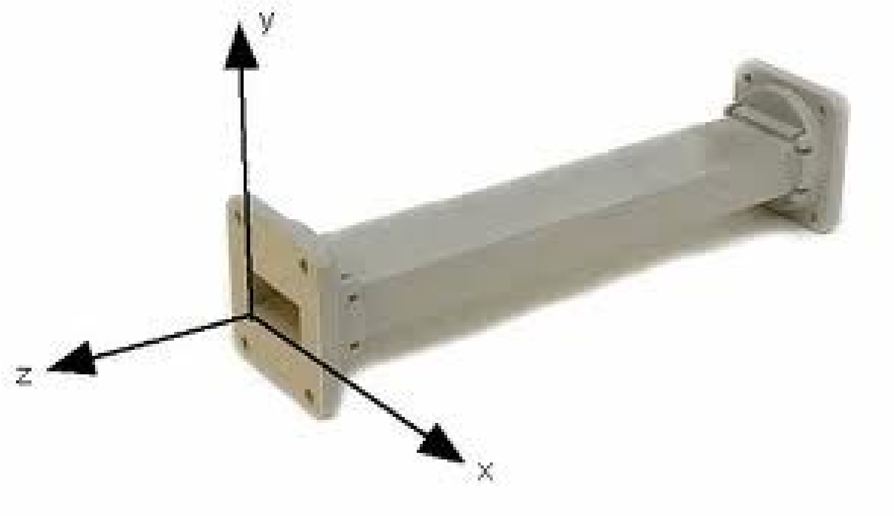
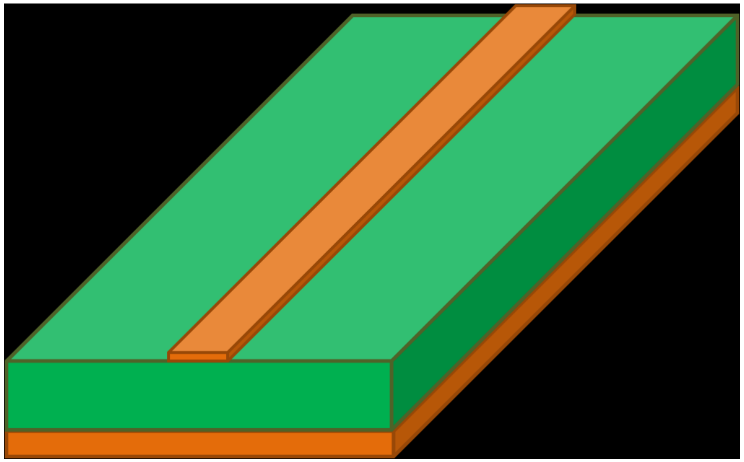
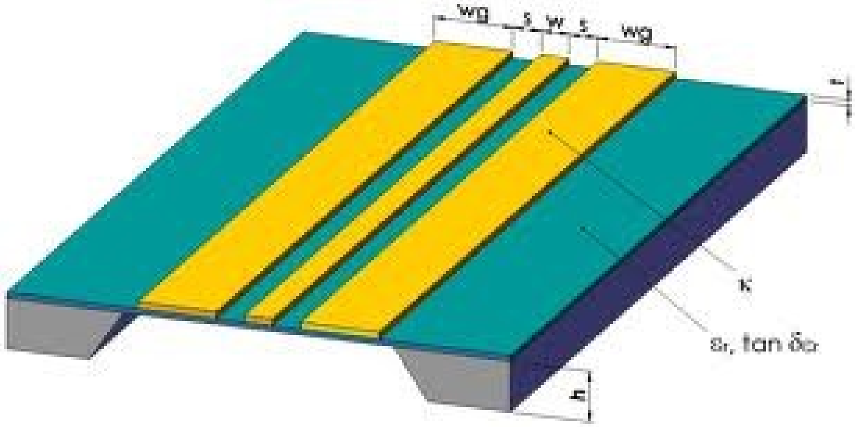
Waveguide Types

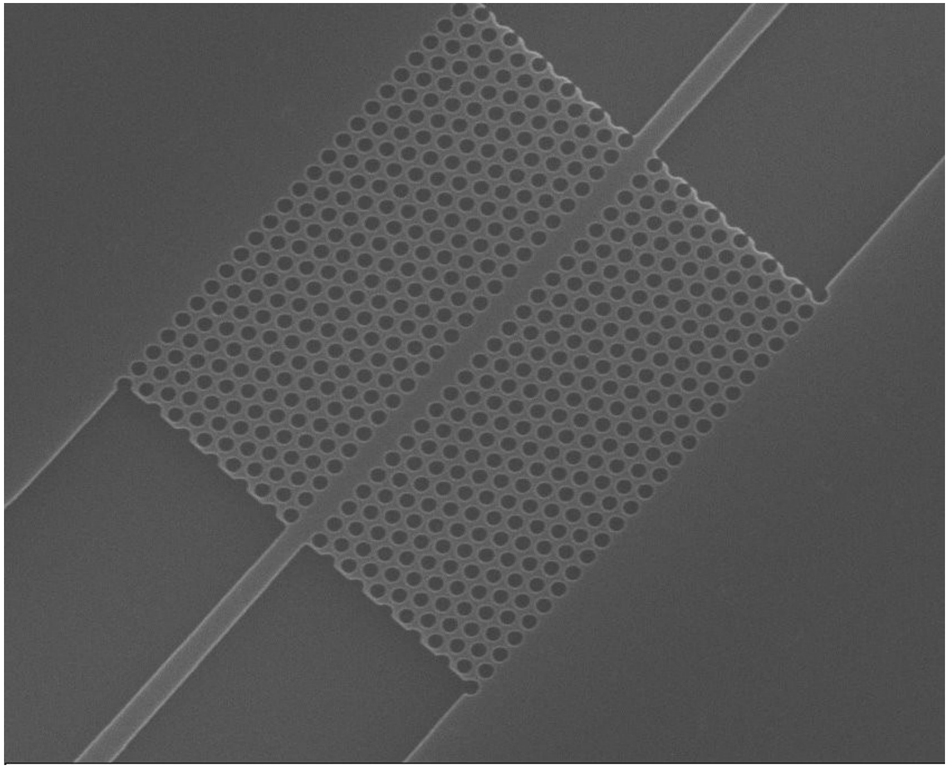


Waveguide Types

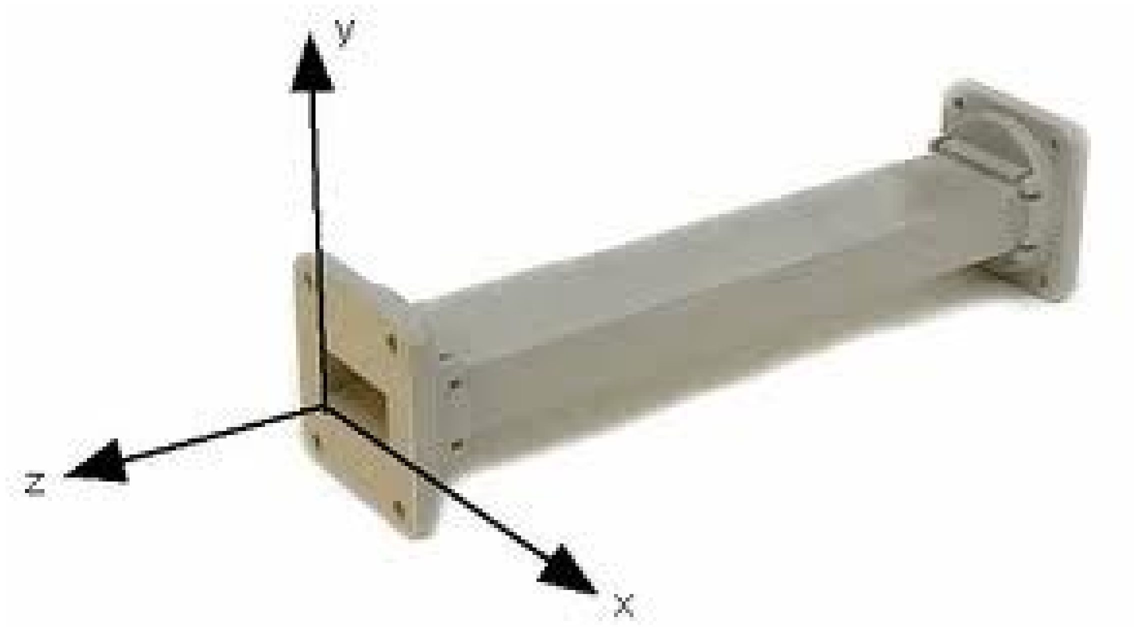




Waveguide Types



Uniform Waveguides



We are interested in finding what electromagnetic field solutions are possible in a uniform infinite waveguide with no sources.

