveguides



TM Modes

$$E_z^{nm}(x,y,z) = A_{mn}\left[sin\left(\frac{n\pi}{a}x\right)\right]\left[sin\left(\frac{m\pi}{b}y\right)\right]e^{-j\beta_z^{mn}z}$$

$$\beta_z^{mn} = \sqrt{\omega^2\mu\varepsilon - \left(\frac{n\pi}{a}\right)^2 - \left(\frac{m\pi}{b}\right)^2}$$

$m \text{ and } n = \pm 1,2,3,\ldots$

$$E_x = \frac{-j}{(k_c{}^2)}\left[\beta_z \frac{\partial E_z}{\partial x}\right]$$

$$H_x = \frac{j}{(k_c{}^2)}\left[\omega\varepsilon \frac{\partial E_z}{\partial y}\right]$$

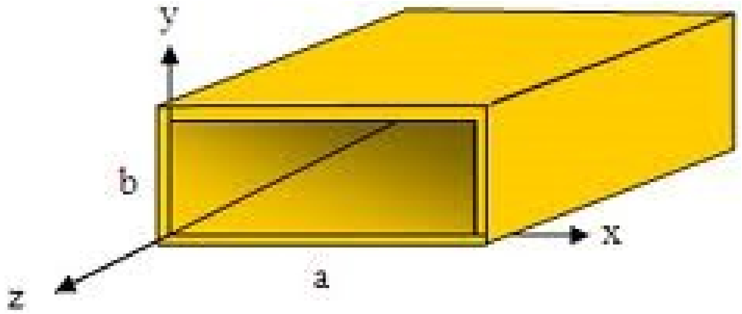
$$E_y = \frac{-j}{(k_c{}^2)}\left[\beta_z \frac{\partial E_z}{\partial y}\right]$$

$$H_y = \frac{-j}{(k_c{}^2)}\left[\omega\varepsilon \frac{\partial E_z}{\partial x}\right]$$

Summary of TM modes

Plane waves in the dielectric medium	Inside the waveguide
$\beta_{PW} = \omega\sqrt{\mu\varepsilon}$	$\beta = \beta_{PW}\sqrt{1-\left[\frac{f_c}{f}\right]^2}$
$\eta_{PW} = \sqrt{\mu / \varepsilon}$	$\eta_{TM} = \eta_{PW}\sqrt{1-\left[\frac{f_c}{f}\right]^2}$
$\nu_p = \omega / \beta_{PW} = f\lambda = 1/\sqrt{\mu\varepsilon} = c$	$\nu_p = \frac{\omega}{\beta_{PW}\sqrt{1-\left[\frac{f_c}{f}\right]^2}}$
$\lambda_{PW} = \frac{c}{f}$	$\lambda = \frac{\lambda_{PW}}{\sqrt{1-\left[\frac{f_c}{f}\right]^2}}$

Rectangular Waveguides



TE Modes

$$H_z = H_o \cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-j\beta_z z}$$

$$E_z = 0$$

$$E_x = \frac{j\omega\mu}{k_c{}^2}\left(\frac{n\pi}{b}\right)H_o \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-\beta_z z}$$

$$E_y = -\frac{j\omega\mu}{k_c{}^2}\left(\frac{m\pi}{a}\right)H_o \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-\beta_z z}$$

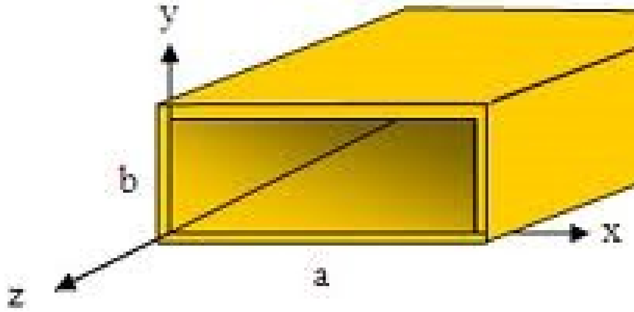
$$H_x = \frac{jk}{k_c{}^2}\left(\frac{m\pi}{a}\right)H_o \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-\beta_z z}$$

$$H_y = \frac{jk}{k_c{}^2}\left(\frac{n\pi}{b}\right)H_o \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-\beta_z z}$$

$$m \ and \ n = \pm 1,2,3,\ldots$$

$$\beta_z{}^2 = \omega^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2$$

Rectangular Waveguides



TE Modes

$$H_z = H_o \cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-j\beta_z z}$$

$$E_z = 0$$

$$E_x = \frac{j\omega\mu}{k_c^{\ 2}}\left(\frac{n\pi}{b}\right)H_o \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-j\beta_z z}$$

$$E_y = -\frac{j\omega\mu}{k_c^{\ 2}}\left(\frac{m\pi}{a}\right)H_o \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-j\beta_z z}$$

