1. The following input/output relations describe systems with input \( v(x) \) and output \( w(x) \). Which systems are linear? Which are space-invariant? [6 pts.]

\[(a) \quad w(x) = a \cdot v(x) + b\]

i. test linearity by testing linear superposition

Let there be two input signals \( v_1(x) \) and \( v_2(x) \)

(1) output as the sum of the outputs from each signal individually

\[
w_1(x) = a \cdot v_1(x) + b \\
w_2(x) = a \cdot v_2(x) + b \\
w(x) = w_1(x) + w_2(x) = a \cdot v_1(x) + b + a \cdot v_2(x) + b = a \cdot (v_1(x) + v_2(x)) + 2b
\]

(2) output as the sum of the two inputs

\[
w(x) = a \cdot (v_1(x) + v_2(x)) + b
\]

Since \( w(x) \neq w'(x) \) the system is NOT linear

ii. test shift invariance by shifting both the output and input and see if the result is the same

(1) shift the output

\[
w_1(x - x_s) = a \cdot v(x - x_s) + b
\]

(2) shift the input

\[
w_2(x - x_s) = a \cdot v(x - x_s) + b
\]

Since the output stays the same in both cases the system IS shift invariant

\[(b) \quad w(x) = a \cdot v(x + b)\]

i. test linearity by testing linear superposition

Let there be two input signals \( v_1(x) \) and \( v_2(x) \)

(1) output as the sum of the outputs from each signal individually

\[
w_1(x) = a \cdot v_1(x + b) \\
w_2(x) = a \cdot v_2(x + b) \\
w(x) = w_1(x) + w_2(x) = a \cdot v_1(x + b) + a \cdot v_2(x + b) + b = a \cdot (v_1(x + b) + v_2(x + b))
\]

(2) output as the sum of the two inputs

\[
w(x) = a \cdot (v_1(x + b) + v_2(x + b))
\]

since \( w(x) = w'(x) \) the system IS linear
ii. Test shift invariance by shifting both the output and input and see if the result is the same

(a) Shift the output
\[ w_1(x - x_o) = a \cdot v(x + b - x_o) \]

(b) Shift the input
\[ w_2(x - x_o) = a \cdot v(x + b - x_o) \]

Since the output stays the same in both cases the system **IS** shift invariant

(c) \( w(x) = \sin(2\pi ax) v(x) \)

i. Test linearity by testing linear superposition

Let there be two input signals \( v_1(x) \) and \( v_2(x) \)

(1) Output as the sum of the outputs from each signal individually
\[ w_1(x) = \sin(2\pi ax) \cdot v_1(x) \]
\[ w_2(x) = \sin(2\pi ax) \cdot v_2(x) \]
\[ w(x) = w_1(x) + w_2(x) = \sin(2\pi ax) \cdot v_1(x) + \sin(2\pi ax) \cdot v_2(x) = \sin(2\pi ax) \cdot (v_1(x) + v_2(x)) \]

(2) Output as the sum of the two inputs
\[ w'(x) = \sin(2\pi ax) \cdot (v_1(x) + v_2(x)) \]

Since \( w'(x) = w(x) \) the system is linear

ii. Test shift invariance by shifting both the output and input and see if the result is the same

(a) Shift the output
\[ w_1(x - x_o) = \sin(2\pi ax(x - x_o)) \cdot v(x - x_o) \]

(b) Shift the input
\[ w_2(x - x_o) = \sin(2\pi ax) \cdot v(x - x_o) \]

Since the outputs are different in the two cases the system **IS NOT** shift invariant
2.10 Given a continuous signal \( f(x,y) = x + y^2 \) evaluate the following: [6 pts.]

