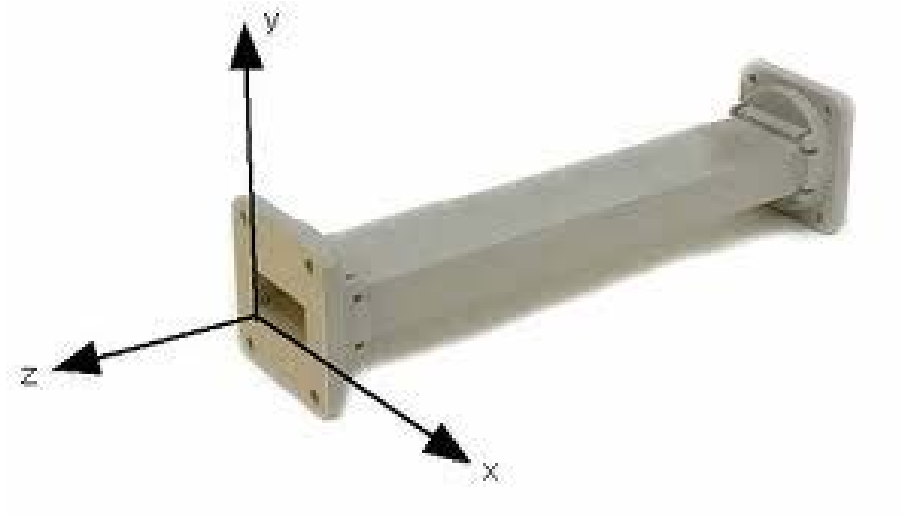
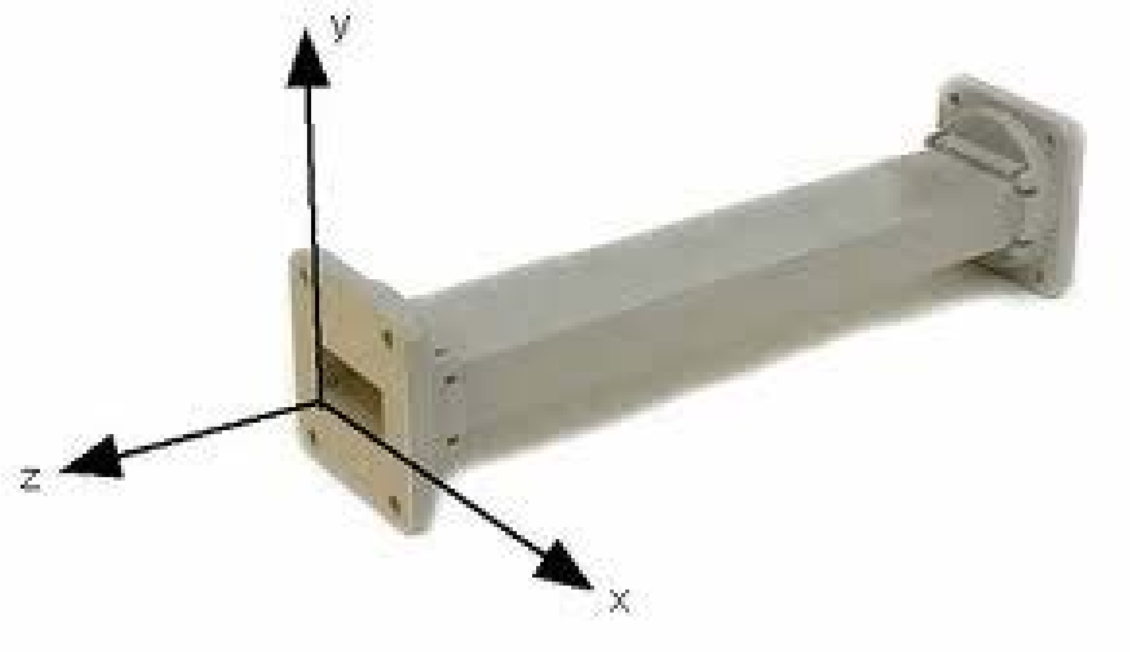
Waveguide Types



Uniform Waveguides



Because the cross section does not change in the z direction

$$\tilde{E}(x, y, z) = \tilde{E}_t(x, y)e^{-j\beta_z z}$$

$$\tilde{H}(x, y, z) = \tilde{H}_t(x, y)e^{-j\beta_z z}$$

Uniform Waveguides

$$E_x = \frac{-j}{(k_c{}^2)}\left[\omega\mu\frac{\partial H_z}{\partial y} + \beta_z\frac{\partial E_z}{\partial x}\right]$$

$$H_x = \frac{j}{(k_c{}^2)}\left[\omega\varepsilon\frac{\partial E_z}{\partial y} - \beta_z\frac{\partial H_z}{\partial x}\right]$$

$$E_y = \frac{j}{(k_c{}^2)}\left[\omega\mu\frac{\partial H_z}{\partial x} - \beta_z\frac{\partial E_z}{\partial y}\right]$$

$$H_y = \frac{-j}{(k_c{}^2)}\left[\omega\varepsilon\frac{\partial E_z}{\partial x} + \beta_z\frac{\partial H_z}{\partial y}\right]$$