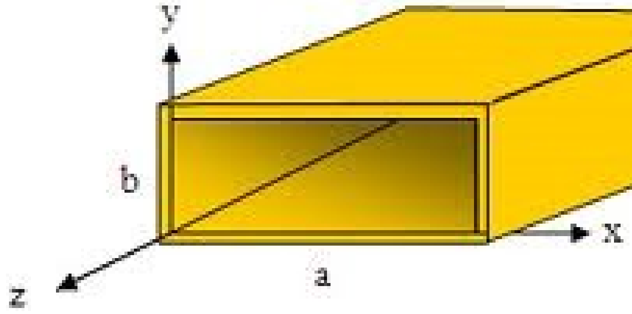
$$H_x = \frac{jk}{k_c^{\ 2}}\left(\frac{m\pi}{a}\right)H_o \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-j\beta_z z}$$

$$H_y = \frac{jk}{k_c^{\ 2}}\left(\frac{n\pi}{b}\right)H_o \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-j\beta_z z}$$

m *and* $n = \pm 1, 2, 3, \ldots$

$$f_c^{nm} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

Rectangular Waveguides



TE Modes   (dominant mode m=1, n=0)

$$H_z = H_o \cos\left(\frac{\pi}{a}x\right)e^{-j\beta_z z}$$

$$f_c^{10} = \frac{1}{2a\sqrt{\mu\varepsilon}} = \frac{c}{2a}$$

$$E_z = 0$$

$$E_x = 0$$

$$E_y = -\frac{j\omega\mu}{k_c{}^2}\left(\frac{\pi}{a}\right)H_o \sin\left(\frac{\pi x}{a}\right)e^{-j\beta_z z}$$

$$H_x = \frac{jk}{k_c{}^2}\left(\frac{\pi}{a}\right)H_o \sin\left(\frac{\pi x}{a}\right)e^{-j\beta_z z}$$