We can always find those solutions by solving:

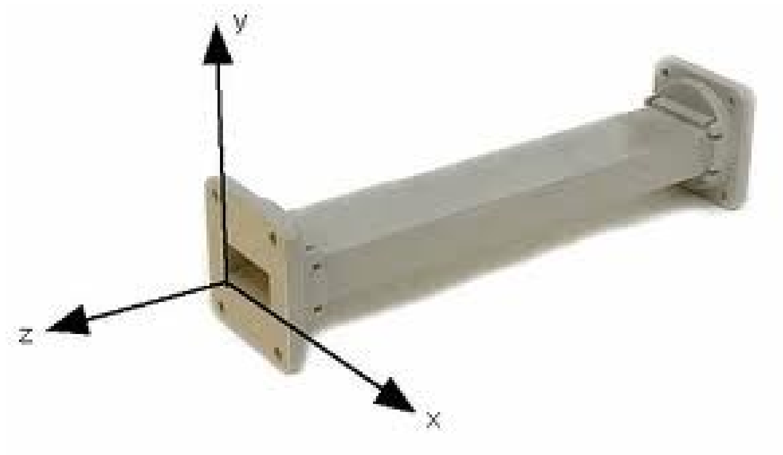
$$\nabla^2 \tilde{E}(x,y,z) + k^2 \tilde{E}(x,y,z) = 0$$

or

$$\nabla^2 \tilde{H}(x,y,z) + k^2 \tilde{H}(x,y,z) = 0$$

Subject to boundary conditions.

Uniform Waveguides



This is really three PDEs

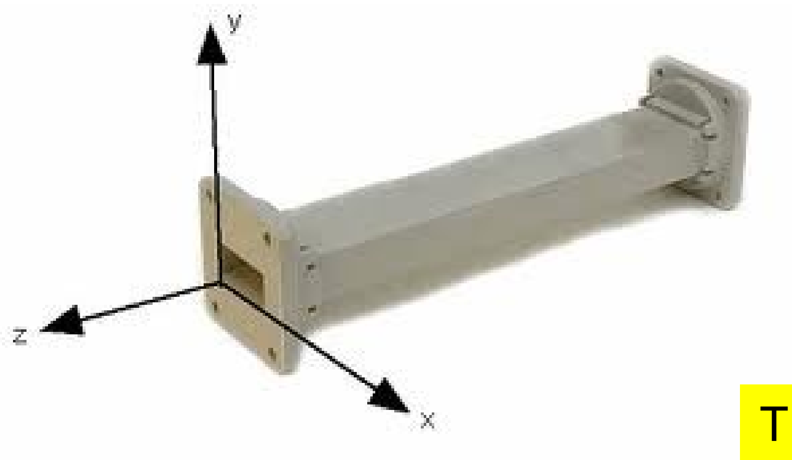
$$\frac{\partial^2}{\partial x^2}E_x(x,y,z)+\frac{\partial^2}{\partial y^2}E_x(x,y,z)+\frac{\partial^2}{\partial z^2}E_x(x,y,z)+k^2E_x(x,y,z)=0$$

$$\frac{\partial^2}{\partial x^2}E_y(x,y,z)+\frac{\partial^2}{\partial y^2}E_y(x,y,z)+\frac{\partial^2}{\partial z^2}E_y(x,y,z)+k^2E_y(x,y,z)=0$$

$$\frac{\partial^2}{\partial x^2}E_z(x,y,z)+\frac{\partial^2}{\partial y^2}E_z(x,y,z)+\frac{\partial^2}{\partial z^2}E_z(x,y,z)+k^2E_z(x,y,z)=0$$

Subject to boundary conditions.

Uniform Waveguides



That's a lot of work!
Are there any shortcuts?

This is really three PDEs

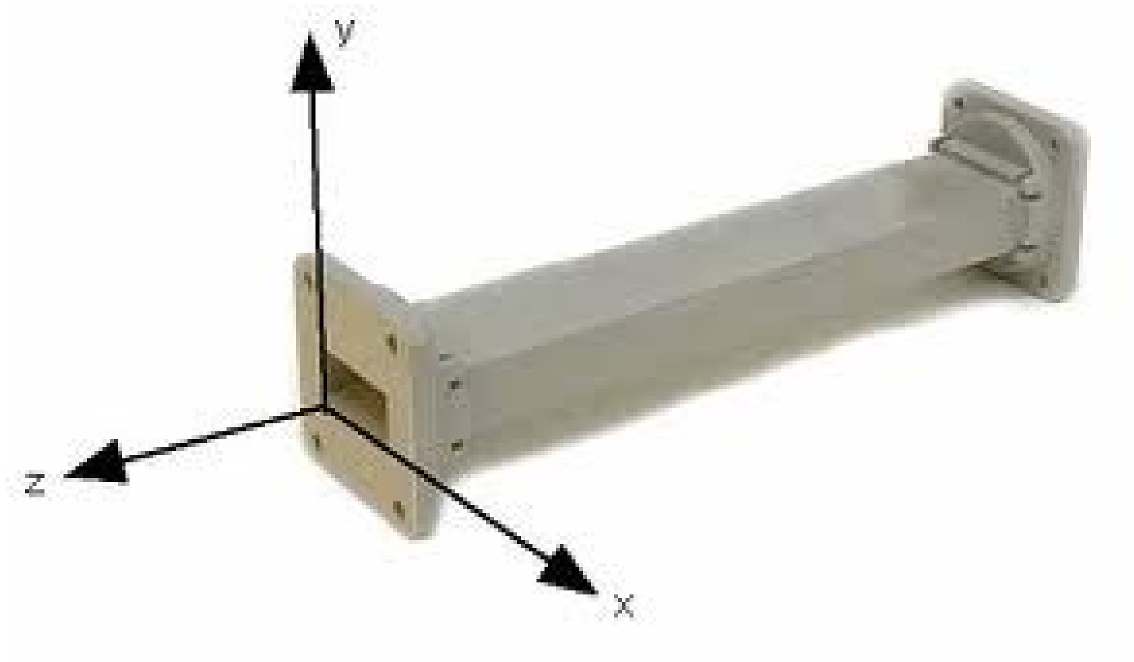
$$\frac{\partial^2}{\partial x^2}E_x(x,y,z)+\frac{\partial^2}{\partial y^2}E_x(x,y,z)+\frac{\partial^2}{\partial z^2}E_x(x,y,z)+k^2E_x(x,y,z)=0$$

$$\frac{\partial^2}{\partial x^2}E_y(x,y,z)+\frac{\partial^2}{\partial y^2}E_y(x,y,z)+\frac{\partial^2}{\partial z^2}E_y(x,y,z)+k^2E_y(x,y,z)=0$$

$$\frac{\partial^2}{\partial x^2}E_z(x,y,z)+\frac{\partial^2}{\partial y^2}E_z(x,y,z)+\frac{\partial^2}{\partial z^2}E_z(x,y,z)+k^2E_z(x,y,z)=0$$

Subject to boundary conditions.

