ELEG 413
Lecture #5

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Wave Equation: Time Harmonic

Time Domain

$$\nabla^2\vec{E} - \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} - \mu\sigma\frac{\partial\vec{E}}{\partial t} = \mu\frac{\partial\vec{J}}{\partial t} + \frac{1}{\varepsilon}\nabla\rho$$

Source Free $\vec{J} = 0, \quad \rho = 0$

$$\nabla^2\vec{E} - \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} - \mu\sigma\frac{\partial\vec{E}}{\partial t} = 0$$

Lossless $\sigma = 0$

$$\nabla^2\vec{E} - \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} = 0$$

Frequency Domain

$$\nabla^2\tilde{E} + \omega^2\mu\varepsilon\tilde{E} - j\omega\mu\sigma\tilde{E} = \mu\tilde{J} + \frac{1}{\varepsilon}\nabla\tilde{\rho}$$

Source Free $\vec{J} = 0, \quad \rho = 0$

$$\nabla^2\tilde{E} + \omega^2\mu\varepsilon\tilde{E} - j\omega\mu\sigma\tilde{E} = 0$$

Lossless $\sigma = 0$

$$\nabla^2\tilde{E} + \omega^2\mu\varepsilon\tilde{E} = 0$$

"Helmholtz Equation"

General Solution Case: Time Harmonic Rectangular Coordinates

Wave Number $\beta = \omega\sqrt{\mu\varepsilon}$

$$\frac{\partial^2 \tilde{E}}{\partial x^2} + \frac{\partial^2 \tilde{E}}{\partial y^2} + \frac{\partial^2 \tilde{E}}{\partial z^2} + \beta^2 \tilde{E} = 0$$

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} + \beta^2 E_x = 0$$

$$\Rightarrow \quad \frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_y}{\partial z^2} + \beta^2 E_y = 0$$

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} + \beta^2 E_z = 0$$

Separation of Variable Solutions

$$\boxed{\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} + \beta^2 E_x = 0}$$

Assume Solution of the form:

$$E_x(x, y, z) = f(x)g(y)h(z)$$

Separation of Variable Solutions

$$\underbrace{\frac{f''(x)}{f(x)}}_{\text{function of x}} + \underbrace{\frac{g''(y)}{g(y)}}_{\text{function of y}} + \underbrace{\frac{h''(z)}{h(z)}}_{\text{function of z}} + \overbrace{\beta^2}^{\text{constant}} = 0$$

(1) $$\frac{f''(x)}{f(x)} = -\beta_x^{\ 2}$$

(2) $$\frac{g''(y)}{g(y)} = -\beta_y^{\ 2}$$

(3) $$\frac{h''(z)}{h(z)} = -\beta_z^{\ 2}$$

(4) $$\beta^2 = \beta_x^{\ 2} + \beta_y^{\ 2} + \beta_z^{\ 2}$$

Separation of Variable Solutions

$$f''(x)+\beta_x^{\,2} f(x)=0$$

$$g''(y)+\beta_y^{\,2} g(y)=0$$

$$h''(z)+\beta_z^{\,2} h(z)=0$$

$$\Rightarrow$$

$$f(x)=A_1 e^{-j\beta_x x}+B_1 e^{j\beta_x x}$$

$$g(y)=A_2 e^{-j\beta_y y}+B_2 e^{j\beta_y y}$$

$$h(z)=A_3 e^{-j\beta_z z}+B_3 e^{j\beta_z z}$$

Separation of Variable Solutions

$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

$$g(y) = A_2 e^{-j\beta_y y} + B_2 e^{j\beta_y y}$$

$$h(z) = A_3 e^{-j\beta_z z} + B_3 e^{j\beta_z z}$$

Lets look at these solutions a bit more carefully. What do they physically represent?

Answer: It depends on these constants β_x, β_y and β_z

Separation of Variable Solutions

$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

Answer: It depends on these constants β_x, β_y and β_z

Case #1: β_x is a real number

$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

$$E(x,t) = \mathrm{Re}\left\{ f(x) \cdot e^{j\omega t} \right\}$$

$$E(x,t) = \mathrm{Re}\left\{ \left[A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x} \right] \cdot e^{j\omega t} \right\}$$

$$E(x,t) = \mathrm{Re}\left\{ A_1 e^{j(\omega t - \beta_x x)} + B_1 e^{j(\omega t + \beta_x x)} \right\}$$

$$E(x,t) = A_1 \cos(\omega t - \beta_x x) + B_1 \cos(\omega t + \beta_x x)$$

Separation of Variable Solutions

$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

Answer: It depends on these constants β_x, β_y and β_z

Case #1: β_x is a real number

$$E(x,t) = A_1 \cos(\omega t - \beta_x x) + B_1 \cos(\omega t + \beta_x x)$$

These solutions represent waves traveling in the x direction called "traveling waves"