Three Cases:

(1) $E_z=0$, $H_z \neq 0$ TE

$$E_x = \frac{-j}{(k_c{}^2)}\left[\omega\mu\frac{\partial H_z}{\partial y}\right]$$

$$E_y = \frac{j}{(k_c{}^2)}\left[\omega\mu\frac{\partial H_z}{\partial x}\right]$$

$$H_x = \frac{-j}{(k_c{}^2)}\left[\beta_z\frac{\partial H_z}{\partial x}\right]$$

$$H_y = \frac{-j}{(k_c{}^2)}\left[\beta_z\frac{\partial H_z}{\partial y}\right]$$

$$\nabla_t{}^2 H_z(x,y) + k_c{}^2 H_z(x,y) = 0$$

(2) $E_z \neq 0$, $H_z = 0$ TM

$$E_x = \frac{-j}{(k_c{}^2)}\left[\beta_z\frac{\partial E_z}{\partial x}\right]$$

$$E_y = \frac{-j}{(k_c{}^2)}\left[\beta_z\frac{\partial E_z}{\partial y}\right]$$

$$H_x = \frac{j}{(k_c{}^2)}\left[\omega\varepsilon\frac{\partial E_z}{\partial y}\right]$$

$$H_y = \frac{-j}{(k_c{}^2)}\left[\omega\varepsilon\frac{\partial E_z}{\partial x}\right]$$

$$\nabla_t{}^2 E_z(x,y) + k_c{}^2 E_z(x,y) = 0$$

Uniform Waveguides

$$E_x = \frac{-j}{(k_c{}^2)}\left[\omega\mu\frac{\partial H_z}{\partial y} + \beta_z\frac{\partial E_z}{\partial x}\right]$$

$$H_x = \frac{j}{(k_c{}^2)}\left[\omega\varepsilon\frac{\partial E_z}{\partial y} - \beta_z\frac{\partial H_z}{\partial x}\right]$$

$$E_y = \frac{j}{(k_c{}^2)}\left[\omega\mu\frac{\partial H_z}{\partial x} - \beta_z\frac{\partial E_z}{\partial y}\right]$$

$$H_y = \frac{-j}{(k_c{}^2)}\left[\omega\varepsilon\frac{\partial E_z}{\partial x} + \beta_z\frac{\partial H_z}{\partial y}\right]$$

Three Cases:

(3) E_z=0, H_z = 0 TEM

$$k_c = \sqrt{k^2 - \beta_z{}^2} = 0$$

$$\Rightarrow \beta_z = k = \omega\sqrt{\varepsilon\mu}$$

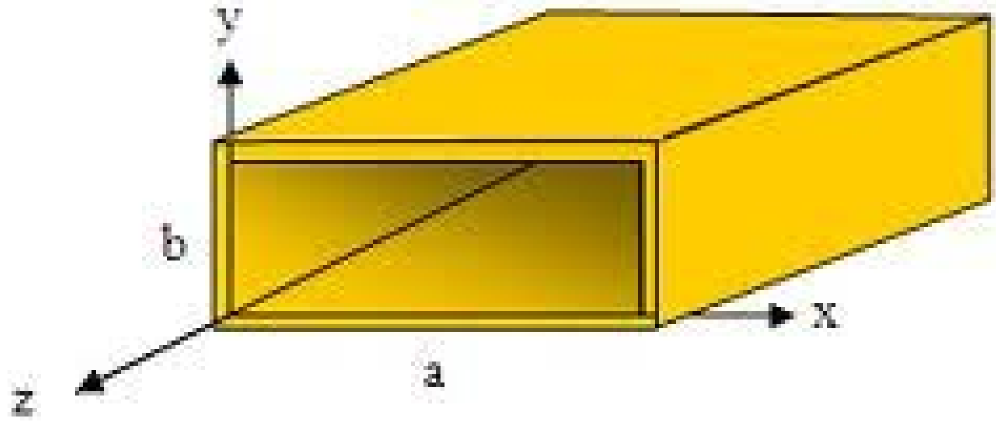
=0

$$\nabla_t{}^2 E_x(x,y) + k_c{}^2 E_x(x,y) = 0$$

$$\Rightarrow \nabla_t{}^2 E_x(x,y) = 0$$

$$\nabla_t{}^2 E_y(x,y) = 0$$

Rectangular Waveguides



TM Modes

$$\nabla_t^{\,2} E_z(x,y) + k_c^{\,2} E_z(x,y) = 0$$

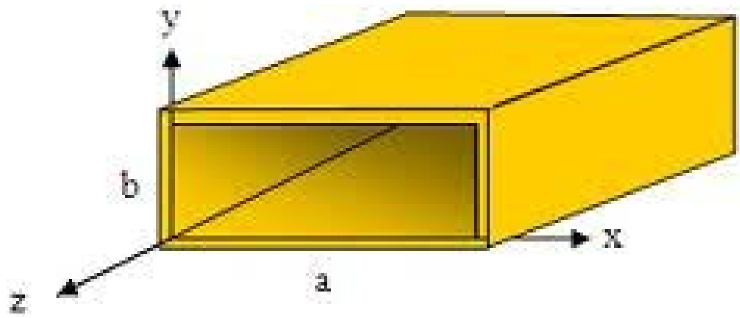
$$E_x = \frac{-j}{(k_c^{\,2})}\left[\beta_z \frac{\partial E_z}{\partial x}\right]$$

$$E_y = \frac{-j}{(k_c^{\,2})}\left[\beta_z \frac{\partial E_z}{\partial y}\right]$$

$$H_x = \frac{j}{(k_c^{\,2})}\left[\omega\varepsilon \frac{\partial E_z}{\partial y}\right]$$

$$H_y = \frac{-j}{(k_c^{\,2})}\left[\omega\varepsilon \frac{\partial E_z}{\partial x}\right]$$

Rectangular Waveguides



TM Modes

$$E_z^{nm}(x,y,z) = A_{mn}\left[\sin\left(\frac{n\pi}{a}x\right)\right]\left[\sin\left(\frac{m\pi}{b}y\right)\right]e^{-j\beta_z^{mn}z}$$

$$\beta_z^{mn} = \sqrt{\omega^2\mu\varepsilon - \left(\frac{n\pi}{a}\right)^2 - \left(\frac{m\pi}{b}\right)^2} \qquad m \text{ and } n = \pm 1,2,3,\ldots$$

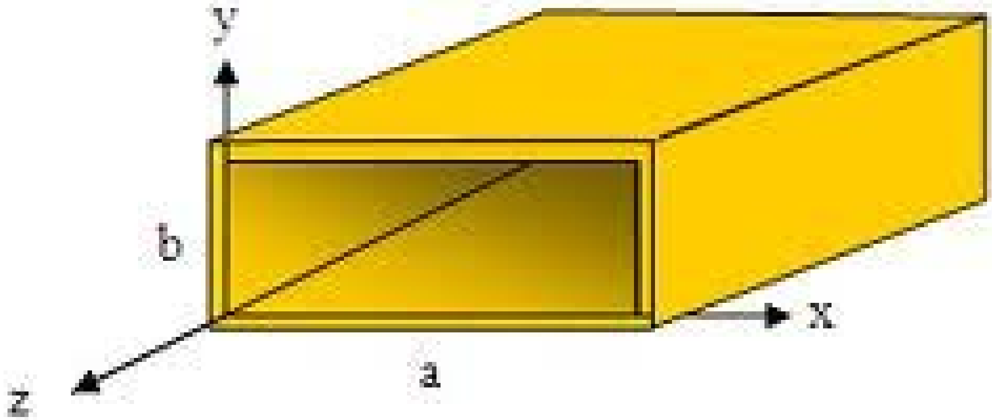
$$E_x = \frac{-j}{(k_c^2)}\left[\beta_z \frac{\partial E_z}{\partial x}\right] \qquad H_x = \frac{j}{(k_c^2)}\left[\omega\varepsilon \frac{\partial E_z}{\partial y}\right]$$

$$E_y = \frac{-j}{(k_c^2)}\left[\beta_z \frac{\partial E_z}{\partial y}\right] \qquad H_y = \frac{-j}{(k_c^2)}\left[\omega\varepsilon \frac{\partial E_z}{\partial x}\right]$$

Summary of TM modes

Plane waves in the dielectric medium	Inside the waveguide
$\beta_{PW} = \omega\sqrt{\mu\varepsilon}$	$\beta = \beta_{PW}\sqrt{1-\left[\frac{f_c}{f}\right]^2}$
$\eta_{PW} = \sqrt{\mu/\varepsilon}$	$\eta_{TM} = \eta_{PW}\sqrt{1-\left[\frac{f_c}{f}\right]^2}$
$v_p = \omega/\beta_{PW} = f\lambda = 1/\sqrt{\mu\varepsilon} = c$	$v_p = \dfrac{\omega}{\beta_{PW}\sqrt{1-\left[\frac{f_c}{f}\right]^2}}$
$\lambda_{PW} = \dfrac{c}{f}$	$\lambda = \dfrac{\lambda_{PW}}{\sqrt{1-\left[\frac{f_c}{f}\right]^2}}$