Uniform Waveguides

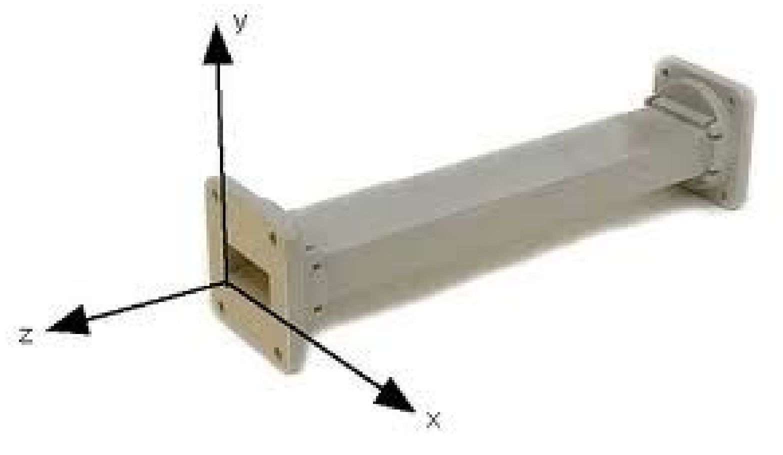


Because the cross section does not change in the z direction

$$\tilde{E}(x,y,z) = \tilde{E}_t(x,y)e^{-j\beta_z z}$$

$$\tilde{H}(x,y,z) = \tilde{H}_t(x,y)e^{-j\beta_z z}$$

Uniform Waveguides



This is really three PDEs

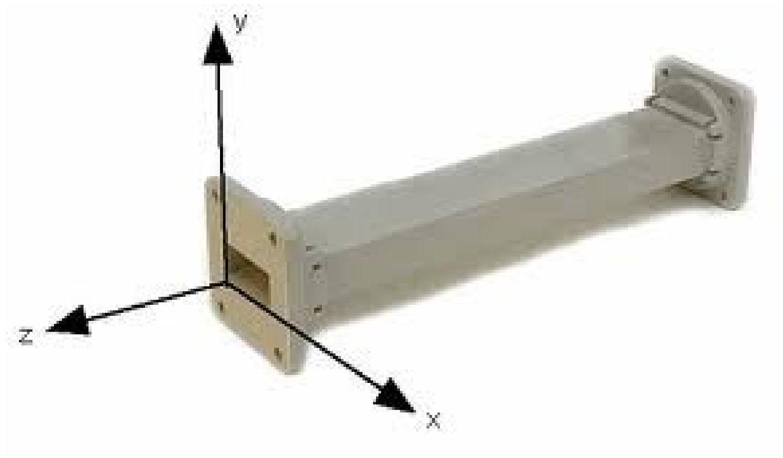
$$\frac{\partial^2}{\partial x^2}E_{tx}(x,y)e^{-j\beta_z z}+\frac{\partial^2}{\partial y^2}E_{tx}(x,y)e^{-j\beta_z z}+\frac{\partial^2}{\partial z^2}E_{tx}(x,y)e^{-j\beta_z z}+k^2E_{tx}(x,y)e^{-j\beta_z z}=0$$

$$\frac{\partial^2}{\partial x^2}E_{ty}(x,y)e^{-j\beta_z z}+\frac{\partial^2}{\partial y^2}E_{ty}(x,y)e^{-j\beta_z z}+\frac{\partial^2}{\partial z^2}E_{ty}(x,y)e^{-j\beta_z z}+k^2E_{ty}(x,y)e^{-j\beta_z z}=0$$

$$\frac{\partial^2}{\partial x^2}E_{tz}(x,y)e^{-j\beta_z z}+\frac{\partial^2}{\partial y^2}E_{tz}(x,y)e^{-j\beta_z z}+\frac{\partial^2}{\partial z^2}E_{tz}(x,y)e^{-j\beta_z z}+k^2E_{tz}(x,y)e^{-j\beta_z z}=0$$

Subject to boundary conditions.

Uniform Waveguides



This is really three PDEs

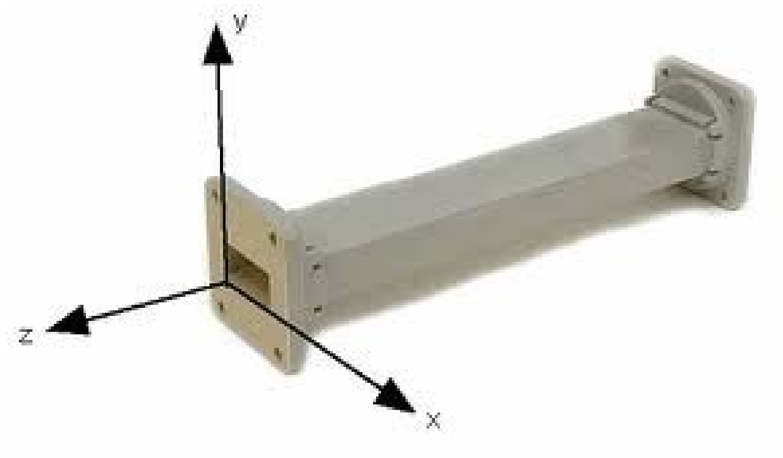
$$\frac{\partial^2}{\partial x^2}E_{tx}(x,y) + \frac{\partial^2}{\partial y^2}E_{tx}(x,y) + \left(k^2 - {\beta_z}^2\right)E_{tx}(x,y) = 0$$

$$\frac{\partial^2}{\partial x^2}E_{ty}(x,y) + \frac{\partial^2}{\partial y^2}E_{ty}(x,y) + \left(k^2 - {\beta_z}^2\right)E_{ty}(x,y) = 0$$

$$\frac{\partial^2}{\partial x^2}E_{tz}(x,y) + \frac{\partial^2}{\partial y^2}E_{tz}(x,y) + \left(k^2 - {\beta_z}^2\right)E_{tz}(x,y) = 0$$

Subject to boundary conditions.

Uniform Waveguides



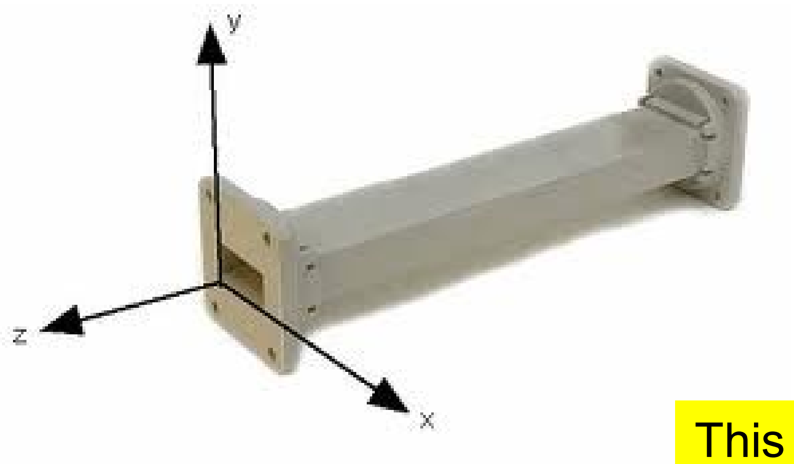
This is really three PDEs

$$\nabla_t^{\,2} E_{tx}(x,y) + k_c^{\,2} E_{tx}(x,y) = 0$$
$$\nabla_t^{\,2} E_{ty}(x,y) + k_c^{\,2} E_{ty}(x,y) = 0$$
$$\nabla_t^{\,2} E_{tz}(x,y) + k_c^{\,2} E_{tz}(x,y) = 0$$

$$k_c = \sqrt{k^2 - \beta_z^{\,2}}$$

Subject to boundary conditions.

Uniform Waveguides



This makes it a bunch easier!

Any other short cuts?

This is really three PDEs

$$\nabla_t^2 E_{tx}(x,y) + k_c^2 E_{tx}(x,y) = 0$$
$$\nabla_t^2 E_{ty}(x,y) + k_c^2 E_{ty}(x,y) = 0$$
$$\nabla_t^2 E_{tz}(x,y) + k_c^2 E_{tz}(x,y) = 0$$

$$k_c = \sqrt{k^2 - \beta_z^2}$$

Subject to boundary conditions.