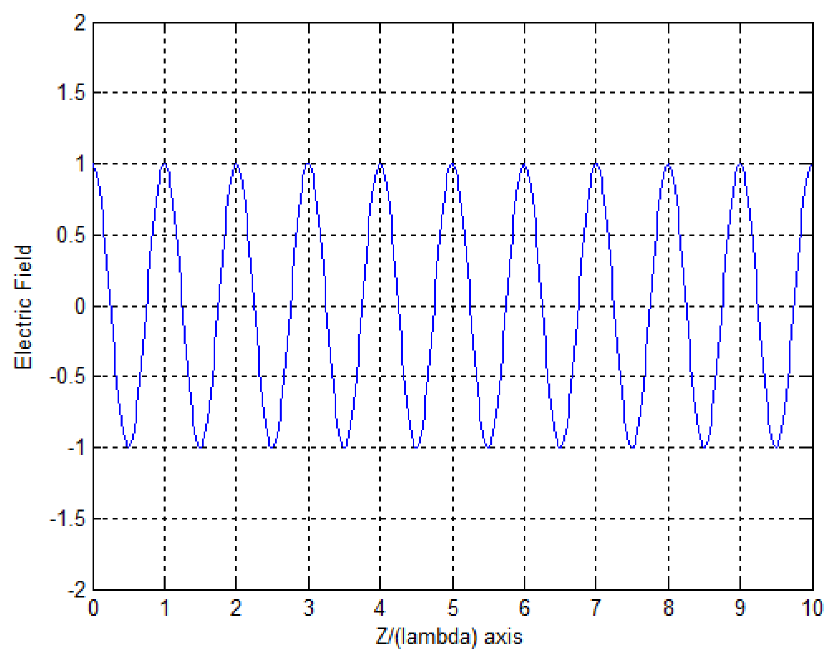
Separation of Variable Solutions

$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

Answer: It depends on these constants β_x, β_y and β_z

Case #1: β_x is a real number

$$E(x,t) = \cos(\omega t - \beta_x x)$$



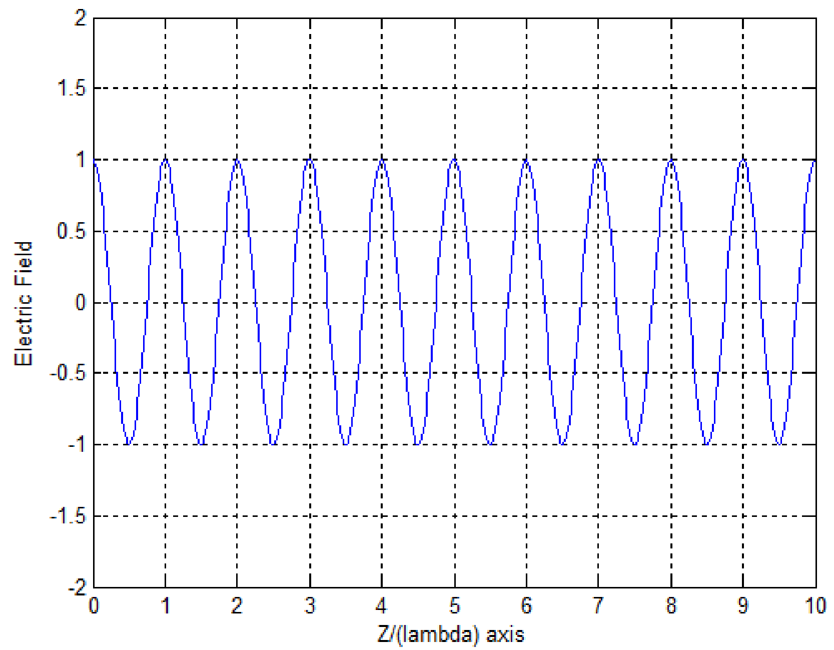
Separation of Variable Solutions

$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

Answer: It depends on these constants β_x, β_y and β_z

Case #1: β_x is a real number

$$E(x,t) = B_1 \cos(\omega t + \beta_x x)$$



Separation of Variable Solutions

$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

Answer: It depends on these constants β_x, β_y and β_z

Case #1: β_x is a real number

$$E(x,t) = \cos(\omega t - \beta_x x) + \cos(\omega t + \beta_x x)$$

?????