(a) \( f(x,y) \delta(x - 1, y - 2) \)
\[
f(x,y) \delta(x-1, y-2) = \begin{cases} 
  f(1,2) \delta(0,0) & x = 1, y = 2 \\
  0 & \text{otherwise}
\end{cases}
\]

(b) \( f(x,y) \ast \delta(x - 1, y - 2) \)
\[
f(x,y) \ast \delta(x-1, y-2) = \int \int [\xi + \eta^2] \delta(x-1 - \xi, y-2 - \eta) d\xi d\eta = x - 1 + (y - 2)^2
\]

(c) \( \int \int \delta(x-1, y-2)f(x,3)dx dy \)
\[
\int \int \delta(x-1, y-2)f(x,3)dx dy = \int \int \delta(x-1, y-2)(x + 9)dx dy = 10
\]

(d) \( f(x+1,y+2) \ast \delta(x - 1, y - 2) \)
\[
f(x+1,y+2) \ast \delta(x-1, y-2) = \int \int [(\xi + 1) + (\eta + 2)^2] \delta(x-1 - \xi, y-1 - \eta) d\xi d\eta = x + y^2
\]
2.13 Find the Fourier transforms of the following continuous signals [10 pts.]

(a) \( \delta(x, y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - m, y - n) \)

\[
F[u, v] = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \mathcal{F}[-2\pi(u mx + v ny)] = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \mathcal{F}[\delta(x - m, y - n)] e^{-j2\pi(u mx + v ny)}
\]

\[
F[u, v] = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(u - m, v - n)
\]

(b) \( \delta(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \delta(x - m\Delta x, y - n\Delta y) \)

\[
F[u, v] = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \mathcal{F}[-2\pi(m \Delta x u + n \Delta y v)] = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \mathcal{F}[\delta(x - m\Delta x, y - n\Delta y)] e^{-j2\pi(m \Delta x u + n \Delta y v)}
\]

\[
F[u, v] = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \delta(u - m / \Delta x, v - n / \Delta y)
\]

(c) \( s(x, y) = \sin(2\pi(u x + v y)) \)

\[
F[u, v] = \mathcal{F}[s(x, y)] = \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{\infty} \sin(2\pi(u x + v y)) e^{-j2\pi(u x + v y)} dx dy
\]

\[
F[u, v] = \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{\infty} \sin(2\pi(u x + v y)) e^{-j2\pi(u x + v y)} dx dy = \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{\infty} \left[ e^{j2\pi(u x + v y)} - e^{-j2\pi(u x + v y)} \right] e^{-j2\pi(u x + v y)} dx dy
\]

\[
F(u, v) = \frac{1}{2j} \left[ e^{j2\pi(u x + v y)} - e^{-j2\pi(u x + v y)} \right] dx dy = \frac{1}{2j} \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{\infty} \left[ e^{j2\pi(u x + v y)} - e^{-j2\pi(u x + v y)} \right] dx dy
\]

\[
F(u, v) = \frac{1}{2j} \left[ \delta(u - u_v, v - v_u) - \delta(u + u_v, v + v_u) \right]
\]
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(d) \( c(x, y) = \cos(2\pi(u_x x + v_y y)) \)

\[
F(u, v) = \mathcal{F}[c(x, y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cos(2\pi(u_x x + v_y y)) e^{-j2\pi(u x + v y)} \, dx \, dy
\]

\[
F(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cos(2\pi(u_x x + v_y y)) e^{-j2\pi(u x + v y)} \, dx \, dy = \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ e^{-j2\pi(u x + v y)} + e^{-j2\pi(u x + v y)} \right] \, dx \, dy
\]

\[
F(u, v) = \frac{1}{2} \left[ \delta(u - u_x, v - v_y) + \delta(u + u_x, v + v_y) \right]
\]

(e) \( f(x, y) = \frac{1}{2\pi\sigma^2} e^{-\frac{(x^2 + y^2)}{2\sigma^2}} \)

\[
F(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma^2} e^{-\frac{(x^2 + y^2)}{2\sigma^2}} e^{-j2\pi(u x + v y)} \, dx \, dy
\]

\[
F(u, v) = \frac{1}{2\pi\sigma^2} \int_{-\infty}^{\infty} e^{-j2\pi y v} \, dy \cdot \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}} e^{-j2\pi u x} \, dx
\]

\[
F(u, v) = \frac{1}{2\pi\sigma^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi\sigma^2}} \, dy \cdot \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}} e^{-j2\pi u x} \, dx
\]

\[
F(u, v) = \frac{1}{2\pi\sigma^2} e^{-\frac{u^2}{4\sigma^2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi\sigma^2}} \, dy \cdot \int_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma^2}} e^{-j2\pi v y} \, dy
\]

\[
F(u, v) = \frac{1}{2\pi\sigma^2} e^{-\frac{u^2}{2\sigma^2}} \sqrt{2\pi\sigma^2} \sqrt{2\pi\sigma^2} = e^{-\frac{u^2}{2\sigma^2}} \sqrt{2\pi\sigma^2}
\]

