Summary from last lecture

- Sampling theory and aliasing
Medical Imaging Systems from a Systems Perspective

Input Image
\[ f(x, y) \]
\[ f(x, y, z) \]

System Function
\[ S \]

Output Image
\[ g(x, y) \]
\[ g(x, y, z) \]

Design Parameters

\[ g = S[f] \]
Sampling Theorem

Sampling an image

\[ f_{\text{sampled}}(x, y) = f(x, y) \cdot \text{comb}_{\Delta x, \Delta y}(x, y) \]

\[ F_{\text{sampled}}(u, v) = F(u, v)** \frac{1}{\Delta x \Delta y} \text{comb}_{1/\Delta x, 1/\Delta y}(u, v) \]

\[ F_{\text{sampled}}(u, v) = F(u, v)** \frac{1}{\Delta x \Delta y} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(u - m / \Delta x, v - n / \Delta y) \]

\[ F_{\text{sampled}}(u, v) = \frac{1}{\Delta x \Delta y} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} F(u - m / \Delta x, v - n / \Delta y) \]
Sampling Theorem

\[ f(x, y) \leftrightarrow comb_{\Delta x, \Delta y}(x, y) \]

\[ F(u, v) \leftrightarrow comb_{1/\Delta x, 1/\Delta y}(u, v) \]
Sampling

\[ F_{\text{sampled}}(u, v) = \frac{1}{\Delta x \Delta y} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} F(u - m/\Delta x, v - n/\Delta y) \]
The original image can be recovered (without loss of information) by using a properly chosen filter.

\[ F_{\text{sampled}}(u, v) = \frac{1}{\Delta x \Delta y} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} F\left(u - \frac{m}{\Delta x}, v - \frac{n}{\Delta y}\right) \]

\[ \approx \frac{1}{\Delta x} \sum_{m=-\infty}^{\infty} F\left(u - \frac{m}{\Delta x}\right) \]

\[ \approx \frac{1}{\Delta y} \sum_{n=-\infty}^{\infty} F\left(v - \frac{n}{\Delta y}\right) \]
Ideal Reconstruction Filter
Interpolation

\[ h(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(u, v) e^{j2\pi(ux+vy)} dudv \]

\[ h(x, y) = \begin{bmatrix} \sin\left(\frac{\pi}{\Delta x} x\right) \\ \Delta x \end{bmatrix} \cdot \begin{bmatrix} \sin\left(\frac{\pi}{\Delta y} y\right) \\ \Delta y \end{bmatrix} \]
Aliasing

\[ F_{\text{sampled}}(u,v) = \frac{1}{\Delta x \Delta y} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} F(u - m / \Delta x, v - n / \Delta y) \]

No aliasing if \( 1/\Delta x > 2W_u \) and \( 1/\Delta y > 2W_y \)

Nyquist Theorem
Today’s lecture

- A couple of odds and ends of linear system theory

Image Quality

- Image contrast
- Image resolution
- Noise
- Artifacts
- Distortion
- Accuracy
Real Medical Imaging Systems are Complicated

- Medical imaging systems are made of many connected subsystems.
- Some of these are analog and some are digital.
- How do we model complicated systems like this?
Real Medical Imaging Systems are Complicated

\[ H(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x, y)e^{-j2\pi(ux+vy)} \, dx \, dy \]

\[ h(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(u, v)e^{j2\pi(ux+vy)} \, dudv \]

\[ f(x_1, y_1) \quad \text{PSF} \quad \text{h} \quad \rightarrow \quad g(x_2, y_2) = f(x_1, y_1)^{**h} \]

or

\[ F(u, v) \quad \text{Transfer function} \quad H(u, v) \quad \rightarrow \quad G(u, v) = F(u, v)H(u, v) \]
Real Medical Imaging Systems are Complicated

Cascaded Systems

\[ g_1 = f^{**h_1} \]
\[ g = g_1^{**h_2} = (f^{**h_1})^{**h_2} \]

\[ f \quad \xrightarrow{\text{PSF} \ h_1} \quad g_1 \quad \xrightarrow{\text{PSF} \ h_2} \quad g \]

\[ f \quad \xrightarrow{\text{PSF} \ h_c = h_1^{**h_2}} \quad g = f^{**h_c} \]
Real Medical Imaging Systems are Complicated