Uniform Waveguides

$$\tilde{E}(x,y,z) = \tilde{E}_t(x,y)e^{-j\beta_z z}$$

$$\tilde{H}(x,y,z) = \tilde{H}_t(x,y)e^{-j\beta_z z}$$

$$\tilde{H}(x,y,z) = \frac{-1}{j\omega\mu}\nabla \times \left[E_t(x,y)e^{-j\beta_z z}\right]$$

$$\tilde{E}(x,y,z) = \frac{1}{j\omega\varepsilon}\nabla \times \left[H_t(x,y)e^{-j\beta_z z}\right]$$

Uniform Waveguides

$$\tilde{E}(x,y,z) = \tilde{E}_t(x,y)e^{-j\beta_z z}$$

$$\widetilde{H}(x,y,z) = \widetilde{H}_t(x,y)e^{-j\beta_z z}$$

$$\widetilde{H}(x,y,z) = \frac{-1}{j\omega\mu}\nabla \times \left[E_t(x,y)e^{-j\beta_z z}\right]$$

$$= \frac{-1}{j\omega\mu}\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_{tx}(x,y)e^{-j\beta_z z} & E_{ty}(x,y)e^{-j\beta_z z} & E_{tz}(x,y)e^{-j\beta_z z} \end{vmatrix}$$

Uniform Waveguides

$$\tilde{H}(x,y,z)=\frac{-1}{j\omega\mu}\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_{tx}(x,y)e^{-j\beta_z z} & E_{ty}(x,y)e^{-j\beta_z z} & E_{tz}(x,y)e^{-j\beta_z z} \end{vmatrix}$$

$$H_x(x,y,z) = \frac{-1}{j\omega\mu}\left[\frac{\partial E_{tz}(x,y)}{\partial y} + j\beta_z E_{ty}(x,y)\right]e^{-j\beta_z z}$$

$$H_y(x,y,z) = \frac{1}{j\omega\mu}\left[\frac{\partial E_{tz}(x,y)}{\partial x} + j\beta_z E_{tx}(x,y)\right]e^{-j\beta_z z}$$

$$H_z(x,y,z) = \frac{-1}{j\omega\mu}\left[\frac{\partial E_{tx}(x,y)}{\partial y} - \frac{\partial E_{ty}(x,y)}{\partial x}\right]e^{-j\beta_z z}$$

Uniform Waveguides

$$\tilde{E}(x,y,z) = \frac{1}{j\omega\varepsilon}\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_{tx}(x,y)e^{-j\beta_z z} & H_{ty}(x,y)e^{-j\beta_z z} & H_{tz}(x,y)e^{-j\beta_z z} \end{vmatrix}$$

$$E_x(x,y,z) = \frac{1}{j\omega\varepsilon}\left[\frac{\partial H_{tz}(x,y)}{\partial y} + j\beta_z H_{ty}(x,y)\right]e^{-j\beta_z z}$$

$$E_y(x,y,z) = \frac{-1}{j\omega\varepsilon}\left[\frac{\partial H_{tz}(x,y)}{\partial x} + j\beta_z H_{tx}(x,y)\right]e^{-j\beta_z z}$$

$$E_z(x,y,z) = \frac{1}{j\omega\varepsilon}\left[\frac{\partial H_{tx}(x,y)}{\partial y} - \frac{\partial H_{ty}(x,y)}{\partial x}\right]e^{-j\beta_z z}$$

Uniform Waveguides

$$j\omega\varepsilon E_{tx}(x,y) = \left[\frac{\partial H_{tz}(x,y)}{\partial y} + j\beta_z H_{ty}(x,y)\right]$$

$$-j\omega\mu H_{tx}(x,y) = \left[\frac{\partial E_{tz}(x,y)}{\partial y} + j\beta_z E_{ty}(x,y)\right]$$

$$-j\omega\varepsilon E_{ty}(x,y) = \left[\frac{\partial H_{tz}(x,y)}{\partial x} + j\beta_z H_{tx}(x,y)\right]$$

$$j\omega\mu H_{ty}(x,y) = \left[\frac{\partial E_{tz}(x,y)}{\partial x} + j\beta_z E_{tx}(x,y)\right]$$

$$j\omega\varepsilon\, E_{tz}(x,y) = \left[\frac{\partial H_{tx}(x,y)}{\partial y} - \frac{\partial H_{ty}(x,y)}{\partial x}\right]$$

$$-j\omega\mu H_{tz}(x,y) = \left[\frac{\partial E_{tx}(x,y)}{\partial y} - \frac{\partial E_{ty}(x,y)}{\partial x}\right]$$

$$E_{tx}(x,y) = \frac{1}{j\omega\varepsilon}\left[\frac{\partial H_{tz}(x,y)}{\partial y} + j\beta_z H_{ty}(x,y)\right] \Leftarrow H_{ty}(x,y) = \frac{1}{j\omega\mu}\left[\frac{\partial E_{tz}(x,y)}{\partial x} + j\beta_z E_{tx}(x,y)\right]$$

$$E_{tx}(x,y) = \frac{1}{j\omega\varepsilon}\left[\frac{\partial H_{tz}(x,y)}{\partial y} + j\beta_z \frac{1}{j\omega\mu}\left[\frac{\partial E_{tz}(x,y)}{\partial x} + j\beta_z E_{tx}(x,y)\right]\right] = \frac{1}{j\omega\varepsilon}\frac{\partial H_{tz}(x,y)}{\partial y} - \frac{j\beta_z}{\omega^2\mu\varepsilon}\left[\frac{\partial E_{tz}(x,y)}{\partial x} + j\beta_z E_{tx}(x,y)\right]$$

$$E_{tx}(x,y)\left(1 - \frac{{\beta_z}^2}{k^2}\right) = \frac{1}{j\omega\varepsilon}\frac{\partial H_{tz}(x,y)}{\partial y} - \frac{j\beta_z}{\omega^2\mu\varepsilon}\frac{\partial E_{tz}(x,y)}{\partial x}$$

Uniform Waveguides

$$j\omega\varepsilon E_{tx}(x,y) = \left[\frac{\partial H_{tz}(x,y)}{\partial y} + j\beta_z H_{ty}(x,y)\right]$$

$$-j\omega\mu H_{tx}(x,y) = \left[\frac{\partial E_{tz}(x,y)}{\partial y} + j\beta_z E_{ty}(x,y)\right]$$

$$-j\omega\varepsilon E_{ty}(x,y) = \left[\frac{\partial H_{tz}(x,y)}{\partial x} + j\beta_z H_{tx}(x,y)\right]$$

$$j\omega\mu H_{ty}(x,y) = \left[\frac{\partial E_{tz}(x,y)}{\partial x} + j\beta_z E_{tx}(x,y)\right]$$

$$j\omega\varepsilon\, E_{tz}(x,y) = \left[\frac{\partial H_{tx}(x,y)}{\partial y} - \frac{\partial H_{ty}(x,y)}{\partial x}\right]$$

$$-j\omega\mu H_{tz}(x,y) = \left[\frac{\partial E_{tx}(x,y)}{\partial y} - \frac{\partial E_{ty}(x,y)}{\partial x}\right]$$

$$E_{tx}(x,y)\left(1 - \frac{{\beta_z}^2}{k^2}\right) = \frac{1}{j\omega\varepsilon}\frac{\partial H_{tz}(x,y)}{\partial y} - \frac{j\beta_z}{\omega^2\mu\varepsilon}\frac{\partial E_{tz}(x,y)}{\partial x}$$

where $k^2 = \omega^2\mu\varepsilon$

$$E_{tx}(x,y) = \frac{-j}{(k^2 - {\beta_z}^2)}\left[\omega\mu\frac{\partial H_{tz}(x,y)}{\partial y} + \beta_z\frac{\partial E_{tz}(x,y)}{\partial x}\right]$$