Rectangular Waveguides



TE Modes

$$\nabla_t^2 H_z(x,y) + k_c^2 H_z(x,y) = 0$$

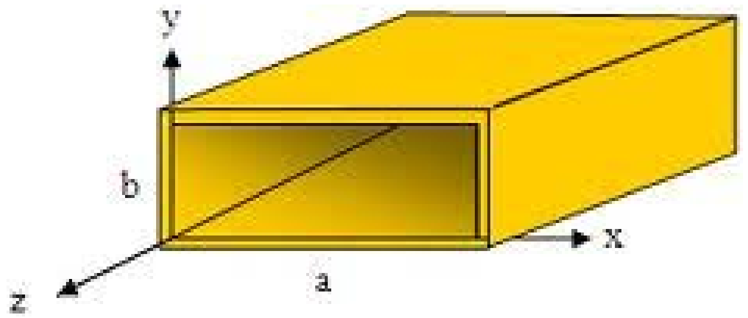
$$E_x = \frac{-j}{(k_c^2)}\left[\omega\mu \frac{\partial H_z}{\partial y}\right]$$

$$E_y = \frac{j}{(k_c^2)}\left[\omega\mu \frac{\partial H_z}{\partial x}\right]$$

$$H_x = \frac{-j}{(k_c^2)}\left[\beta_z \frac{\partial H_z}{\partial x}\right]$$

$$H_y = \frac{-j}{(k_c^2)}\left[\beta_z \frac{\partial H_z}{\partial y}\right]$$

Rectangular Waveguides



TE Modes

$$H_z(x,y,z) = \left[A_1 \cos(\frac{n\pi}{a}x) \right] \left[B_1 \cos(\frac{m\pi}{b}y) \right] e^{-j\beta_z z}$$

$$H_z^{nm}(x,y,z) = A_{nm} \cos(\frac{n\pi}{a}x) \cos(\frac{m\pi}{b}y) e^{-j\beta_z z}$$

How do we find βz ?

$$\beta_z = \sqrt{\omega^2 \mu\varepsilon - \beta_x^2 - \beta_y^2}$$

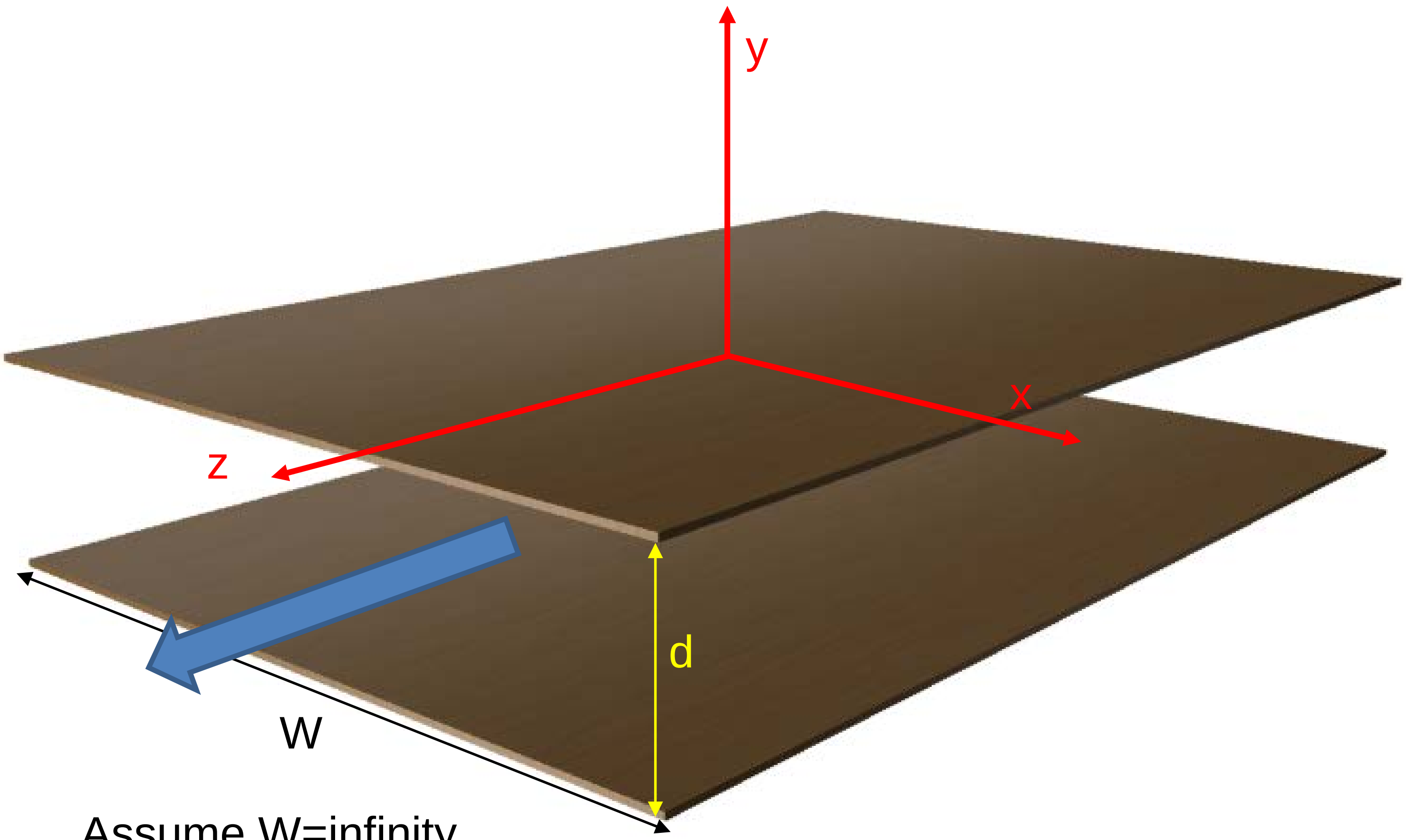
$$\beta_z = \sqrt{\omega^2 \mu\varepsilon - \left(\frac{n\pi}{a}\right)^2 - \left(\frac{m\pi}{b}\right)^2}$$