Parallel Systems

\[ g_1 = f^{**h_1} \]
\[ g_2 = f^{**h_2} \]
\[ g = g_1 + g_2 = f^{**h_1} + f^{**h_2} = f^{**(h_1 + h_2)} \]

\[ f \rightarrow PSF_{h_1} \rightarrow g_1 = f^{**h_1} \rightarrow + \rightarrow g = g_1 + g_2 = f^{**h_1} + f^{**h_2} = f^{**(h_1 + h_2)} \]

\[ f \rightarrow PSF_{h_2} \rightarrow g_2 = f^{**h_2} \rightarrow + \rightarrow g = g_1 + g_2 = f^{**h_1} + f^{**h_2} = f^{**(h_1 + h_2)} \]

\[ g = f^{**h_c} \]

\[ f \rightarrow PSF_{h_c} = h_1 + h_2 \rightarrow g = f^{**h_c} \]
Real Medical Imaging Systems are Complicated

$h_c = h_1 + h_2$

$g = f**h_c$

Complicated Subsystem Example
Real Medical Imaging Systems are Complicated

Complicated Subsystem Example

\[ f \xrightarrow{h_1} PSF h_1 \xrightarrow{h_2} PSF h_2 \xrightarrow{h_3} PSF h_3 \xrightarrow{h_4} PSF h_4 \xrightarrow{h_5 + h_6} + g \]

\[ f \xrightarrow{?} PSF ? \xrightarrow{} g \]
Real Medical Imaging Systems are Complicated

Complicated Subsystem Example

\[ f \rightarrow \text{PSF } h_1 \ast h_3 \rightarrow \text{PSF } h_5 + h_6 \rightarrow g \]

\[ f \rightarrow \text{PSF} \ ? \rightarrow g \]

\[ f \rightarrow \text{PSF } h_1 \ast h_3 \rightarrow \text{PSF } h_5 + h_6 \rightarrow g \]
Real Medical Imaging Systems are Complicated

Complicated Subsystem Example

\[ f \]

\[ (h_1 ** h_3)^{**(h_5 + h_6)} \]

\[ = \]

\[ (h_1 ** h_3)^{**(h_5 + h_6)} + h_1 ** h_3 \]

\[ g \]
Properties of the Fourier Transform Pair

\[ H(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x, y) e^{-j2\pi(ux+vy)} \, dx \, dy \]

\[ h(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(u, v) e^{j2\pi(ux+vy)} \, du \, dv \]
Scaling Property

If $F(u,v)$ is the Fourier transform of an image $f(x,y)$ and $f_{ab}$ is a scaled version of $f(x,y)$

$$f_{ab} = f(ax, by)$$

Where $a, b$ are nonzero constants then

$$F_{ab}(u, v) = \frac{1}{|ab|} F\left(\frac{u}{a}, \frac{v}{b}\right)$$
Product Property

If $F(u,v)$ is the Fourier transform of an image $f(x,y)$ and $G(u,v)$ is the Fourier transform of an image $g(x,y)$ then

$$\mathcal{F}(f \cdot g) = F(u,v) \ast G(u,v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(\xi,\eta)F(u-\xi,v-\eta)d\xi d\eta$$

and

$$\mathcal{F}(f \ast g) = F(u,v)G(u,v)$$
Translation Property

If $F(u,v)$ is the Fourier transform of an image $f(x,y)$ and if $f$ is shifted by $x_o, y_o$ then

$$f_{\text{shifted}}(x, y) = f(x - x_o, y - y_o)$$

then

$$F_{\text{shifted}}(u, v) = e^{-j2\pi(ux_o + vy_o)} F(u, v)$$
Rotation Property

If $F(u,v)$ is the Fourier transform of an image $f(x,y)$ and if $f$ is rotated by the angle $\theta$ then

$$f_{rotated}(x,y) = f(x \cos(\theta) - y \sin(\theta), x \sin(\theta) + y \cos(\theta))$$

then

$$F_{rotated}(u,v) = F(u \cos(\theta) - v \sin(\theta), u \sin(\theta) + v \cos(\theta))$$
Parseval’s Theorem

The energy or power present in a signal or image is given by:

\[ P = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x, y)|^2 \, dx \, dy \]

According to Parseval’s theorem this can also be given by:

\[ P = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x, y)|^2 \, dx \, dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F(u, v)|^2 \, du \, dv \]
Measures of Quality

- Physics-oriented issues:
  - contrast, resolution
  - noise, artifacts, distortion
  - Quantitative accuracy

- Task-oriented issues:
  - sensitivity, specificity
  - diagnostic accuracy
Image Contrast

- Image contrast is the difference between the image intensity of an object and surrounding objects or background. Vitally important in medical imaging.
- What do we mean by a medical imaging system having good or poor image contrast? We know it when we see it but how can we quantify it?