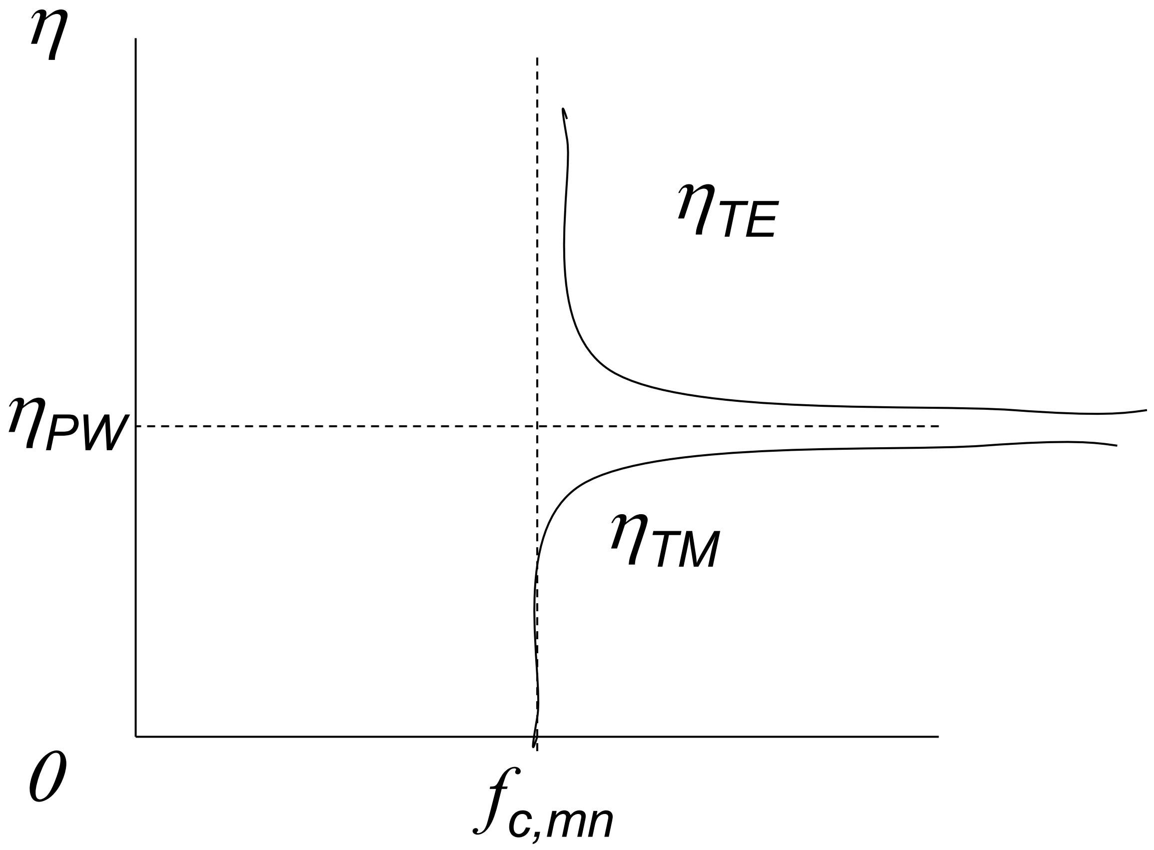
$$H_y = 0$$

Summary of TE modes

Plane waves in the dielectric medium	Inside the waveguide
$\beta_{PW} = \omega\sqrt{\mu\varepsilon}$	$\beta = \beta_{PW}\sqrt{1-\left[\frac{f_c}{f}\right]^2}$
$\eta_{PW} = \sqrt{\mu/\varepsilon}$	$\eta_{TE} = \frac{\eta_{PW}}{\sqrt{1-\left[\frac{f_c}{f}\right]^2}}$
$v_p = \omega/\beta_{PW} = f\lambda = 1/\sqrt{\mu\varepsilon} = c$	$v_p = \frac{\omega}{\beta_{PW}\sqrt{1-\left[\frac{f_c}{f}\right]^2}}$
$\lambda_{PW} = \frac{c}{f}$	$\lambda = \frac{\lambda_{PW}}{\sqrt{1-\left[\frac{f_c}{f}\right]^2}}$

Variation of wave impedance

- Wave impedance varies with frequency and mode



Power transmission

- The average Poynting vector for the waveguide fields is

$$\mathscr{P}_{ave} = \frac{1}{2}\mathrm{Re}\left[E \times H^*\right] = \frac{1}{2}\mathrm{Re}\left[E_x H_y{}^* - E_y H_x{}^*\right]$$

$$= \frac{\left|E_x\right|^2 + \left|E_y\right|^2}{2\eta}\hat{z} \qquad \text{[W/m}^2\text{]}$$

- where $\eta = \eta_{TE}$ or η_{TM} depending on the mode

$$P_{ave} = \int \mathscr{P}_{ave} \cdot dS = \int_{x=0}^{a}\int_{y=0}^{b} \frac{\left|E_x\right|^2 + \left|E_y\right|^2}{2\eta}\, dy\, dx \qquad \text{[W]}$$

Attenuation in Lossy waveguide

- When dielectric inside guide is lossy, and walls are not perfect conductors, power is lost as it travels along guide.

$$P_{ave} = P_o e^{-2\alpha z}$$

- The lost power is $P_L = -\frac{dP_{ave}}{dz} = 2\alpha P_{ave}$

- Where $\alpha = \alpha_c + \alpha_d$ are the attenuation due to ohmic (conduction) and dielectric losses

- Usually $\alpha_c >> \alpha_d$

Attenuation for TE_{10}

- Dielectric attenuation, Np/m

$$\alpha_d = -\frac{\sigma \eta_{PW}}{2\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

Dielectric conductivity!

Attenuation for TE_{10}

Conductor attenuation, Np/m

If there are losses in the conductor, our solution is not strictly correct since the boundary condition has changed.