Uniform Waveguides

$$j\omega\varepsilon E_{tx}(x,y) = \left[\frac{\partial H_{tz}(x,y)}{\partial y} + j\beta_z H_{ty}(x,y)\right]$$

$$-j\omega\mu H_{tx}(x,y) = \left[\frac{\partial E_{tz}(x,y)}{\partial y} + j\beta_z E_{ty}(x,y)\right]$$

$$-j\omega\varepsilon E_{ty}(x,y) = \left[\frac{\partial H_{tz}(x,y)}{\partial x} + j\beta_z H_{tx}(x,y)\right]$$

$$j\omega\mu H_{ty}(x,y) = \left[\frac{\partial E_{tz}(x,y)}{\partial x} + j\beta_z E_{tx}(x,y)\right]$$

$$j\omega\varepsilon\, E_{tz}(x,y) = \left[\frac{\partial H_{tx}(x,y)}{\partial y} - \frac{\partial H_{ty}(x,y)}{\partial x}\right]$$

$$-j\omega\mu H_{tz}(x,y) = \left[\frac{\partial E_{tx}(x,y)}{\partial y} - \frac{\partial E_{ty}(x,y)}{\partial x}\right]$$

$$E_{tx}(x,y) = \frac{-j}{(k^2 - \beta_z{}^2)}\left[\omega\mu \frac{\partial H_{tz}(x,y)}{\partial y} + \beta_z \frac{\partial E_{tz}(x,y)}{\partial x}\right]$$

$$H_{tx}(x,y) = \frac{j}{(k^2 - \beta_z{}^2)}\left[\omega\varepsilon \frac{\partial E_{tz}(x,y)}{\partial y} - \beta_z \frac{\partial H_{tz}(x,y)}{\partial x}\right]$$

$$E_{ty}(x,y) = \frac{j}{(k^2 - \beta_z{}^2)}\left[\omega\mu \frac{\partial H_{tz}(x,y)}{\partial x} - \beta_z \frac{\partial E_{tz}(x,y)}{\partial y}\right]$$

$$H_{ty}(x,y) = \frac{-j}{(k^2 - \beta_z{}^2)}\left[\omega\varepsilon \frac{\partial E_{tz}(x,y)}{\partial x} + \beta_z \frac{\partial H_{tz}(x,y)}{\partial y}\right]$$

Uniform Waveguides

$$E_x = \frac{-j}{(k_c^2)}\left[\omega\mu\frac{\partial H_z}{\partial y} + \beta_z\frac{\partial E_z}{\partial x}\right]$$

$$H_x = \frac{j}{(k_c^2)}\left[\omega\varepsilon\frac{\partial E_z}{\partial y} - \beta_z\frac{\partial H_z}{\partial x}\right]$$

$$E_y = \frac{j}{(k_c^2)}\left[\omega\mu\frac{\partial H_z}{\partial x} - \beta_z\frac{\partial E_z}{\partial y}\right]$$

$$H_y = \frac{-j}{(k_c^2)}\left[\omega\varepsilon\frac{\partial E_z}{\partial x} + \beta_z\frac{\partial H_z}{\partial y}\right]$$

Three Cases:

(1) $E_z=0$, $H_z \neq 0$ TE

$$E_x = \frac{-j}{(k_c^2)}\left[\omega\mu\frac{\partial H_z}{\partial y}\right]$$

$$E_y = \frac{j}{(k_c^2)}\left[\omega\mu\frac{\partial H_z}{\partial x}\right]$$

$$H_x = \frac{-j}{(k_c^2)}\left[\beta_z\frac{\partial H_z}{\partial x}\right]$$

$$H_y = \frac{-j}{(k_c^2)}\left[\beta_z\frac{\partial H_z}{\partial y}\right]$$

$$\nabla_t^2 H_z(x,y) + k_c^2 H_z(x,y) = 0$$

(2) $E_z \neq 0$, $H_z = 0$ TM

$$E_x = \frac{-j}{(k_c^2)}\left[\beta_z\frac{\partial E_z}{\partial x}\right]$$

$$E_y = \frac{-j}{(k_c^2)}\left[\beta_z\frac{\partial E_z}{\partial y}\right]$$

$$H_x = \frac{j}{(k_c^2)}\left[\omega\varepsilon\frac{\partial E_z}{\partial y}\right]$$

$$H_y = \frac{-j}{(k_c^2)}\left[\omega\varepsilon\frac{\partial E_z}{\partial x}\right]$$

$$\nabla_t^2 E_z(x,y) + k_c^2 E_z(x,y) = 0$$

Uniform Waveguides

$$E_x = \frac{-j}{(k_c{}^2)}\left[\omega\mu\frac{\partial H_z}{\partial y} + \beta_z\frac{\partial E_z}{\partial x}\right]$$

$$H_x = \frac{j}{(k_c{}^2)}\left[\omega\varepsilon\frac{\partial E_z}{\partial y} - \beta_z\frac{\partial H_z}{\partial x}\right]$$

$$E_y = \frac{j}{(k_c{}^2)}\left[\omega\mu\frac{\partial H_z}{\partial x} - \beta_z\frac{\partial E_z}{\partial y}\right]$$

$$H_y = \frac{-j}{(k_c{}^2)}\left[\omega\varepsilon\frac{\partial E_z}{\partial x} + \beta_z\frac{\partial H_z}{\partial y}\right]$$

Three Cases:

(3) E_z=0, H_z = 0 TEM

What do we do with this?
Looks like all the fields are zero!

Uniform Waveguides

$$E_x = \frac{-j}{(k_c{}^2)}\left[\omega\mu\frac{\partial H_z}{\partial y} + \beta_z\frac{\partial E_z}{\partial x}\right]$$

$$H_x = \frac{j}{(k_c{}^2)}\left[\omega\varepsilon\frac{\partial E_z}{\partial y} - \beta_z\frac{\partial H_z}{\partial x}\right]$$

$$E_y = \frac{j}{(k_c{}^2)}\left[\omega\mu\frac{\partial H_z}{\partial x} - \beta_z\frac{\partial E_z}{\partial y}\right]$$

$$H_y = \frac{-j}{(k_c{}^2)}\left[\omega\varepsilon\frac{\partial E_z}{\partial x} + \beta_z\frac{\partial H_z}{\partial y}\right]$$

Three Cases:

(3) E_z=0, H_z = 0 TEM

What do we do with this?
Looks like all the fields are zero!

The only way the fields are not all zero is if

$$k_c = \sqrt{k^2 - \beta_z{}^2} = 0$$

$$\Rightarrow \quad \beta_z = k = \omega\sqrt{\varepsilon\mu}$$

Uniform Waveguides

$$E_x = \frac{-j}{(k_c^2)}\left[\omega\mu\frac{\partial H_z}{\partial y} + \beta_z\frac{\partial E_z}{\partial x}\right]$$

$$H_x = \frac{j}{(k_c^2)}\left[\omega\varepsilon\frac{\partial E_z}{\partial y} - \beta_z\frac{\partial H_z}{\partial x}\right]$$

$$E_y = \frac{j}{(k_c^2)}\left[\omega\mu\frac{\partial H_z}{\partial x} - \beta_z\frac{\partial E_z}{\partial y}\right]$$

$$H_y = \frac{-j}{(k_c^2)}\left[\omega\varepsilon\frac{\partial E_z}{\partial x} + \beta_z\frac{\partial H_z}{\partial y}\right]$$

Three Cases:

(3) $E_z=0$, $H_z = 0$ TEM

$$k_c = \sqrt{k^2 - \beta_z{}^2} = 0$$

$$\Rightarrow \quad \beta_z = k = \omega\sqrt{\varepsilon\mu}$$

=0

$$\nabla_t{}^2 E_x(x,y) + k_c{}^2 E_x(x,y) = 0$$

$$\Rightarrow \quad \nabla_t{}^2 E_x(x,y) = 0$$

$$\nabla_t{}^2 E_y(x,y) = 0$$

Uniform Waveguides

$$E_x = \frac{-j}{(k_c{}^2)}\left[\omega\mu\frac{\partial H_z}{\partial y} + \beta_z\frac{\partial E_z}{\partial x}\right]$$

$$H_x = \frac{j}{(k_c{}^2)}\left[\omega\varepsilon\frac{\partial E_z}{\partial y} - \beta_z\frac{\partial H_z}{\partial x}\right]$$

$$E_y = \frac{j}{(k_c{}^2)}\left[\omega\mu\frac{\partial H_z}{\partial x} - \beta_z\frac{\partial E_z}{\partial y}\right]$$

$$H_y = \frac{-j}{(k_c{}^2)}\left[\omega\varepsilon\frac{\partial E_z}{\partial x} + \beta_z\frac{\partial H_z}{\partial y}\right]$$