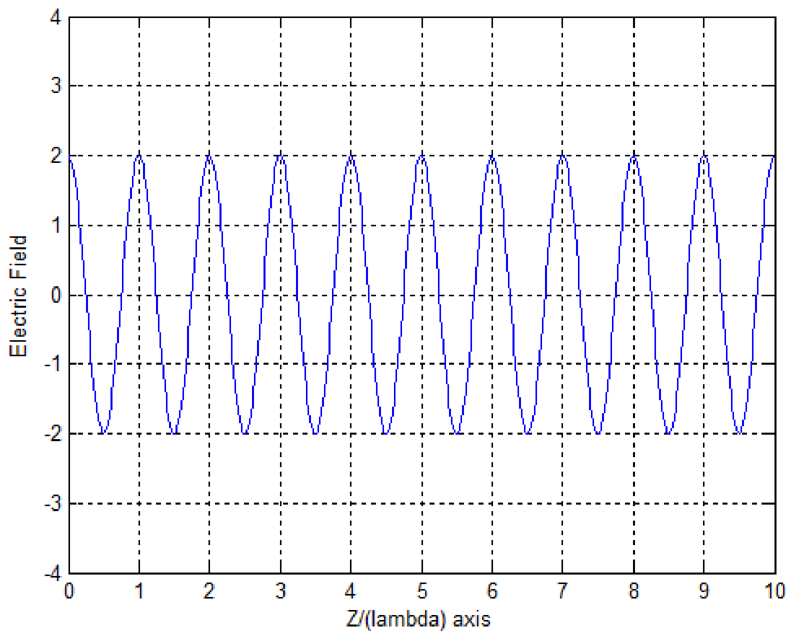
Separation of Variable Solutions

$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

Answer: It depends on these constants β_x, β_y and β_z

Case #1: β_x is a real number

$$E(x,t) = \cos(\omega t - \beta_x x) + \cos(\omega t + \beta_x x)$$



Separation of Variable Solutions

$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

Answer: It depends on these constants β_x, β_y and β_z

Case #2: β_x is a purely imaginary number

Separation of Variable Solutions

$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

Answer: It depends on these constants β_x, β_y and β_z

Case #2: β_x is a purely imaginary number

Let $\beta_x = j\alpha$ where α is purely real

Separation of Variable Solutions

$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

Answer: It depends on these constants β_x, β_y and β_z

Case #2: β_x is a purely imaginary number

Let $\beta_x = j\alpha$ where α is purely real

$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

$$f(x) = A_1 e^{-j \cdot j\alpha x} + B_1 e^{j \cdot j\alpha x}$$

$$f(x) = A_1 e^{\alpha x} + B_1 e^{-\alpha x}$$

Separation of Variable Solutions

$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

Answer: It depends on these constants β_x, β_y and β_z

Case #2: β_x is a purely imaginary number

Let $\beta_x = j\alpha$ where α is purely real

$$f(x) = A_1 e^{\alpha x} + B_1 e^{-\alpha x}$$

$$E(x,t) = \mathrm{Re}\left\{ f(x) \cdot e^{j\omega t} \right\}$$

$$E(x,t) = \mathrm{Re}\left\{ \left[A_1 e^{\alpha x} + B_1 e^{-\alpha x} \right] \cdot e^{j\omega t} \right\}$$

$$E(x,t) = \mathrm{Re}\left\{ A_1 e^{\alpha x} e^{j\omega t} + B_1 e^{-\alpha x} e^{j\omega t} \right\}$$

$$E(x,t) = A_1 e^{\alpha x}\, \mathrm{Re}\left\{ e^{j\omega t} \right\} + B_1 e^{-\alpha x}\, \mathrm{Re}\left\{ e^{j\omega t} \right\}$$

Separation of Variable Solutions

$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

Answer: It depends on these constants β_x, β_y and β_z

Case #2: β_x is a purely imaginary number

Let $\beta_x = j\alpha$ where α is purely real

$$f(x) = A_1 e^{\alpha x} + B_1 e^{-\alpha x}$$

$$E(x,t) = A_1 e^{\alpha x} \operatorname{Re}\left\{e^{j\omega t}\right\} + B_1 e^{-\alpha x} \operatorname{Re}\left\{e^{j\omega t}\right\}$$

$$E(x,t) = A_1 e^{\alpha x} \cos(\omega t) + B_1 e^{-\alpha x} \cos(\omega t)$$

Separation of Variable Solutions

$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

Answer: It depends on these constants β_x, β_y and β_z

Case #2: β_x is a purely imaginary number

Let $\beta_x = j\alpha$ where α is purely real

$$f(x) = A_1 e^{\alpha x} + B_1 e^{-\alpha x}$$

$$E(x,t) = A_1 e^{\alpha x} \cos(\omega t) + B_1 e^{-\alpha x} \cos(\omega t)$$

These solutions represent waves that do not travel but instead either grow or attenuate in space. These solutions are called “evanescent waves”