Nonetheless, we can use our current solution with the perturbation theory we discussed for transmission lines.

$$P_L = -\frac{dP_{ave}}{dz} = 2\alpha P_{ave} \quad \Longrightarrow \quad \alpha = \frac{P_L}{2P_{ave}}$$

Attenuation for TE$_{10}$

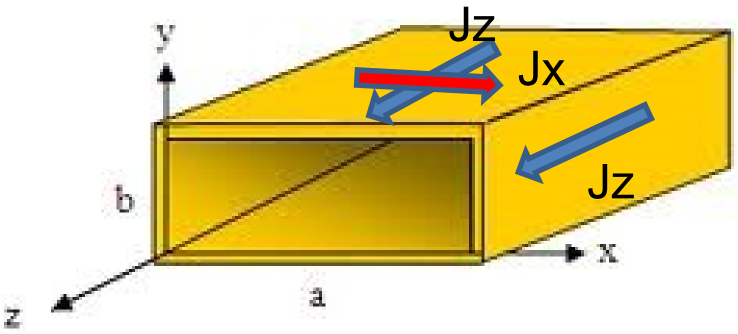
Conductor attenuation, Np/m

How do we find the power lost?

$$\alpha = \frac{P_L}{2P_{ave}}$$

$$H_z = H_o \cos\left(\frac{\pi}{b} y\right) e^{-j\beta_z z}$$

$$H_x = \frac{jk}{k_c^{\,2}}\left(\frac{\pi}{a}\right) H_o \sin\left(\frac{\pi x}{a}\right) e^{-j\beta_z z}$$



Magnetic fields on the walls create surface currents.
If the walls have a resistance they will absorb energy.

Attenuation for TE$_{10}$

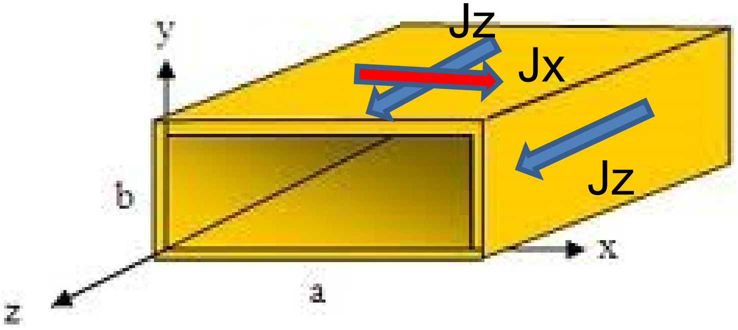
Conductor attenuation, Np/m

How do we find the power lost?

$$\alpha = \frac{P_L}{2P_{ave}}$$

$$H_z = H_o \cos\left(\frac{\pi}{a}x\right)e^{-j\beta_z z}$$

$$H_x = \frac{jk}{k_c^{\,2}}\left(\frac{\pi}{a}\right)H_o \sin\left(\frac{\pi x}{a}\right)e^{-j\beta_z z}$$



$$J_x = H_x = \frac{jk}{k_c^{\,2}}\left(\frac{\pi}{a}\right)H_o \sin\left(\frac{\pi x}{a}\right)$$

$$J_z = H_z = H_o \cos\left(\frac{\pi}{a}x\right)$$

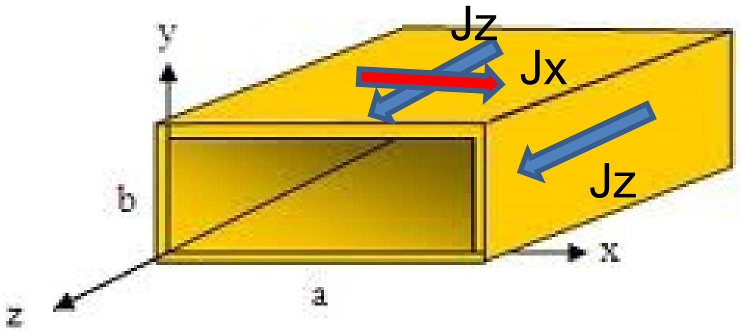
y=0 and y=b

$$J_z = H_z = H_o$$

x=0 and x=a

Attenuation for TE$_{10}$

Conductor attenuation, Np/m



y=0 and y=b

$$J_x = H_x = \frac{jk}{k_c^{\,2}}\left(\frac{\pi}{a}\right)H_o \sin\left(\frac{\pi x}{a}\right)$$