Summary of TE modes

Plane waves in the dielectric medium	Inside the waveguide
$\beta_{PW} = \omega\sqrt{\mu\varepsilon}$	$\beta = \beta_{PW}\sqrt{1-\left[\frac{f_c}{f}\right]^2}$
$\eta_{PW} = \sqrt{\mu/\varepsilon}$	$\eta_{TE} = \frac{\eta_{PW}}{\sqrt{1-\left[\frac{f_c}{f}\right]^2}}$
$v_p = \omega/\beta_{PW} = f\lambda = 1/\sqrt{\mu\varepsilon} = c$	$v_p = \frac{\omega}{\beta_{PW}\sqrt{1-\left[\frac{f_c}{f}\right]^2}}$
$\lambda_{PW} = \frac{c}{f}$	$\lambda = \frac{\lambda_{PW}}{\sqrt{1-\left[\frac{f_c}{f}\right]^2}}$

Parallel Plate Waveguides

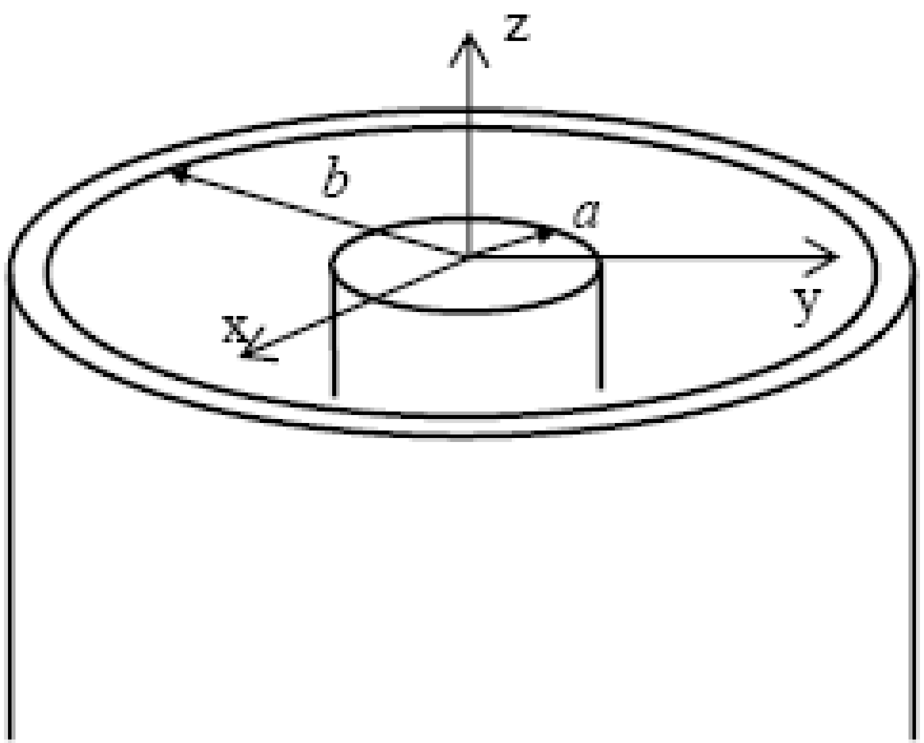
TM Modes (propagation in z direction)



Assume W=infinity

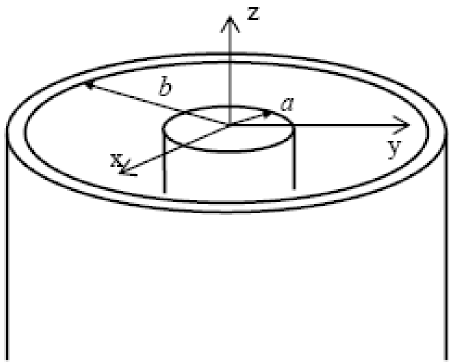
Coaxial Waveguides

TEM Modes (propagation in z direction)



Coaxial Waveguides

TEM Modes (propagation in z direction)