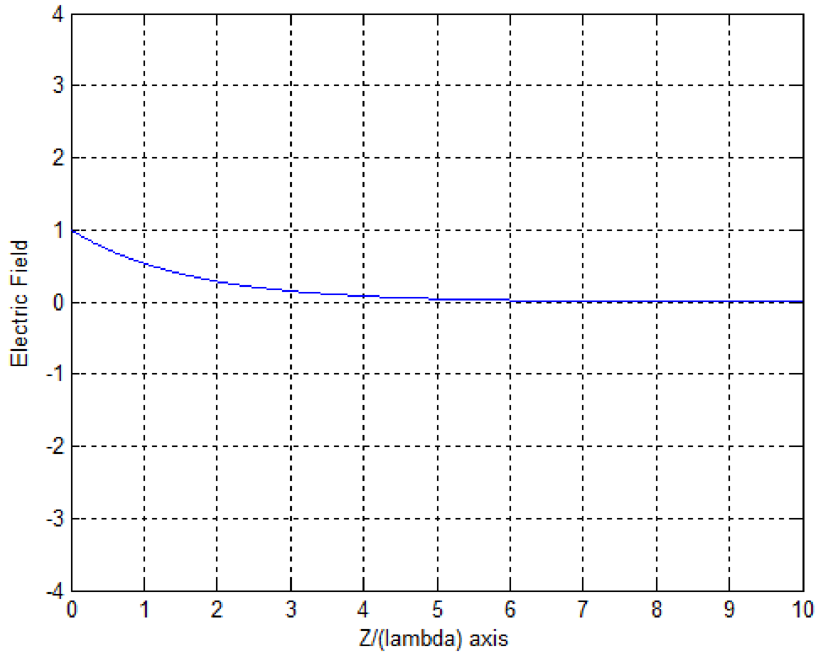
Separation of Variable Solutions

$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

Answer: It depends on these constants β_x, β_y and β_z

Case #2: β_x is a purely imaginary number

$$E(x,t) = e^{-\alpha x}\cos(\omega t)$$



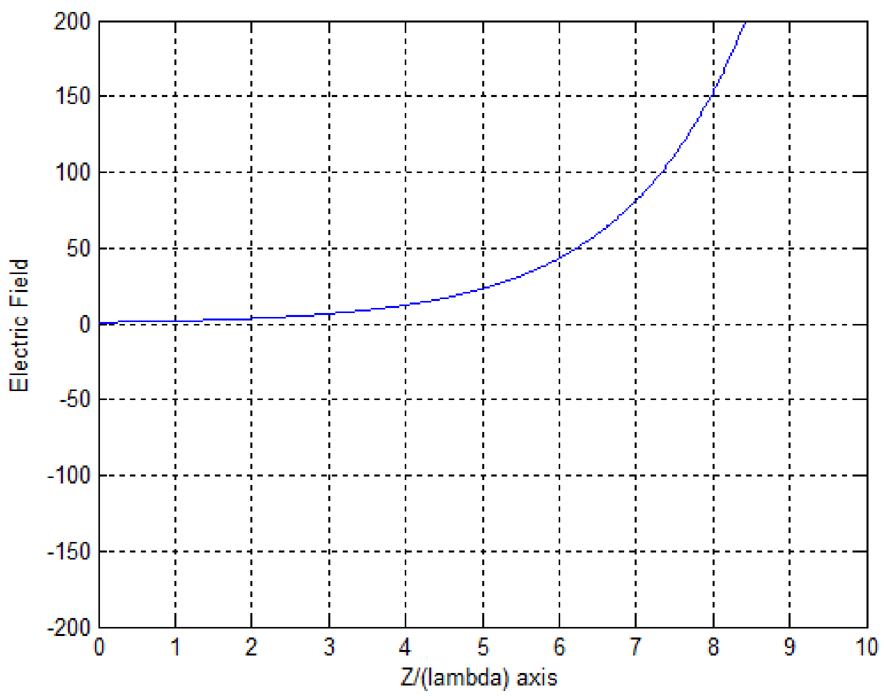
Separation of Variable Solutions

$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

Answer: It depends on these constants β_x, β_y and β_z

Case #2: β_x is a purely imaginary number

$$E(x,t) = e^{\alpha x}\cos(\omega t)$$



Separation of Variable Solutions

$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

Answer: It depends on these constants β_x, β_y and β_z

Case #3: β_x is a complex number

Let $\beta_x = \pm\gamma \pm j\alpha$ where α and γ are purely real

Separation of Variable Solutions

$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

Answer: It depends on these constants β_x, β_y and β_z

Case #3: β_x is a complex number

Let $\beta_x = \pm\gamma \pm j\alpha$ where α and γ are purely real

$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

$$f(x) = A_1 e^{-j\cdot(\pm\gamma + j\alpha)x} + B_1 e^{j(\pm\gamma - j\alpha)x}$$

$$f(x) = A_1 e^{\alpha x} e^{-j\gamma x} + A_1 e^{-\alpha x} e^{-j\gamma x} + B_1 e^{-\alpha x} e^{j\gamma x} + B_1 e^{\alpha x} e^{j\gamma x}$$

Separation of Variable Solutions

$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

Answer: It depends on these constants β_x, β_y and β_z

Case #3: β_x is a complex number

Let $\beta_x = \gamma \pm j\alpha$ where α and γ are purely real

$$f(x) = A_1 e^{\alpha x} e^{-j\gamma x} + A_1 e^{-\alpha x} e^{-j\gamma x} + B_1 e^{-\alpha x} e^{j\gamma x} + B_1 e^{\alpha x} e^{j\gamma x}$$

$$E(x,t) = \mathrm{Re}\left\{ f(x) \cdot e^{j\omega t} \right\}$$

$$E(x,t) = \mathrm{Re}\left\{ \left[A_1 e^{\alpha x} e^{-j\gamma x} + B_1 e^{-\alpha x} e^{j\gamma x} + A_1 e^{-\alpha x} e^{-j\gamma x} + B_1 e^{\alpha x} e^{j\gamma x} \right] \cdot e^{j\omega t} \right\}$$

$$E(x,t) = A_1 e^{\alpha x}\,\mathrm{Re}\left\{ e^{j(\omega t - \gamma x)} \right\} + A_1 e^{-\alpha x}\,\mathrm{Re}\left\{ e^{j(\omega t - \gamma x)} \right\} + B_1 e^{-\alpha x}\,\mathrm{Re}\left\{ e^{j(\omega t + \gamma x)} \right\} + B_1 e^{\alpha x}\,\mathrm{Re}\left\{ e^{j(\omega t + \gamma x)} \right\}$$

Separation of Variable Solutions

$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

Answer: It depends on these constants β_x, β_y and β_z

Case #3: β_x is a complex number

Let $\beta_x = \gamma \pm j\alpha$ where α and γ are purely real

$$f(x) = A_1 e^{\alpha x} e^{-j\gamma x} + A_1 e^{-\alpha x} e^{-j\gamma x} + B_1 e^{-\alpha x} e^{j\gamma x} + B_1 e^{\alpha x} e^{j\gamma x}$$

$$E(x,t) = A_1 e^{\alpha x} \operatorname{Re}\left\{e^{j(\omega t - \gamma x)}\right\} + A_1 e^{-\alpha x} \operatorname{Re}\left\{e^{j(\omega t - \gamma x)}\right\} + B_1 e^{-\alpha x} \operatorname{Re}\left\{e^{j(\omega t + \gamma x)}\right\} + B_1 e^{\alpha x} \operatorname{Re}\left\{e^{j(\omega t + \gamma x)}\right\}$$

$$E(x,t) = A_1 e^{\alpha x} \cos(\omega t - \gamma x) + A_1 e^{-\alpha x} \cos(\omega t - \gamma x)$$
$$+ B_1 e^{\alpha x} \cos(\omega t + \gamma x) + B_1 e^{-\alpha x} \cos(\omega t + \gamma x)$$

Separation of Variable Solutions

$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

Answer: It depends on these constants β_x, β_y and β_z

Case #3: β_x is a complex number

Let $\beta_x = \gamma \pm j\alpha$ where α and γ are purely real

$$E(x,t) = A_1 e^{\alpha x}\cos(\omega t - \gamma x) + A_1 e^{-\alpha x}\cos(\omega t - \gamma x)$$
$$+ B_1 e^{\alpha x}\cos(\omega t + \gamma x) + B_1 e^{-\alpha x}\cos(\omega t + \gamma x)$$

What do these solutions look like physically?

Separation of Variable Solutions

$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

Answer: It depends on these constants β_x, β_y and β_z

Case #3: β_x is a complex number

Let $\beta_x = \gamma \pm j\alpha$ where α and γ are purely real

$$E(x,t) = A_1 e^{\alpha x}\cos(\omega t - \gamma x) + A_1 e^{-\alpha x}\cos(\omega t - \gamma x)$$
$$+ B_1 e^{\alpha x}\cos(\omega t + \gamma x) + B_1 e^{-\alpha x}\cos(\omega t + \gamma x)$$

These solutions represent waves that travel and either grow or attenuate as they travel.