$$J_z = H_z = H_o \cos\left(\frac{\pi}{a}x\right)$$

$$P_d = \frac{1}{2}\int_{x=0}^{x=a} R_s\left[\left|J_x\right|^2 + \left|J_z\right|^2\right]dx = \frac{R_s}{2}\int_{x=0}^{x=a}\left[\left(\frac{k}{k_c^{\,2}}\left(\frac{\pi}{a}\right)H_o \sin\left(\frac{\pi x}{a}\right)\right)^2 + \left(H_o \sin\left(\frac{\pi x}{a}\right)^2\right)\right]dx$$

x=0 and x=a

$$J_z = H_z = H_o$$

$$P_d = \frac{1}{2}\int_{x=0}^{x=a} R_s\left[\left|J_z\right|^2\right]dx = \frac{R_s}{2}\int_{x=0}^{x=a} H_o^{\,2}\, dx$$

Where the surface resistance is

$$R_s = \sqrt{\frac{\omega\mu}{2\sigma}}$$

Attenuation for TE$_{10}$

Conductor attenuation, Np/m

y=0 and y=b

$$P_d^{\,y=0,y=b} = \frac{1}{2}\int_{x=0}^{x=a} R_s\left[\left|J_x\right|^2 + \left|J_z\right|^2\right]dx = \frac{R_s}{2}\int_{x=0}^{x=a}\left[\left(\frac{k}{k_c^{\,2}}\left(\frac{\pi}{a}\right)H_o\sin\left(\frac{\pi x}{a}\right)\right)^2 + \left(H_o\sin\left(\frac{\pi x}{a}\right)^2\right)\right]dx$$

x=0 and x=a

$$P_d^{\,x=0,x=a} = \frac{1}{2}\int_{y=0}^{y=b} R_s\left[\left|J_z\right|^2\right]dy = \frac{R_s}{2}\int_{y=0}^{y=b} H_o^{\,2}\,dy$$

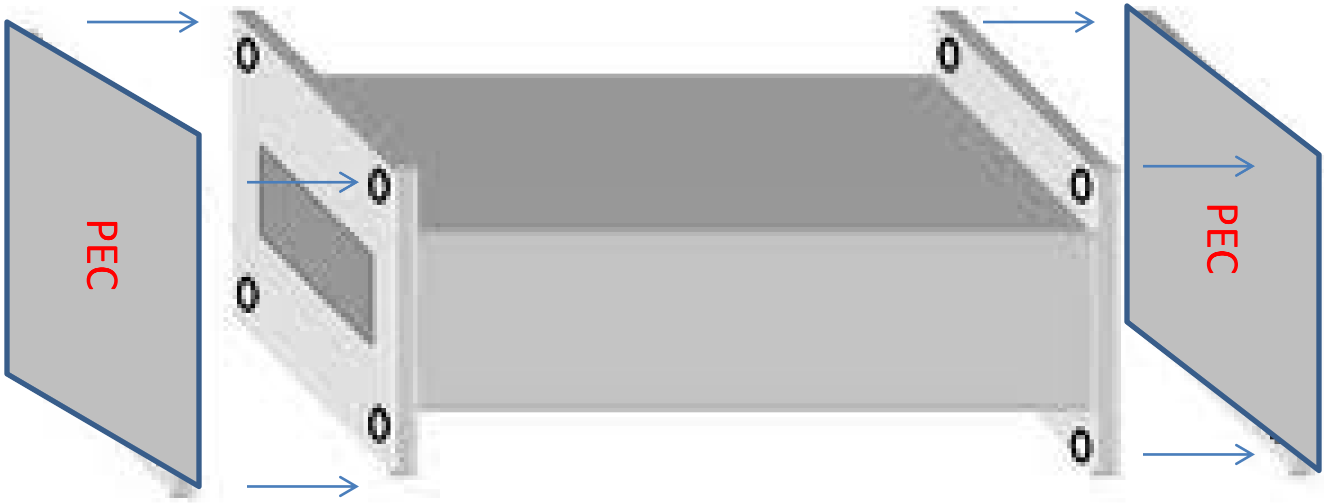
$$P_{ave} = \int \mathcal{P}_{ave}\cdot dS = \int_{x=0}^{a}\int_{y=0}^{b}\frac{\left|E_x\right|^2 + \left|E_y\right|^2}{2\eta}\,dy\,dx$$

$$\alpha = \frac{P_L}{2P_{ave}} = \frac{P_d^{\,y=0,y=b} + P_d^{\,x=0,x=a}}{2P_{ave}}$$