![Image contrast examples](a) (b) (c)

Figure 1.4

Medical Imaging Signals and Systems, by Jerry L. Prince and Jonathan Links.
Image Contrast

- Contrast: Difference between image characteristics of an object of interest and surrounding objects or background

- General definition
  - \( f_{\text{max}}, f_{\text{min}} \): maximum and minimum values of the signal in an image

\[
\text{Contrast} = \text{modulation} = m_f = \frac{\text{amplitude}}{\text{average}} = \frac{f_{\text{max}} - f_{\text{min}}}{f_{\text{max}} + f_{\text{min}}}
\]

- For a sinusoidal signal

\[
f(x, y) = A + B \sin(2\pi u_0 x)
\]
Image Contrast

- For a sinusoidal signal

\[ f(x, y) = A + B \sin(2\pi u_0 x) \quad A \geq B \]

\[ m_f = \frac{f_{\text{max}} - f_{\text{min}}}{f_{\text{max}} + f_{\text{min}}} = \frac{(A + B) - (A - B)}{(A + B) + (A - B)} = \frac{B}{A} \]
Image Contrast

- For a sinusoidal signal

\[ f(x, y) = A + B \sin(2\pi u_0 x) \quad m_f = \frac{B}{A} \]
How does an image passing through a medical imaging system effect its contrast?

\[
\begin{align*}
\text{Input Image} & \quad f(x, y) \\
& \quad f(x, y, z) \\
\text{System Function} & \quad S \\
\text{Design Parameters} & \\
\text{Output Image} & \quad g(x, y) \\
& \quad g(x, y, z) \\
\end{align*}
\]

\[g = S[f]\]
How does an image passing through a medical imaging system effect its contrast?

The actual signal being imaged can be decomposed into many sinusoidal signals with different frequencies:

\[ f(x, y) = A + \sum_k B_k \sin(2\pi u_k x + 2\pi v_k y); \quad m_{f,k} = \frac{B_k}{A} \]
How does an image passing through a medical imaging system effect its contrast?

The actual signal being imaged can be decomposed into many sinusoidal signals with different frequencies

\[ f(x, y) = A + \sum_k B_k \sin(2\pi u_k x + 2\pi v_k y); \quad m_{f,k} = \frac{B_k}{A} \]

Suppose the imaging system can be considered as a LSI system with frequency response \( H(u, v) \)

How can we calculate the output \( g(x, y) \)?
How does an image passing through a medical imaging system effect its contrast?

![Diagram showing the process of an image passing through a system function to produce an output image.]

The actual signal being imaged can be decomposed into many sinusoidal signals with different frequencies:

\[ f(x, y) = A + \sum_k B_k \sin(2\pi u_k x + 2\pi v_k y); \quad m_{f,k} = \frac{B_k}{A} \]

Suppose the imaging system can be considered as a LSI system with frequency response \( H(u,v) \)

How can we calculate the output \( g(x,y) \)?

Imaged signal is

\[ g(x, y) = H(0,0)A + \sum_k H(u_k, v_k)B_k \sin(2\pi u_k x + 2\pi v_k y); \quad m_{g,k} = \frac{|H(u_k, v_k)|B_k}{H(0,0)A} \]
Modulation Transfer Function (MTF)

\[ f(x, y) = A + \sum_k B_k \sin(2\pi u_k x + 2\pi v_k y); \quad m_{f,k} = \frac{B_k}{A} \]

Suppose the imaging system can be considered as a LSI system with frequency response \( H(u, v) \)

\[ g(x, y) = H(0,0)A + \sum_k H(u_k, v_k)B_k \sin(2\pi u_k x + 2\pi v_k y); \quad m_{g,k} = \frac{|H(u_k, v_k)|B_k}{H(0,0)A} \]

The MTF refers to the ratio of the contrast (or modulation) of the imaged signal to the contrast of the original signal at different frequencies

\[ MTF(u, v) = \frac{m_{g,u,v}}{m_{f,u,v}} = \frac{|H(u, v)|}{H(0,0)} \]
Modulation Transfer Function (MTF)