Separation of Variable Solutions

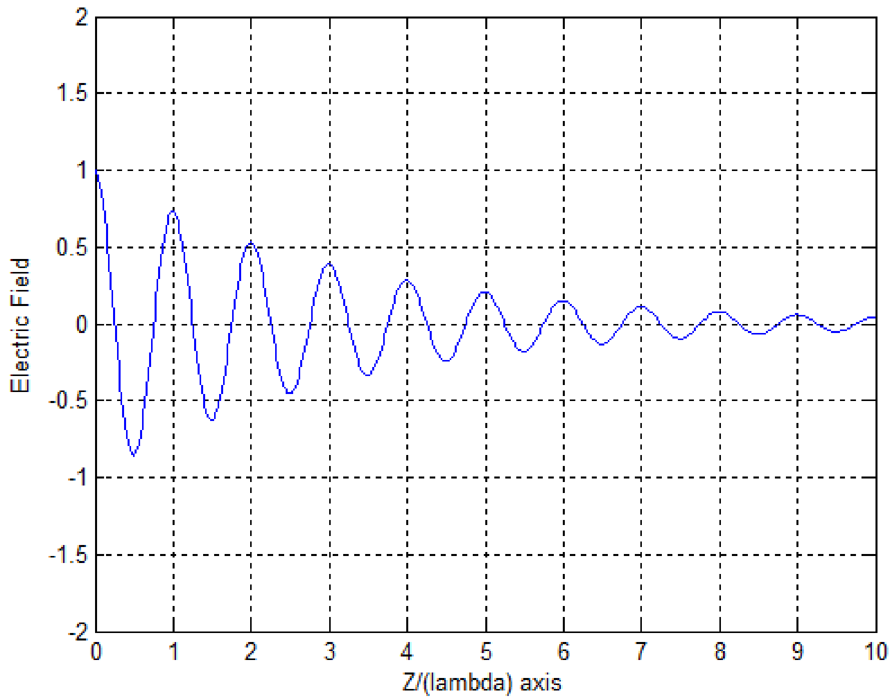
$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

Answer: It depends on these constants β_x, β_y and β_z

Case #3: β_x is a complex number

Let $\beta_x = \gamma \pm j\alpha$ where α and γ are purely real

$$E(x,t) = e^{-\alpha x}\cos(\omega t - \gamma x)$$



Separation of Variable Solutions

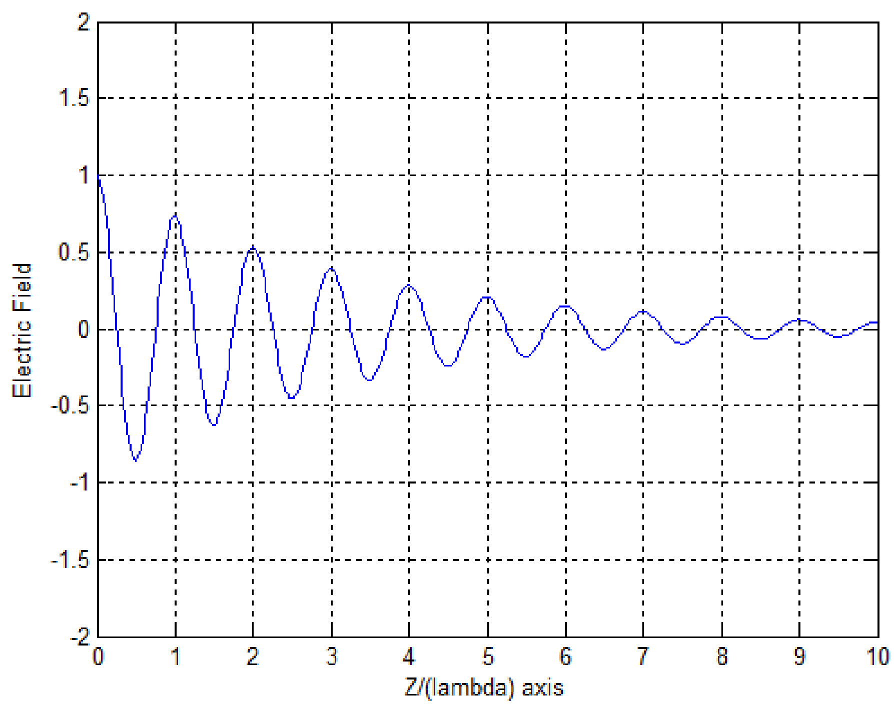
$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