⇨

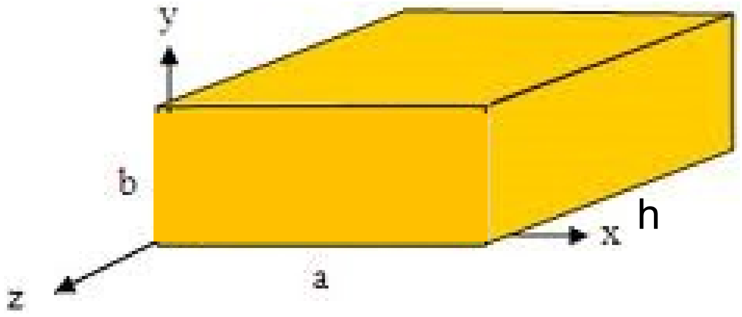
$$\alpha = \frac{2R_s}{b\eta\sqrt{1-\left(\frac{f_c}{f}\right)^2}}\left(1+\frac{2b}{a}\left(\frac{f_c}{f}\right)^2\right)$$

Rectangular Cavity Resonator



We can form a rectangular cavity by placing metal caps on two ends of a rectangular waveguide.

Rectangular Cavity Resonator

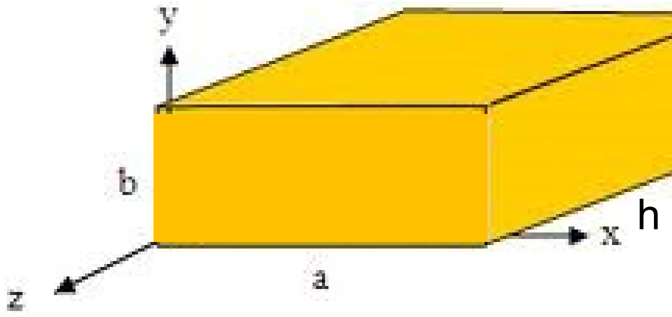


We are interested in finding what electromagnetic field solutions are possible in a resonant cavity with no sources.

We can always find those solutions by solving:

$$\nabla^2 \tilde{E}(x,y,z) + k^2 \tilde{E}(x,y,z) = 0$$
$$\nabla^2 \tilde{H}(x,y,z) + k^2 \tilde{H}(x,y,z) = 0$$

Rectangular Cavity



We can solve these!

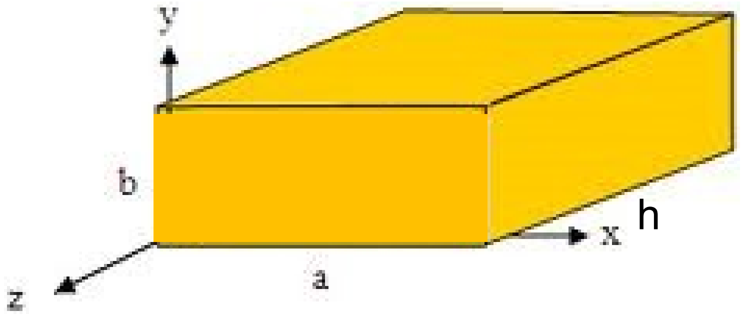
$$\frac{\partial^2}{\partial x^2}E_x(x,y,z)+\frac{\partial^2}{\partial y^2}E_x(x,y,z)+\frac{\partial^2}{\partial z^2}E_x(x,y,z)+k^2E_x(x,y,z)=0$$

$$\frac{\partial^2}{\partial x^2}E_y(x,y,z)+\frac{\partial^2}{\partial y^2}E_y(x,y,z)+\frac{\partial^2}{\partial z^2}E_y(x,y,z)+k^2E_y(x,y,z)=0$$

$$\frac{\partial^2}{\partial x^2}E_z(x,y,z)+\frac{\partial^2}{\partial y^2}E_z(x,y,z)+\frac{\partial^2}{\partial z^2}E_z(x,y,z)+k^2E_z(x,y,z)=0$$

Subject to boundary conditions.