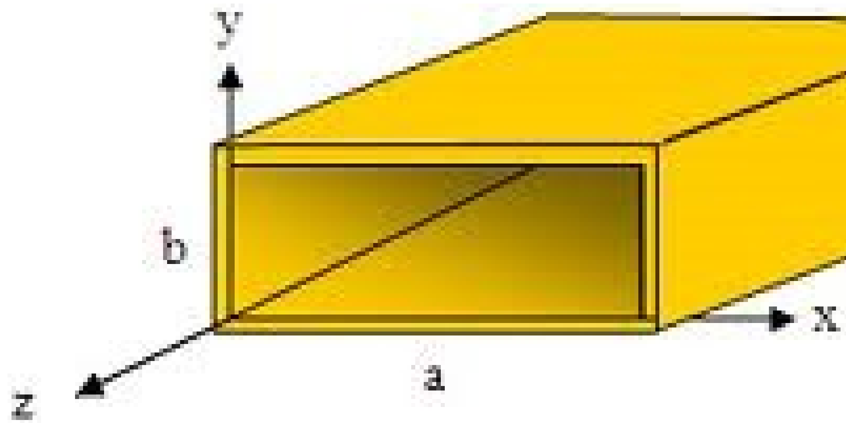
Three Cases:

(3) E_z=0, H_z = 0 TEM

$$\nabla_t{}^2 E_x(x,y) = 0$$

$$\nabla_t{}^2 E_y(x,y) = 0$$

Rectangular Waveguides



TM Modes

$$\nabla_t{}^2 E_z(x,y) + k_c{}^2 E_z(x,y) = 0$$

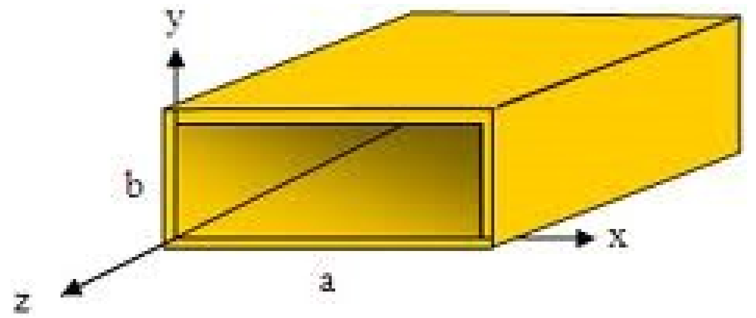
$$E_x = \frac{-j}{(k_c{}^2)}\left[\beta_z \frac{\partial E_z}{\partial x}\right]$$

$$E_y = \frac{-j}{(k_c{}^2)}\left[\beta_z \frac{\partial E_z}{\partial y}\right]$$

$$H_x = \frac{j}{(k_c{}^2)}\left[\omega\varepsilon \frac{\partial E_z}{\partial y}\right]$$

$$H_y = \frac{-j}{(k_c{}^2)}\left[\omega\varepsilon \frac{\partial E_z}{\partial x}\right]$$

Rectangular Waveguides



TM Modes

$$\nabla_t{}^2 E_z(x,y) + k_c{}^2 E_z(x,y) = 0$$

$$\frac{\partial^2}{\partial x^2} E_z(x,y) + \frac{\partial^2}{\partial y^2} E_z(x,y) + k_c{}^2 E_z(x,y) = 0$$

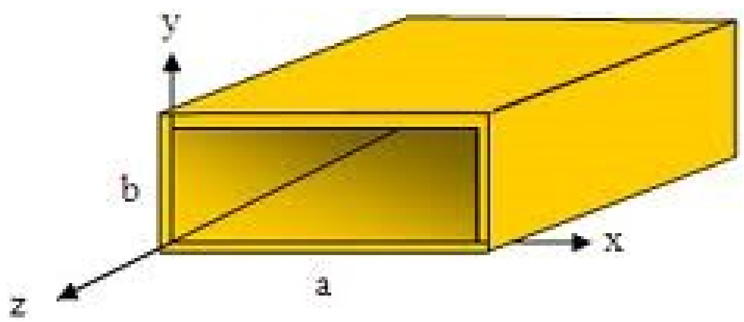
$$E_z(x,y) = X(x)Y(y)$$

$$X''Y + Y''X + k_c{}^2 XY = 0$$

$$\frac{X''Y + Y''X + k_c{}^2 XY}{XY} = 0$$

$$\frac{X''}{X} + \frac{Y''}{Y} + k_c{}^2 = 0$$

Rectangular Waveguides



TM Modes

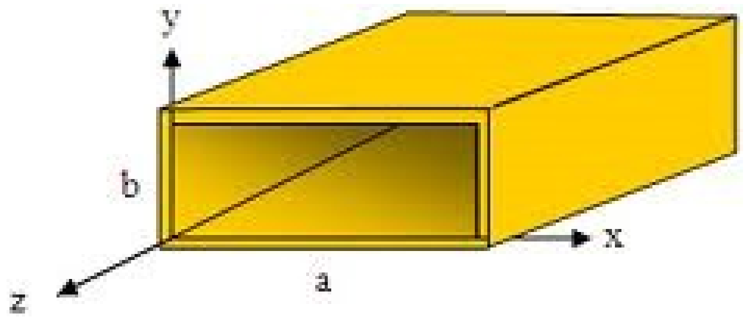
$$\nabla_t{}^2 E_z(x,y) + k_c{}^2 E_z(x,y) = 0$$

$$\frac{\partial^2}{\partial x^2} E_z(x,y) + \frac{\partial^2}{\partial y^2} E_z(x,y) + k_c{}^2 E_z(x,y) = 0$$

$$\frac{X''}{X} + \frac{Y''}{Y} + k_c{}^2 = 0$$

$$\frac{X''}{X} = -\beta_x{}^2 \qquad \frac{Y''}{Y} = -\beta_y{}^2 \qquad k_c{}^2 = \beta_x{}^2 + \beta_y{}^2$$

Rectangular Waveguides



TM Modes

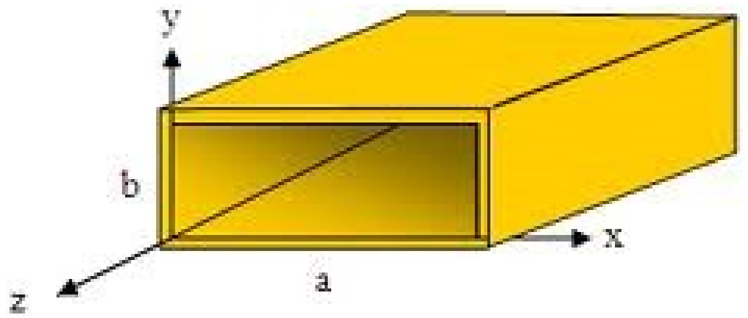
$$\nabla_t^{\,2} E_z(x,y) + k_c^{\,2} E_z(x,y) = 0$$

$$\frac{X''}{X} = -\beta_x^{\,2} \quad \Rightarrow \quad X(x) = A_1 cos(\beta_x x) + A_2 sin(\beta_x x)$$

$$\frac{Y''}{Y} = -\beta_y^{\,2} \quad \Rightarrow \quad Y(y) = B_1 cos(\beta_y y) + B_2 sin(\beta_y y)$$

$$\Rightarrow \quad k_c^{\,2} = \beta_x^{\,2} + \beta_y^{\,2}$$

Rectangular Waveguides



TM Modes

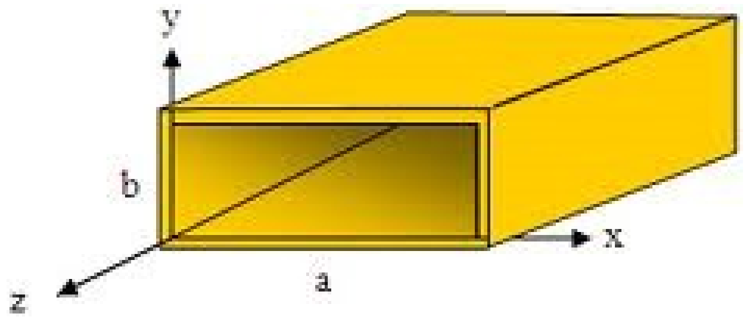
$$\nabla_t^{\,2} E_z(x,y) + k_c^{\,2} E_z(x,y) = 0$$

$$E_z(x,y) = [A_1 cos(\beta_x x) + A_2 sin(\beta_x x)]\left[B_1 cos\left(\beta_y y\right) + B_2 sin\left(\beta_y y\right)\right]$$