Answer: It depends on these constants β_x, β_y and β_z

Case #3: β_x is a complex number

Let $\beta_x = \gamma \pm j\alpha$ where α and γ are purely real

$$E(x,t) = e^{-\alpha x}\cos(\omega t + \gamma x)$$



Separation of Variable Solutions

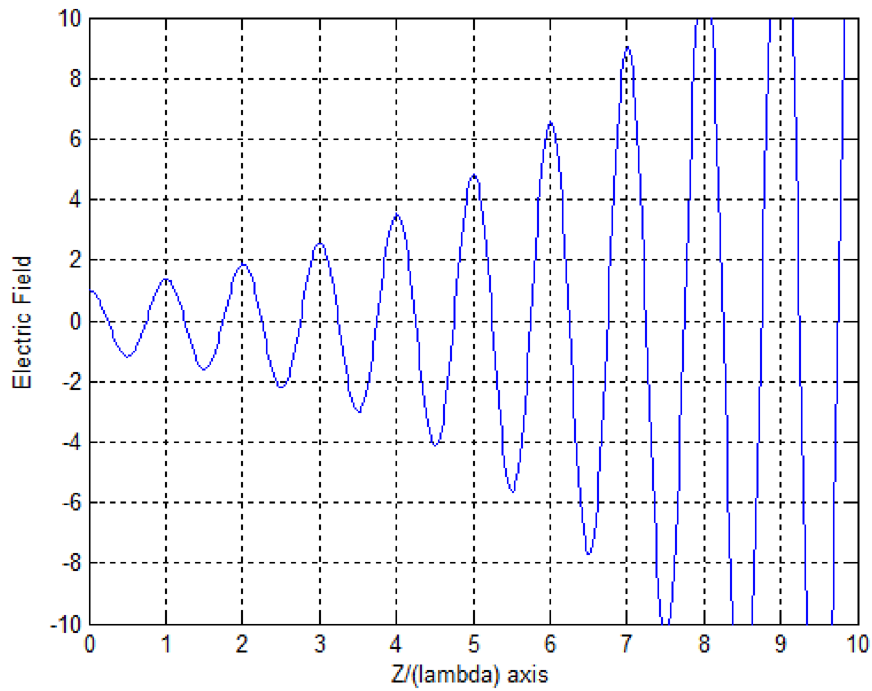
$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

Answer: It depends on these constants β_x, β_y and β_z

Case #3: β_x is a complex number

Let $\beta_x = \gamma \pm j\alpha$ where α and γ are purely real

$$E(x,t) = e^{\alpha x}\cos(\omega t - \gamma x)$$



Separation of Variable Solutions

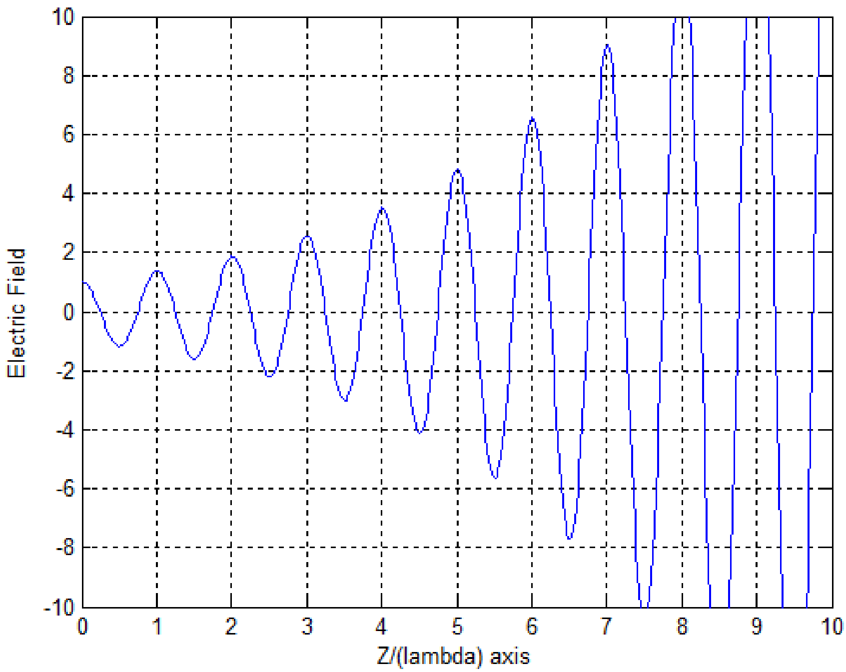
$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

Answer: It depends on these constants β_x, β_y and β_z

Case #3: β_x is a complex number

Let $\beta_x = \gamma \pm j\alpha$ where α and γ are purely real

$$E(x,t) = e^{\alpha x}\cos(\omega t + \gamma x)$$



Separation of Variable Solutions

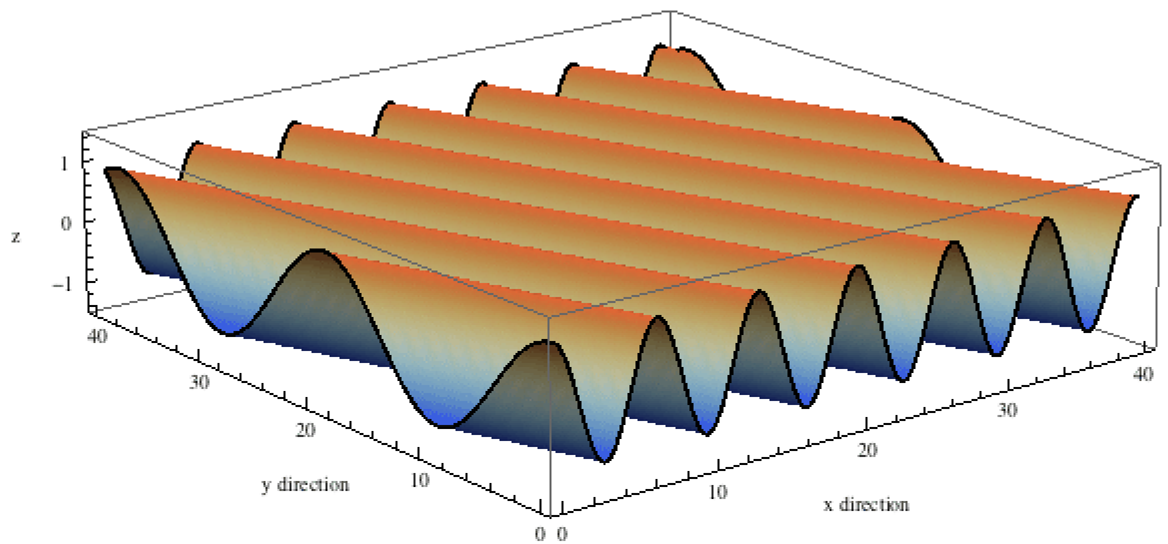
β purely real	β purely imaginary	β complex
Traveling and standing waves $f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$ $E(x,t) = A_1 \cos(\omega t - \beta_x x)$ $+ B_1 \cos(\omega t + \beta_x x)$	**Evanescent waves** $f(x) = A_1 e^{\alpha x} + B_1 e^{-\alpha x}$ $E(x,t) = A_1 e^{\alpha x} \cos(\omega t)$ $+ B_1 e^{-\alpha x} \cos(\omega t)$	**Exponentially modulated traveling wave** $f(x) = A_1 e^{\alpha x} e^{-j\gamma x} + A_1 e^{-\alpha x} e^{-j\gamma x}$ $+ B_1 e^{-\alpha x} e^{j\gamma x} + B_1 e^{\alpha x} e^{j\gamma x}$ $E(x,t) = A_1 e^{\alpha x} \cos(\omega t - \gamma x) + A_1 e^{-\alpha x} \cos(\omega t - \gamma x)$ $+ B_1 e^{\alpha x} \cos(\omega t + \gamma x) + B_1 e^{-\alpha x} \cos(\omega t + \gamma x)$