The MTF refers to the ratio of the contrast (or modulation) of the imaged signal to the contrast of the original signal at different frequencies

\[ MTF(u, v) = \frac{m_{g,u,v}}{m_{f,u,v}} = \frac{|H(u, v)|}{H(0,0)} \]

- MTF characterizes how the contrast (or modulation) of a signal component at a particular frequency changes after imaging
- MTF = magnitude of the frequency response of the imaging system (normalized by \( H(0,0) \))
- Typically \( 0 \leq MTF(u, v) \leq MTF(0,0) = 1 \)

Decreasing MTF at higher frequencies causes the blurring of high frequency features in an image.
Modulation Transfer Function (MTF)
Modulation Transfer Function (MTF)

Example PSFs

Corresponding MTFs

How do we get from PSF to MTF?
Local Contrast

A target is an object of interest in an image. Eg. a tumor (target) in a liver (background).

\[ C = \frac{f_t - f_b}{f_b} \]

\( f_t \) = intensity (or average intensity) within a target
\( f_b \) = intensity (or average intensity) within the background

Figure 3.5

Local Contrast

\[ C = \frac{f_t - f_b}{f_b} \]

- \( f_t \): intensity (or average intensity) within a target
- \( f_b \): intensity (or average intensity) within the background
Resolution

- The ability of a system to depict spatial details.
- Which image below has higher resolution?

(a)  
(b)  
(c)  

Figure I.4

*Medical Imaging Signals and Systems*, by Jerry L. Prince and Jonathan Links.
Resolution

Two primary ways to define resolution:

(1) We characterize resolution of an imaging system by looking at either the PSF or the line spread function (LSF). The LSF tells us how wide a very thin line becomes after imaging.

(2) The second method for characterizing the resolution of an imaging system is by its MTF.
Resolution

Two primary ways to define resolution:

(1) We characterize resolution of an imaging system by looking at either the PSF or the line spread function (LSF). The LSF tells us how wide a very thin line becomes after imaging.

Assume we are trying to image two points (delta functions) separated by a distance $dx$. After imaging the two points will be smeared out by the PSF of the system.

Question: How close can those two points be placed together and still resolved after imaging?
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**Question:** How close can those two points be placed together and still resolved after imaging?
Resolution

Assume we are trying to image two points (delta functions) separated by a distance $dx$. After imaging the two points will be smeared out by the PSF of the system.

Question: How close can those two points be placed together and still resolved after imaging?

Assume each point after imaging is described by a smeared as shown here.

The full width half maximum (FWHM) distance is defined as the width in which the maximum intensity drops by 50%.

The smaller the FWHM the smaller the spot size.
Resolution

Assume we are trying to image two points (delta functions) separated by a distance $dx$. After imaging the two points will be smeared out by the PSF of the system.

Question: How close can those two points be placed together and still resolved after imaging?

If the separation between two points is less than the FWHM we cannot resolve them!

The FWHM of the PSF or LSF is then defined as the minimum resolution of our imaging system.
Resolution and MTF

Two primary ways to define resolution:

(2) The second method for characterizing the resolution of an imaging system is by its MTF.

- A pure vertical sinusoidal pattern can be thought of as the blurred image of uniformly spaced vertical lines
- The distance between lines is equal to distance between maxima
- If the frequency $= u_0$, the distance $= 1/ u_0$

$$f(x, y) = A + B \sin(2\pi u_0 x)$$

$$g(x, y) = H(0,0) A + H(u_0,0) \sin(2\pi u_0 x)$$

$$= H(0,0) A + MTF(u_0,0) H(0,0) \sin(2\pi u_0 x)$$
Resolution and MTF

- A pure vertical sinusoidal pattern can be thought of as the blurred image of uniformly spaced vertical lines.
- The distance between lines is equal to distance between maxima.
- If the frequency = $u_0$, the distance = $1/u_0$.

$$f(x, y) = A + B \sin(2\pi u_0 x)$$
$$g(x, y) = H(0,0)A + H(u_0,0) \sin(2\pi u_0 x)$$
$$= H(0,0)A + \text{MTF}(u_0,0)H(0,0) \sin(2\pi u_0 x)$$

- If $\text{MTF}(u_0)=0$, the sinusoidal patterns become all constant and one cannot see different lines.
- If $\text{MTF}(u)$ first becomes 0 at frequency $u_c$, the minimum distance between distinguishable lines = $1/u_c$.
- Resolution is directly proportional to the stopband edge in MTF.
Resolution and MTF

- Which system below has better contrast and resolution?
How can we measure resolution?