Boundary Conditions

$$E_z(0,y) = E_z(a,y) = 0 \qquad E_z(x,0) = E_z(x,b) = 0$$

Rectangular Waveguides



TM Modes

$$E_z(x,y) = [A_1 cos(\beta_x x) + A_2 sin(\beta_x x)][B_1 cos(\beta_y y) + B_2 sin(\beta_y y)]$$

Boundary Conditions $\quad E_z(0,y) = 0$

$$E_z(0,y) = [A_1 cos(\beta_x 0) + A_2 sin(\beta_x 0)][B_1 cos(\beta_y y) + B_2 sin(\beta_y y)] = 0$$

$$[A_1][B_1 cos(\beta_y y) + B_2 sin(\beta_y y)] = 0 \quad \Rightarrow \quad A_1 = 0$$

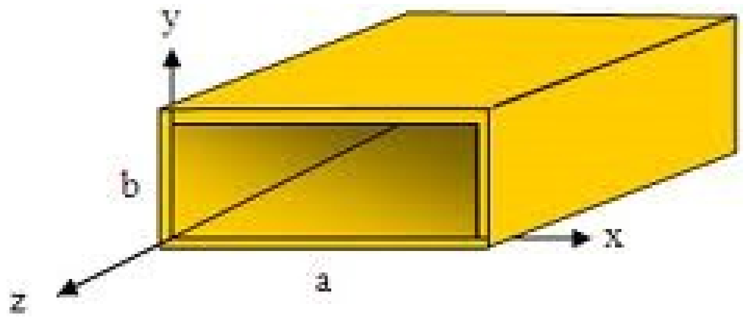
Boundary Conditions $\quad E_z(a,y) = 0$

$$E_z(a,y) = [A_2 sin(\beta_x a)][B_1 cos(\beta_y y) + B_2 sin(\beta_y y)] = 0$$

$$[A_2 sin(\beta_x a)][B_1 cos(\beta_y y) + B_2 sin(\beta_y y)] = 0 \quad \Rightarrow \quad \beta_x = \frac{n\pi}{a}$$

$$n = \pm 1,2,3,\ldots$$

Rectangular Waveguides



TM Modes

$$E_z(x,y) = \left[A_2 sin\left(\frac{n\pi}{a}\,x\right)\right]\left[B_1 cos(\beta_y y) + B_2 sin(\beta_y y)\right]$$

Boundary Conditions $\quad E_z(x,0) = 0$

$$E_z(0,y) = \left[A_2 sin\left(\frac{n\pi}{a}\,x\right)\right]\left[B_1 cos(\beta_y 0) + B_2 sin(\beta_y 0)\right] = 0$$

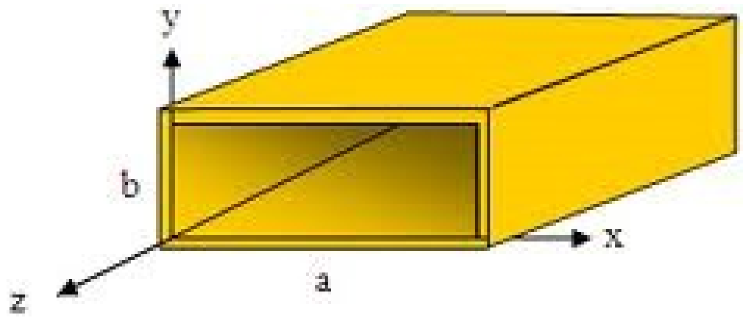
$$\left[A_2 sin\left(\frac{n\pi}{a}\,x\right)\right][B_1] = 0 \quad \Rightarrow \quad B_1 = 0$$

Boundary Conditions $\quad E_z(x,b) = 0$

$$E_z(a,y) = \left[A_2 sin\left(\frac{n\pi}{a}x\right)\right]\left[B_2 sin(\beta_y b)\right] = 0$$

$$\left[A_2 sin\left(\frac{n\pi}{a}x\right)\right]\left[B_2 sin(\beta_y b)\right] = 0 \quad \Rightarrow \quad \beta_y = \frac{m\pi}{b} \qquad m = \pm 1,2,3,\ldots$$

Rectangular Waveguides



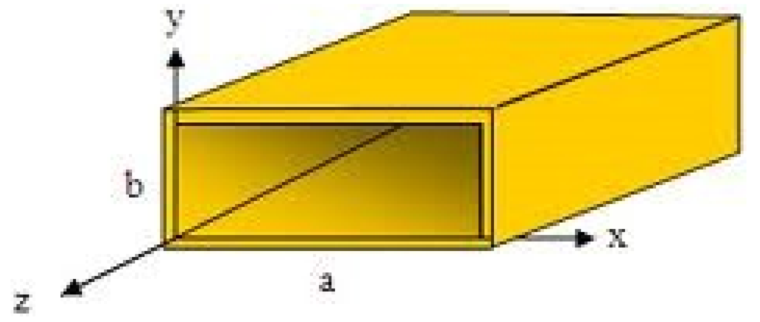
TM Modes

$$E_z(x,y) = A_{mn}\left[sin\left(\frac{n\pi}{a}x\right)\right]\left[sin\left(\frac{m\pi}{b}y\right)\right] \qquad m\ and\ n = \pm 1,2,3,\ldots$$

$$E_z(x,y,z) = E_z(x,y)e^{-j\beta_z z} = A_{mn}\left[sin\left(\frac{n\pi}{a}x\right)\right]\left[sin\left(\frac{m\pi}{b}y\right)\right]e^{-j\beta_z z}$$

How do we find βz ?

Rectangular Waveguides



TM Modes

$$E_z(x,y) = A_{mn}\left[\sin\left(\frac{n\pi}{a}x\right)\right]\left[\sin\left(\frac{m\pi}{b}y\right)\right] \qquad m \text{ and } n = \pm 1,2,3,\ldots$$

$$E_z(x,y,z) = E_z(x,y)e^{-j\beta_z z} = A_{mn}\left[\sin\left(\frac{n\pi}{a}x\right)\right]\left[\sin\left(\frac{m\pi}{b}y\right)\right]e^{-j\beta_z z}$$

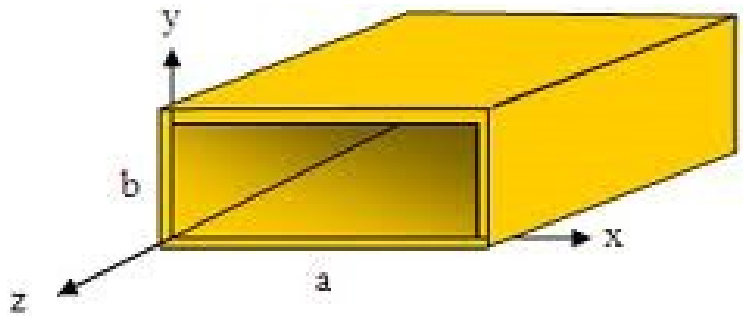
How do we find βz ?