Separation of Variable Solutions: Examples

<u>case a. β_x, β_y, β_z all real (forward traveling waves)</u>

$$E_x(x,y,z) = f(x)g(y)h(z) = E_{xo}e^{-j\beta_x x}e^{-j\beta_y y}e^{-j\beta_z z}$$

Plane Waves



Separation of Variable Solutions: Examples

Guided Waves

case b. β_x real (forward standing wave), β_x (real standing wave), β_z real (traveling wave)

$$E_x(x,y,z) = E_{xo} e^{-j\beta_z z}(A_1\cos(\beta_x x) + B_1\sin(\beta_x x))(C_1\cos(\beta_y y) + D_1\sin(\beta_y y))$$

$$E_y(x,y,z) = E_{yo} e^{-j\beta_z z}(A_1\cos(\beta_x x) + B_1\sin(\beta_x x))(C_1\cos(\beta_y y) + D_1\sin(\beta_y y))$$

$$E_z(x,y,z) = E_{zo} e^{-j\beta_z z}(A_1\cos(\beta_x x) + B_1\sin(\beta_x x))(C_1\cos(\beta_y y) + D_1\sin(\beta_y y))$$

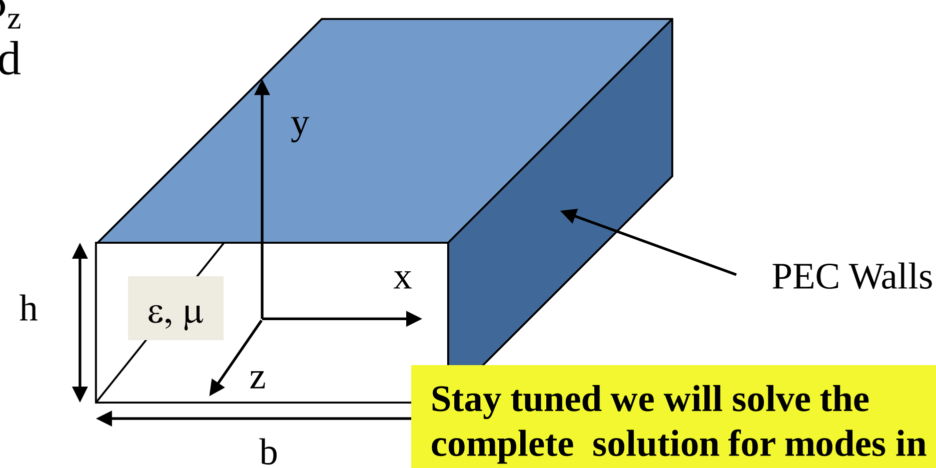
Unknown constants A_1, B_1, C_1, D_1, β_x, β_y, β_z Found by applying boundary conditions and dispersion relation. Namely:

$$E_x(x, y=\pm h/2, z) = 0$$

$$E_y(x=\pm b/2, y, z) = 0$$

$$E_z(x=\pm b/2, y=\pm h/2, z) = 0$$

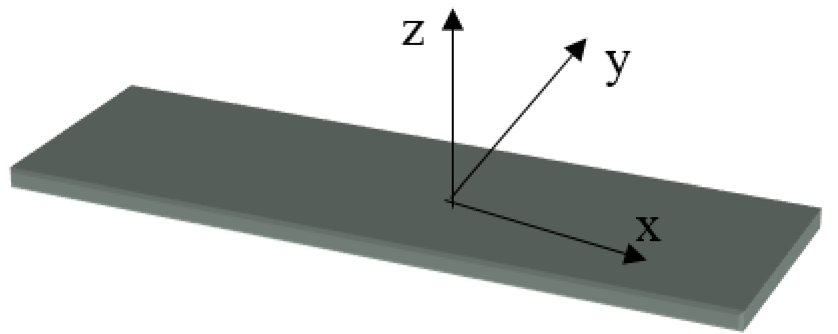
$$\beta_x^{\,2} + \beta_y^{\,2} + \beta_z^{\,2} = \beta^2 = \omega^2\varepsilon\mu$$



Stay tuned we will solve the complete solution for modes in a rectangular waveguide in a later lecture.