Solve the electrostatic problem in the transverse plane

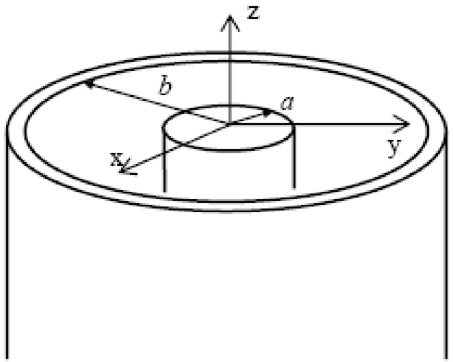
$$\tilde{E}(r,\phi,z) = \tilde{E}_t(r,\phi)e^{-jkz}$$

$$\tilde{E}_t(r,\phi) = -\nabla_t\Phi_t(r,\phi)$$

$$\nabla_t^2\Phi_t(r,\phi) = 0$$

Coaxial Waveguides

TEM Modes (propagation in z direction)

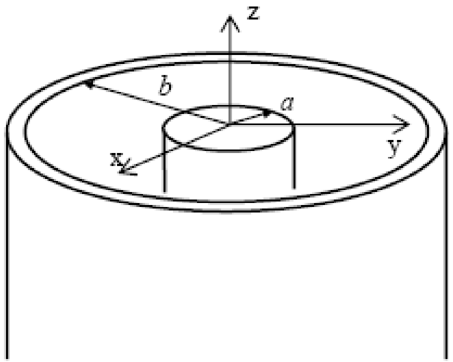


$$\nabla_t^2 \Phi_t(r,\phi) = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi_t}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \Phi_t}{\partial \phi^2} = 0$$

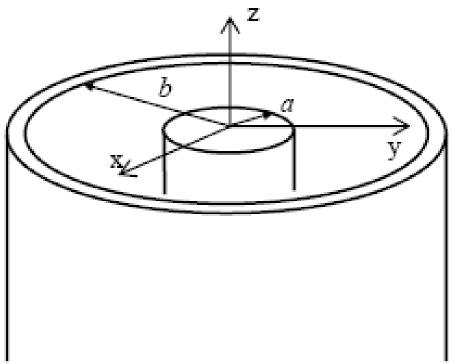
Coaxial Waveguides

TEM Modes (propagation in z direction)



Coaxial Waveguides

TEM Modes (propagation in z direction)



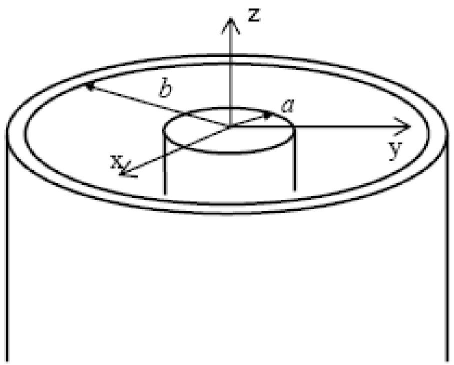
Coaxial Waveguides

TEM Modes (propagation in z direction)

$$\nabla \times \tilde{A}(r,\phi,z) = \left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{a}_r$$

$$+\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{a}_r$$

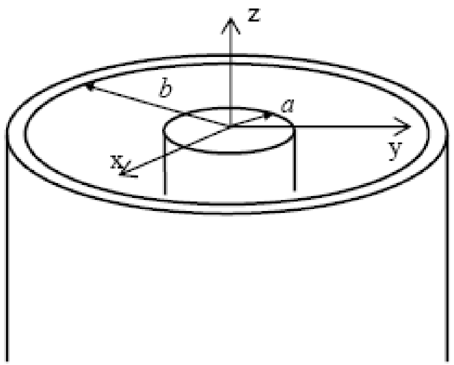
$$+\frac{1}{r}\left(\frac{\partial (rA_\phi)}{\partial r} - \frac{\partial A_r}{\partial \phi}\right)\hat{a}_z$$