Commented [K2]: Corrected Answer
2.21 A new imaging system with which you are experimenting has anisotropic properties. You measure the impulse response function as \( h(x, y) = e^{-\pi(x^2 + y^2)/4} \) [6 pts]

(a) Sketch the impulse response

(b) What is the transfer function

\[
H(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x, y)e^{-j2\pi(u x + v y)} \, dx \, dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\pi(x^2 + y^2)/4} e^{-j2\pi(u x + v y)} \, dx \, dy
\]

\[
H(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\pi(x^2 + y^2)/4} e^{-j2\pi(u x + v y)} \, dx \, dy = 4e^{-\pi(u^2 + v^2)}
\]
2.25. We want to sample the 2-D continuous signal \( f(x, y) = e^{-x^2 + y^2} \) by means of a rectangular sampling scheme to obtain 1.5 samples per millimeter. Determine the PSF \( h(x, y) \) of an ideal low-pass anti-aliasing filter with the maximum possible frequency content. What percentage of the spectrum energy of \( f(x, y) \) is preserved by this filter? In practice, can we sample \( f(x, y) \) alias-free without using an anti-aliasing filter? [6 pts.]

(a) The sampling rate is given as 1.5 samples per millimeter. Assume \( x \) and \( y \) are in units of millimeters.

This would mean \( \Delta x = \Delta y = \frac{1}{1.5} = 0.667 \text{ mm} \)

To avoid aliasing

\[
\Delta x \leq \frac{1}{2u_{\text{max}}} \quad \Delta y \leq \frac{1}{2v_{\text{max}}}
\]

this means that the maximum spatial frequency of the signals are

\[
u_{\text{max}} = \frac{1}{2\Delta x} = 0.750 \frac{1}{\text{mm}}, \quad v_{\text{max}} = \frac{1}{2\Delta y} = 0.750 \frac{1}{\text{mm}}
\]

To ensure there is no aliasing we would need to implement an anti-aliasing low-pass filter with the maximum spatial frequencies shown above.

\[
h(x, y) = \left[ \frac{\sin(\frac{\pi}{2\Delta x} x)}{\frac{\pi}{2\Delta x} x} \right] \cdot \left[ \frac{\sin(\frac{\pi}{2\Delta y} y)}{\frac{\pi}{2\Delta y} y} \right]
\]

(b) What percentage of the spectral energy of \( f(x, y) \) is preserved by this filter

\[
E_{\text{total}} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F(u, v)|^2 \, du \, dv
\]

\[
E_{\text{filtered}} = \int_{-v_{\text{max}}}^{v_{\text{max}}} \int_{-u_{\text{max}}}^{u_{\text{max}}} |F(u, v)|^2 \, du \, dv
\]

\[
F(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2 + y^2} e^{-j2\pi(u x + v y)} \, dx \, dy = e^{-x^2 + y^2}
\]

\[
E_{\text{total}} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F(u, v)|^2 \, du \, dv = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-2x^2 + y^2} \, dx \, dy = \frac{1}{2}
\]

\[
E_{\text{filtered}} = \int_{-v_{\text{max}}}^{v_{\text{max}}} \int_{-u_{\text{max}}}^{u_{\text{max}}} |F(u, v)|^2 \, du \, dv = \int_{-v_{\text{max}}}^{v_{\text{max}}} \int_{-u_{\text{max}}}^{u_{\text{max}}} e^{-2u^2 + v^2} \, du \, dv
\]

\[
\%\text{preserved} = 100 \times \frac{E_{\text{filtered}}}{E_{\text{total}}} = 200 \int_{-v_{\text{max}}}^{v_{\text{max}}} \int_{-u_{\text{max}}}^{u_{\text{max}}} e^{-2u^2 + v^2} \, du \, dv = 200 \int_{-0.750}^{0.750} \int_{-0.750}^{0.750} e^{-2u^2 + v^2} \, du \, dv = 98.44\%
\]
6. Matlab assignment

Write a Matlab script file that does the following:
(1) Loads in the input image found in homer.mat (file on the website)
(2) Compute the 2D Fourier Transform (using the fft2 command)
(3) Create a modified version of the Fourier Transform by forcing coefficients within certain regions (described in the figures below) to be zero
(4) Calculate the reconstructed image using the modified Fourier Transform and plot the result.