Bar Phantom

- The resolution of an imaging system can be evaluated by imaging a bar phantom.
- The resolution is the frequency (in lp/mm) of the finest line group that can be resolved after imaging.
  - Gamma camera: 2-3 lp/cm
  - CT: 2 lp/mm
  - chest x-ray: 6-8 lp/mm
What is Noise

- Random fluctuations in image intensity that are not due to actual signal
- The source of noise in an imaging system depends on the physics and instrumentation of the imaging modality
- Which image below is most noisy?

(a)  (b)  (c)

Figure 1.4
What is Noise

Figure 3.10

Medical Imaging Signals and Systems, by Jerry L. Prince and Jonathan Links.
White vs. Correlated Noise

- Model of a typical imaging system

\[ g(x, y) = f(x, y) * h(x, y) + N(x, y) \]

\( N(x, y) \) is noise
\( N(x, y) \) is a random variable at each \((x, y)\)
\( N(x, y) \) could be continuous or discrete

- White Noise: Noise values at different positions are independent of each other, and position independent
  - Mean and variance at different \((x, y)\) are same

- Correlated noise: noise at adjacent positions are correlated
  - Described by the correlation function \( R(x,y) \), whose Fourier transform is the noise power spectrum density \( NPSD(u,v) \) or simply \( NPS(u,v) \)
  - White noise has a PSD = constant = variance
Signal to Noise Ratio

- Amplitude SNR
  \[ SNR_a = \frac{\text{amplitude}(f)}{\text{amplitude}(N)} \]

- Power SNR
  \[ SNR_p = \frac{\text{power}(f)}{\text{power}(N)} \]

- Signal power:
  \[ \text{power}(f) = \iint_{x,y} |h(x, y) * f(x, y)|^2 \, dx \, dy = \iint_{u,v} |H(u, v)F(u, v)|^2 \, du \, dv \]

  Approximation: \( \text{power}(f) = A^2 \), \( A \) is the average value of the signal
  Approximation: \( \text{power}(f) = \sigma_f^2 \), variance of the signal

- Noise power:
  \[ \text{power}(N) = \iint_{u,v} NPS(u, v) \, du \, dv \]

- For white noise:
  \[ \text{power}(N) = \sigma_N^2 \]
Artifacts, distortion & accuracy

- **Artifacts:**
  - Some imaging systems can create image features that do not represent a valid object in the imaged patient, or false shapes/textures.

- **Distortion**
  - Some imaging system may distort the actual shape/position and other geometrics of imaged object.

- **Accuracy**
  - Conformity to truth and clinical utility
Image Artifacts

- Artifacts: image features that do not correspond to a real object, and are not due to noise
  - Motion artifacts: blurring or streaks due to patient motion
  - Star artifact: in CT, due to presence of metallic material in a patient
  - Beam hardening artifact: broad dark bands or streaks, due to significant beam attenuation caused by certain materials
  - Ring artifact: because detectors are out of calibration
Image Artifacts

(a) Motion artifact

(b) Star artifact

(c) Beam hardening

(d) Ring artifact
Geometric Distortion

Figure 3.13


• In (a): two objects with different sizes appear to have the same size
• In (b): two objects with same shape appear to have different shapes
Accuracy

(1) Quantitative accuracy - how well does the imaging system accurately depict the objects being imaged.

(2) Clinical accuracy – how accurate is a diagnosis based on the images
Diagnostic Accuracy

- Contingency Table

<table>
<thead>
<tr>
<th>Test</th>
<th>Disease</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>+</td>
<td></td>
</tr>
<tr>
<td>+</td>
<td>a</td>
<td>b</td>
</tr>
<tr>
<td></td>
<td>c</td>
<td>d</td>
</tr>
</tbody>
</table>

- \( a = \# \text{ w/ disease & test says disease} \)
- \( b = \# \text{ w/o disease & test says disease} \)
- \( c = \# \text{ w/ disease & test says normal} \)
- \( d = \# \text{ w/o disease & test says normal} \)

\[
\text{sensitivity} = \frac{a}{a + c}
\]
\[
\text{specificity} = \frac{d}{b + d}
\]
\[
\text{diagnostic accuracy} = \frac{a + d}{a + b + c + d}
\]