Rectangular Cavity



TM Modes in Z or TM$_z$

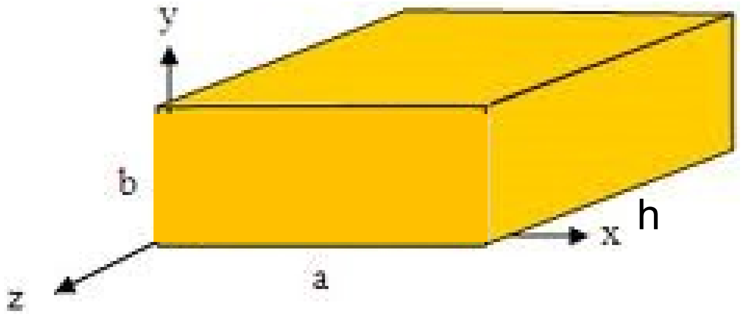
$$\frac{\partial^2}{\partial x^2}E_z(x,y,z)+\frac{\partial^2}{\partial y^2}E_z(x,y,z)+\frac{\partial^2}{\partial z^2}E_z(x,y,z)+k^2E_z(x,y,z)=0$$

Solve using separation of variables

$$E_z(x,y,z)=X(x)Y(y)Z(z)$$

$$\frac{X''(x)}{X(x)}+\frac{Y''(y)}{Y(y)}+\frac{Z''(z)}{Z(z)}+k^2=0$$

Rectangular Cavity



TM Modes in Z

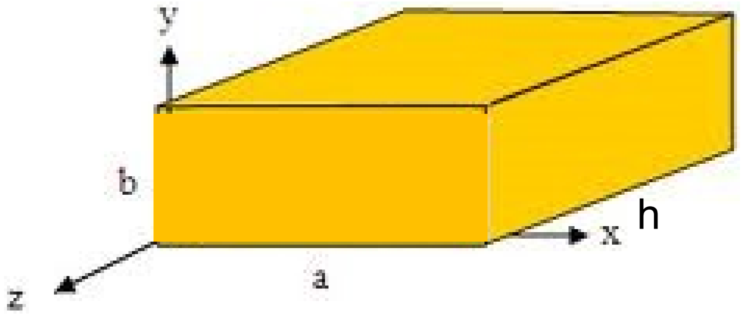
$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} + k^2 = 0$$

$$X(x) = A_1 \cos(\beta_x x) + A_2 \sin(\beta_x x)$$

$$Y(y) = B_1 \cos(\beta_y y) + B_2 \sin(\beta_y y)$$

$$Z(z) = C_1 \cos(\beta_z z) + C_2 \sin(\beta_z z)$$

Rectangular Cavity



TM Modes in Z

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} + k^2 = 0$$

$$X(x) = A_1 \cos(\beta_x x) + A_2 \sin(\beta_x x)$$

$$Y(y) = B_1 \cos(\beta_y y) + B_2 \sin(\beta_y y)$$

$$Z(z) = C_1 \cos(\beta_z z) + C_2 \sin(\beta_z z)$$

$$k^2 = \beta_x^{\ 2} + \beta_y^{\ 2} + \beta_z^{\ 2}$$

TM$_{mnp}$ Boundary Conditions

From these, we conclude:

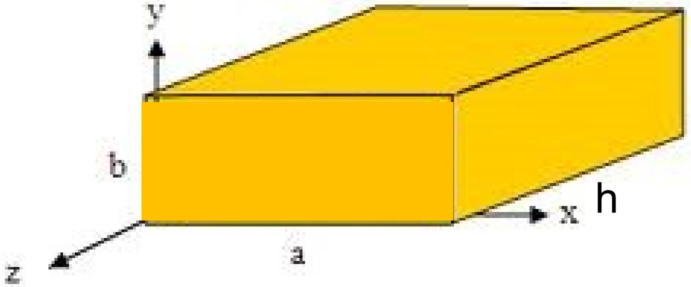
$$E_z = 0 \text{ at } y = 0, b$$
$$E_z = 0 \text{ at } x = 0, a$$
$$E_y = E_x = 0, \text{ at } z = 0, h$$

$$\beta_x = m\pi/a$$
$$\beta_y = n\pi/b$$
$$\beta_z = p\pi/h$$

$$E_z = E_o \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{p\pi z}{h}\right)$$

where

$$k^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{h}\right)^2 = \omega^2\mu\varepsilon$$