$$\tilde{H} = \frac{-1}{j\omega\mu}\nabla \times \tilde{E}$$

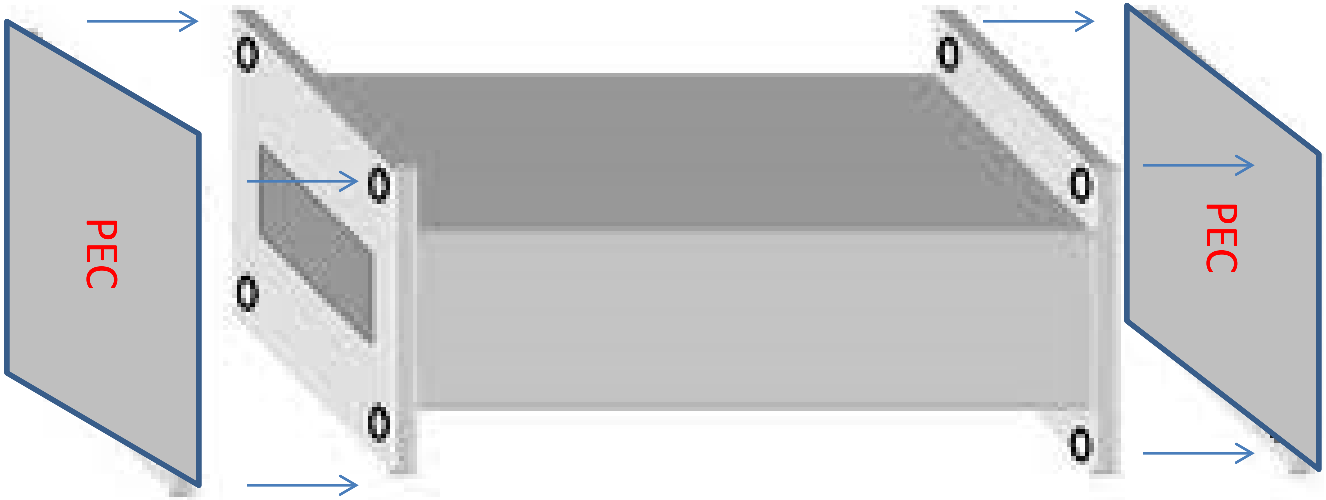
Coaxial Waveguides

TEM Modes (propagation in z direction)



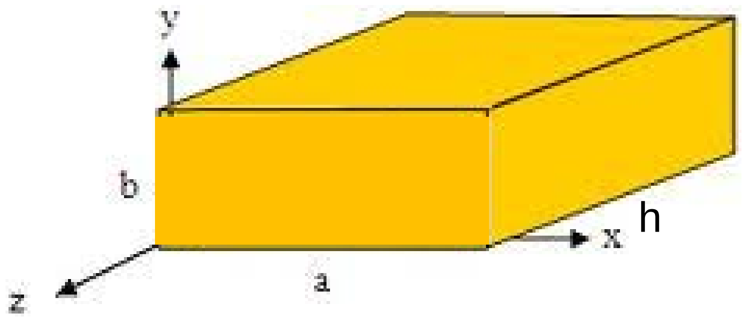
$$Z = ?$$

Rectangular Cavity Resonator



We can form a rectangular cavity by placing metal caps on two ends of a rectangular waveguide.

Rectangular Cavity Resonator

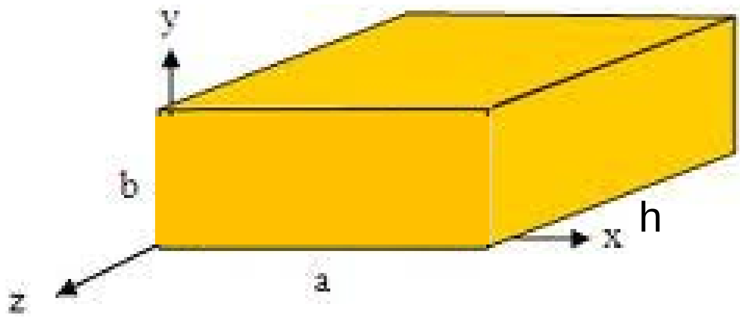


We are interested in finding what electromagnetic field solutions are possible in a resonant cavity with no sources.

We can always find those solutions by solving:

$$\nabla^2 \tilde{E}(x, y, z) + k^2 \tilde{E}(x, y, z) = 0$$
$$\nabla^2 \tilde{H}(x, y, z) + k^2 \tilde{H}(x, y, z) = 0$$

Rectangular Cavity



We can solve these!

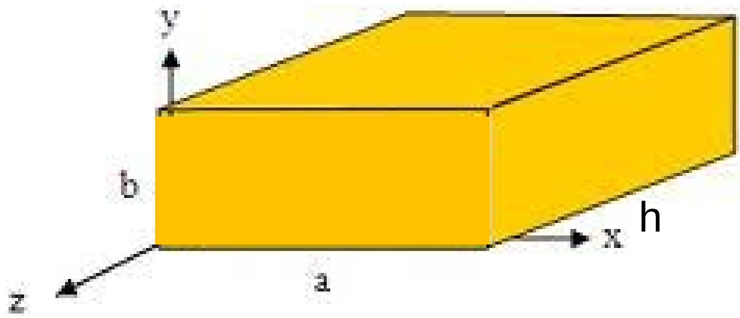
$$\frac{\partial^2}{\partial x^2}E_x(x,y,z) + \frac{\partial^2}{\partial y^2}E_x(x,y,z) + \frac{\partial^2}{\partial z^2}E_x(x,y,z) + k^2E_x(x,y,z) = 0$$

$$\frac{\partial^2}{\partial x^2}E_y(x,y,z) + \frac{\partial^2}{\partial y^2}E_y(x,y,z) + \frac{\partial^2}{\partial z^2}E_y(x,y,z) + k^2E_y(x,y,z) = 0$$

$$\frac{\partial^2}{\partial x^2}E_z(x,y,z) + \frac{\partial^2}{\partial y^2}E_z(x,y,z) + \frac{\partial^2}{\partial z^2}E_z(x,y,z) + k^2E_z(x,y,z) = 0$$

Subject to boundary conditions.