In your homework you should provide: a) commented copy of your Matlab code, b) imaging results for the 3 windows shown on the next page. [16 pts]

Assume everything that is black is where I would like you to make the Fourier Transform zero.
Problem 6:

0.25 Low Pass Frequency Domain

0.25 Low Pass Filter
% This script file is for BMEG/ELEG 479/679 Problem 6

% load an image to decompose into a bunch of spatial frequencies
% This file has 3 variables: Xn (2D matrix that holds the image)
% xi (1D vector of x coordinates), yi (1D vector of y coordinates)
load homer.mat
% determine the distance between x coordinates
dx=xi(2)-xi(1);
% determine the distance between y coordinates
dy=yi(2)-yi(1);

% plot the original image, Xn
figure(1)
colormap('gray');
imagesc(Xn)

% find the Fourier coefficients of the image Xn. We use the 2D FFT
% function to do this called fft2. This function is the 2D Fourier
% Transform of the original image
F=fft2(Xn);
% This function moves the frequencies around so that they are easier to
% work with
F=fftsplit(F);

% create vectors of spatial frequencies in the u,v plane
u=linspace(-.5,.5,Nx)/dx;
v=linspace(-.5,.5,Ny)/dy;

% plot the Fourier transform of the original image
figure(2)
colormap('gray');
imagesc(u,v,log10(abs(F)));
exlabel('u');
ylabel('v');
title('log(abs(F))')

% Copy the FT so we can modify it for the three conditions
fftLowPass1 = F;
fftLowPass2 = F;
fftHighPass = F;

% Loop through the 2D matrices of frequencies and remove the desired
% frequency content
for i = 1:length(u)
  uValue = u(i);
  for j = 1:length(v)
    vValue = v(j);
    % 0 out all high freq values to make the .25x.25 square
    if (abs(uValue) > 0.125) || (abs(vValue) > 0.125)
      fftLowPass1(1:2,1:2) = 0;
      fftLowPass2(1:2,1:2) = 0;
      fftHighPass(1:2,1:2) = 0;
    end
  end
end
fftLowPass1(j, i) = 0;
end

% 0 out all high freq values to make the .125x.125 square
if ((abs(uValue) > 0.0625) || (abs(vValue) > 0.0625))
fftLowPass2(j, i) = 0;
end

% 0 out all low freq values to remove the .25x.25 square
if ((abs(uValue) < 0.125) && (abs(vValue) < 0.125))
fftHighPass(j, i) = 0;
end
end

% Plot FT for the .25x.25 square
figure(3)
colormap('gray');
imagesc(u, v, log10(abs(fftLowPass1)));
title('0.25 Low Pass Frequency Domain')
xlabel('u');
ylabel('v');

% Plot reconstructed image after removing frequencies
figure(4)
colormap('gray');
% ifftshift undoes fftshift
imagesc(ifft2(ifftshift(fftLowPass1)));
title('0.25 Low Pass Filter')

% Plot FT for the .125x.125 square
figure(5)
colormap('gray');
imagesc(u, v, log10(abs(fftLowPass2)));
title('0.125 Low Pass Frequency Domain')
xlabel('u');
ylabel('v');

% Plot reconstructed image after removing frequencies
figure(6)
colormap('gray');
% ifftshift undoes fftshift
imagesc(ifft2(ifftshift(fftLowPass2)));
title('0.125 Low Pass Filter')

% Plot the FT for the the high pass filter
figure(7)
colormap('gray');
imagesc(u, v, log10(abs(fftHighPass)));
title('High Pass Frequency Domain')
xlabel('u');
ylabel('v');

% Plot reconstructed image after removing frequencies
figure(8)
colormap('gray');
% ifftshift undoes fftshift
imagesc(ifft2(ifftshift(fftHighPass)))
title('High Pass Filter')