Resonant frequency

- The resonant frequency is the same for TM or TE modes, except that the lowest-order TM is TM_{111} and the lowest-order in TE is TE_{101}.

$$f_r = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{p}{h}\right)^2}$$

Cavity TE Mode to z

Solving by Separation of Variables :

$H_z(x, y, z) = X(x)Y(y)Z(z)$

from where we obtain :

$$X(x) = A_1 \cos \beta_x x + A_2 \sin \beta_x x$$

$$Y(y) = B_1 \cos \beta_y y + B_2 \sin \beta_y y$$

$$Z(z) = C_1 \cos \beta_z z + C_2 \sin \beta_z z$$

$$where \quad k^2 = \beta_x^2 + \beta_y^2 + \beta_z{}^2$$

TE$_{mnp}$ Boundary Conditions

$$H_z = 0 \text{ at } z = 0, h$$

$$E_y = 0 \text{ at } x = 0, a$$

$$E_x = 0, \text{ at } y = 0, b$$

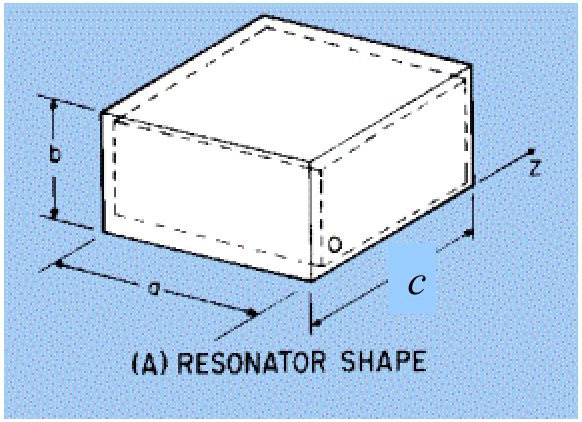
From these, we conclude:

$\beta_x = m\pi/a$

$\beta_y = n\pi/b$

$\beta_z = p\pi/h$

$$H_z = H_o \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\sin\left(\frac{p\pi y}{h}\right)$$



(A) RESONATOR SHAPE

Quality Factor, Q

- The cavity has walls with finite conductivity and is therefore losing stored energy.
- The quality factor, *Q*, characterized the loss and also the bandwidth of the cavity resonator.
- Dielectric cavities are used for resonators, amplifiers and oscillators at microwave frequencies.

Quality Factor, Q

- Is defined as

$$Q = 2\pi \frac{Time\ average\ energy\ stored}{loss\ energy\ per\ cycle\ of\ oscillation}$$

$$= 2\pi \frac{W}{P_L}$$

For the dominant mode TE_{101}

$$Q_{TE_{101}} = \frac{\left(a^2 + h^2\right)abh}{\delta\left[2b\left(a^3 + h^3\right) + ah\left(a^2 + h^2\right)\right]}$$

where

$$\delta = \frac{1}{\sqrt{\pi f_{101} \mu_o \sigma_c}}$$

Example

For a cavity of dimensions; 3cm x 2cm x 7cm filled with air and made of copper (σ_c=5.8 x 10^7)

- Find the resonant frequency and the quality factor for the dominant mode.

Answer:

$$f_{r110} = \frac{3 \cdot 10^{10}}{2}\sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{0}{7}\right)^2} = 9GHz$$

$$f_r = \frac{3 \cdot 10^{10}}{2}\sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{0}{2}\right)^2 + \left(\frac{1}{7}\right)^2} = 5.44GHz$$

$$\delta = \frac{1}{\sqrt{(5.44 \cdot 10^9)\mu_o \sigma_c}} = 1.6 \cdot 10^{-6}$$

$$Q_{TE_{101}} = \frac{\left(3^2 + 7^2\right)3 \cdot 2 \cdot 7}{\delta\left[2 \cdot 2\left(3^3 + 7^3\right) + 3 \cdot 7\left(3^2 + 7^2\right)\right]} = 568{,}378$$