$$k_c^{\,2} = \beta_x^{\,2} + \beta_y^{\,2}$$

$$k_c^{\,2} = k^2 - \beta_z^{\,2} = \beta_x^{\,2} + \beta_y^{\,2}$$

$$\beta_z = \sqrt{k^2 - \beta_x^{\,2} - \beta_y^{\,2}} = \sqrt{\omega^2\mu\varepsilon - \left(\frac{n\pi}{a}\right)^2 - \left(\frac{m\pi}{b}\right)^2}$$

Rectangular Waveguides



TM Modes

$$E_z^{nm}(x,y,z) = A_{mn}\left[sin\left(\frac{n\pi}{a}x\right)\right]\left[sin\left(\frac{m\pi}{b}y\right)\right]e^{-j\beta_z{}^{mn}z}$$

$$\beta_z{}^{mn} = \sqrt{\omega^2\mu\varepsilon - \left(\frac{n\pi}{a}\right)^2 - \left(\frac{m\pi}{b}\right)^2}$$

$m\ and\ n = \pm 1,2,3,\ldots$

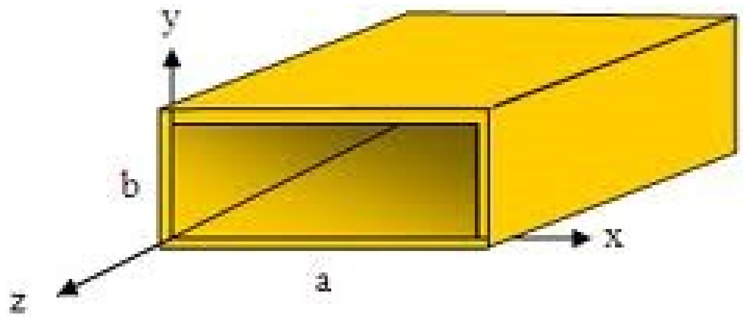
$$E_x = \frac{-j}{(k_c{}^2)}\left[\beta_z\frac{\partial E_z}{\partial x}\right]$$

$$H_x = \frac{j}{(k_c{}^2)}\left[\omega\varepsilon\frac{\partial E_z}{\partial y}\right]$$

$$E_y = \frac{-j}{(k_c{}^2)}\left[\beta_z\frac{\partial E_z}{\partial y}\right]$$

$$H_y = \frac{-j}{(k_c{}^2)}\left[\omega\varepsilon\frac{\partial E_z}{\partial x}\right]$$

Rectangular Waveguides



TM Modes

$$E_z^{nm}(x,y,z) = A_{mn}\left[sin\left(\frac{n\pi}{a}x\right)\right]\left[sin\left(\frac{m\pi}{b}y\right)\right]e^{-j\beta_z^{mn}z}$$

$$\beta_z^{mn} = \sqrt{\omega^2\mu\varepsilon - \left(\frac{n\pi}{a}\right)^2 - \left(\frac{m\pi}{b}\right)^2}$$

$m \; and \; n = \pm 1,2,3,\ldots$

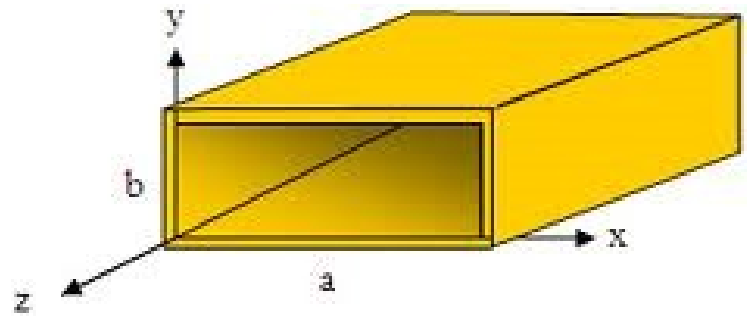
Case I: $\omega^2\mu\varepsilon - \left(\frac{n\pi}{a}\right)^2 - \left(\frac{m\pi}{b}\right)^2 > 0$

β_z^{mn} is real and the mode propagates without attenuation

Case II: $\omega^2\mu\varepsilon - \left(\frac{n\pi}{a}\right)^2 - \left(\frac{m\pi}{b}\right)^2 < 0$

β_z^{mn} is imaginary and the wave decays exponentially

Rectangular Waveguides



TM Modes

$$E_z^{nm}(x,y,z) = A_{mn}\left[sin\left(\frac{n\pi}{a}x\right)\right]\left[sin\left(\frac{m\pi}{b}y\right)\right]e^{-j\beta_z{}^{mn}z}$$

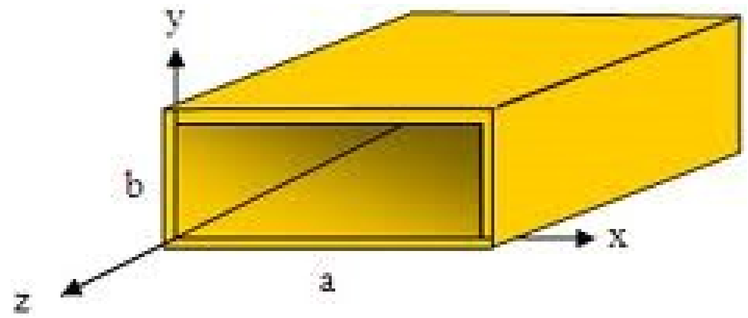
$$\beta_z{}^{mn} = \sqrt{\omega^2\mu\varepsilon - \left(\frac{n\pi}{a}\right)^2 - \left(\frac{m\pi}{b}\right)^2} \qquad m \ and \ n = \pm 1,2,3,\ldots$$

At the point $\omega_c{}^2\mu\varepsilon - \left(\frac{n\pi}{a}\right)^2 - \left(\frac{m\pi}{b}\right)^2 = 0$

the mode changes from evanescent to propagating.

$$f_c{}^{nm} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}$$

Rectangular Waveguides



TM Modes

$$E_z{}^{nm}(x,y,z) = A_{mn}\left[sin\left(\frac{n\pi}{a}x\right)\right]\left[sin\left(\frac{m\pi}{b}y\right)\right]e^{-j\beta_z{}^{mn}z}$$

$$\beta_z{}^{mn} = \sqrt{\omega^2\mu\varepsilon - \left(\frac{n\pi}{a}\right)^2 - \left(\frac{m\pi}{b}\right)^2}$$

$$m \; and \; n = \pm 1,2,3,\ldots$$

$$f_c{}^{nm} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}$$

Rectangular Waveguides

Dominant mode: (a>b)

$$f_c^{\,10} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{1\pi}{a}\right)^2 + \left(\frac{0\pi}{b}\right)^2} = \frac{1}{2a\sqrt{\mu\varepsilon}}$$

Guide wavelength:

$$\lambda_g = \frac{2\pi}{\beta_z} = \frac{2\pi}{\sqrt{k^2 - \left(\frac{n\pi}{a}\right)^2 - \left(\frac{m\pi}{b}\right)^2}} = \frac{2\pi}{k\sqrt{1 - \frac{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}{k^2}}} = \frac{\lambda}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}}$$

\

$$Z_w = \frac{E_x}{H_y} = -\frac{E_y}{H_x} = \frac{\omega\mu}{\sqrt{k^2 - \left(\frac{n\pi}{a}\right)^2 - \left(\frac{m\pi}{b}\right)^2}} = \frac{\sqrt{\mu/\varepsilon}}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}}$$

Rectangular Waveguides

Phase velocity:

$$v_p = \frac{\omega}{\beta_z} = \frac{\omega}{\sqrt{k^2 - \left(\frac{n\pi}{a}\right)^2 - \left(\frac{m\pi}{b}\right)^2}} = \frac{c}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}} > c$$

Group velocity:

$$v_g = \frac{1}{\frac{d\beta_z}{d\omega}} = \frac{1}{\frac{d\left(\sqrt{k^2 - \left(\frac{n\pi}{a}\right)^2 - \left(\frac{m\pi}{b}\right)^2}\right)}{d\omega}} = c\sqrt{1 - \left(\frac{f_c}{f}\right)^2} < c$$