Rectangular Cavity



TM Modes in Z or TM_z

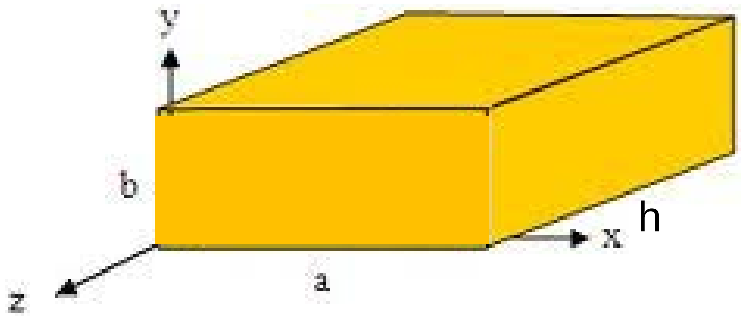
$$\frac{\partial^2}{\partial x^2}E_z(x,y,z)+\frac{\partial^2}{\partial y^2}E_z(x,y,z)+\frac{\partial^2}{\partial z^2}E_z(x,y,z)+k^2E_z(x,y,z)=0$$

Solve using separation of variables

$$E_z(x,y,z)=X(x)Y(y)Z(z)$$

$$\frac{X''(x)}{X(x)}+\frac{Y''(y)}{Y(y)}+\frac{Z''(z)}{Z(z)}+k^2=0$$

Rectangular Cavity



TM Modes in Z

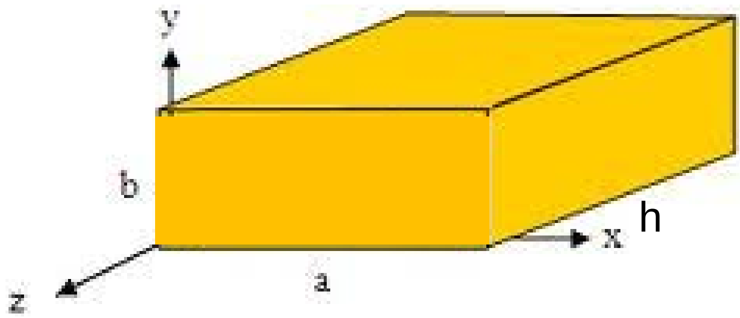
$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} + k^2 = 0$$

$$X(x) = A_1 \cos(\beta_x x) + A_2 \sin(\beta_x x)$$

$$Y(y) = B_1 \cos(\beta_y y) + B_2 \sin(\beta_y y)$$

$$Z(z) = C_1 \cos(\beta_z z) + C_2 \sin(\beta_z z)$$

Rectangular Cavity



TM Modes in Z

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} + k^2 = 0$$

$$X(x) = A_1 \cos(\beta_x x) + A_2 \sin(\beta_x x)$$

$$Y(y) = B_1 \cos(\beta_y y) + B_2 \sin(\beta_y y)$$

$$Z(z) = C_1 \cos(\beta_z z) + C_2 \sin(\beta_z z)$$

$$k^2 = \beta_x^{\,2} + \beta_y^{\,2} + \beta_z^{\,2}$$

TM_{mnp} Boundary Conditions

From these, we conclude:

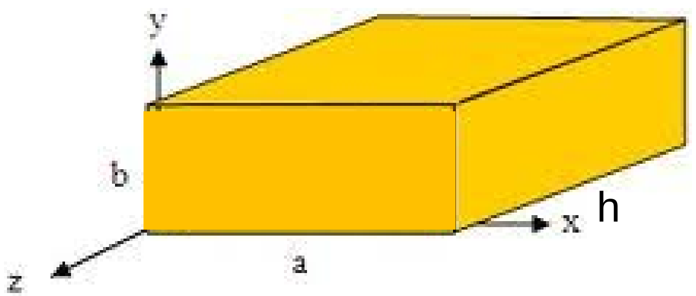
$$\beta_x = m\pi/a$$
$$\beta_y = n\pi/b$$
$$\beta_z = p\pi/h$$

$$E_z = E_o \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{p\pi z}{h}\right)$$

where

$$k^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{h}\right)^2 = \omega^2\mu\varepsilon$$

$$E_z = 0 \text{ at } y = 0, b$$
$$E_z = 0 \text{ at } x = 0, a$$
$$E_y = E_x = 0, \text{ at } z = 0, h$$



Resonant frequency

- The <u>resonant frequency</u> is the same for TM or TE modes, except that the lowest-order TM is TM_{111} and the lowest-order in TE is TE_{101}.

$$f_r = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{p}{h}\right)^2}$$