Separation of Variable Solutions: Examples

case c. β_x real (forward traveling wave), β_x (real standing wave), β_z imaginary (evanescent wave)



Surface Waves

$$E_x(x,y,z) = f(x)g(y)h(z) = E_{xo}e^{-j\beta_x x}(C_1\cos(\beta_y y) + D_1\sin(\beta_y y))e^{-\alpha z}$$

General Solution Case: Time Harmonic Cylindrical Coordinates

$$\nabla^2 \tilde{E} + \omega^2 \mu\varepsilon \tilde{E} = 0 \quad \Rightarrow \quad \nabla^2 \tilde{E} + \beta^2 \tilde{E} = 0$$

$$\beta = \omega\sqrt{\mu\varepsilon}$$ **Wave Number**

In cylindrical coordinates:

$$\nabla^2 \tilde{E} + \beta^2 \tilde{E} = \nabla(\nabla \cdot \tilde{E}) - \nabla \times \nabla \times \tilde{E} + \beta^2 \tilde{E} = 0$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial E_\rho}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 E_\rho}{\partial\phi^2} + \frac{\partial^2 E_\rho}{\partial z^2} - \frac{E_\rho}{\rho^2} - \frac{2}{\rho^2}\frac{\partial E_\phi}{\partial\phi} + \beta^2 E_\rho = 0$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial E_\phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 E_\phi}{\partial\phi^2} + \frac{\partial^2 E_\phi}{\partial z^2} - \frac{E_\phi}{\rho^2} + \frac{2}{\rho^2}\frac{\partial E_\rho}{\partial\phi} + \beta^2 E_\phi = 0$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial E_z}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 E_z}{\partial\phi^2} + \frac{\partial^2 E_z}{\partial z^2} + \beta^2 E_z = 0$$

Separation of Variable Solutions

$$\boxed{\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial E_z}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 E_z}{\partial\phi^2}+\frac{\partial^2 E_z}{\partial z^2}+\beta^2 E_z=0}$$

Assume Solution of the form: $E_z(\rho,\phi,z)=f(\rho)g(\phi)h(z)$

$$g(\phi)h(z)\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f(\rho)}{\partial\rho}\right)+f(\rho)h(z)\frac{1}{\rho^2}\frac{\partial^2 g(\phi)}{\partial\phi^2}+f(\rho)g(\phi)\frac{\partial^2 h(z)}{\partial z^2}+\beta^2 f(\rho)g(\phi)h(z)=0$$

$$\frac{1}{f(\rho)g(\phi)h(z)}\left[g(\phi)h(z)\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f(\rho)}{\partial\rho}\right)+f(\rho)h(z)\frac{1}{\rho^2}\frac{\partial^2 g(\phi)}{\partial\phi^2}+f(\rho)g(\phi)\frac{\partial^2 h(z)}{\partial z^2}+\beta^2 f(\rho)g(\phi)h(z)\right]=0$$

$$\underbrace{\frac{1}{f(\rho)\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f(\rho)}{\partial\rho}\right)}_{\textbf{function of }\rho}+\underbrace{\frac{1}{g(\phi)}\frac{1}{\rho^2}\frac{\partial^2 g(\phi)}{\partial\phi^2}}_{\textbf{function of }\phi,\,\rho}+\underbrace{\frac{1}{h(z)}\frac{\partial^2 h(z)}{\partial z^2}}_{\textbf{function of z}}+\overbrace{\beta^2}^{\textbf{constant}}=0$$

Separation of Variable Solutions

$$\underbrace{\frac{1}{f(\rho)\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f(\rho)}{\partial\rho}\right)}_{\textbf{function of }\rho}+\underbrace{\frac{1}{g(\phi)}\frac{1}{\rho^2}\frac{\partial^2 g(\phi)}{\partial\phi^2}}_{\textbf{function of }\phi,\,\rho}+\underbrace{\frac{1}{h(z)}\frac{\partial^2 h(z)}{\partial z^2}}_{\textbf{function of z}}+\overbrace{\beta^2}^{\textbf{constant}}=0$$

$$\Rightarrow \frac{1}{h(z)}\frac{\partial^2 h(z)}{\partial z^2}=-\beta_z{}^2$$

$$\underbrace{\frac{\rho}{f(\rho)}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f(\rho)}{\partial\rho}\right)}_{\textbf{function of }\rho}+\underbrace{\frac{1}{g(\phi)}\frac{\partial^2 g(\phi)}{\partial\phi^2}}_{\textbf{function of }\phi}+\rho^2\left(\beta^2-\beta_z{}^2\right)=0$$

$$\Rightarrow \frac{1}{g(\phi)}\frac{\partial^2 g(\phi)}{\partial\phi^2}=-m^2 \qquad \text{and} \qquad \rho\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f(\rho)}{\partial\rho}\right)+\left((\rho\beta_\rho)^2-m^2\right)f(\rho)=0$$

where $\beta_\rho{}^2=\beta^2-\beta_z{}^2$

Separation of Variable Solutions

Three ordinary differential equations:

(1) $$\frac{\partial^2 h(z)}{\partial z^2} + \beta_z{}^2 h(z) = 0$$

(2) $$\frac{\partial^2 g(\phi)}{\partial \phi^2} + m^2 g(\phi) = 0$$

(3) $$\rho^2 \frac{\partial^2 f(\rho)}{\partial \rho^2} + \rho \frac{\partial f(\rho)}{\partial \rho} + \left((\rho\beta_\rho)^2 - m^2\right) f(\rho) = 0$$

where $\beta^2 = \beta_\rho{}^2 + \beta_z{}^2$

Solutions:

(1) $h(z) = A_1 e^{-j\beta_z z} + B_1 e^{j\beta_z z}$

(2) $g(\phi) = A_2 \cos(m\phi) + B_2 \sin(m\phi)$

(3) $f(\rho) = A_3 J_m(\beta_\rho \rho) + B_3 Y_m(\beta_\rho \rho)$ or $f(\rho) = A_3 H_m^{(1)}(\beta_\rho \rho) + B_3 H_m^{(2)}(\beta_\rho \rho)$

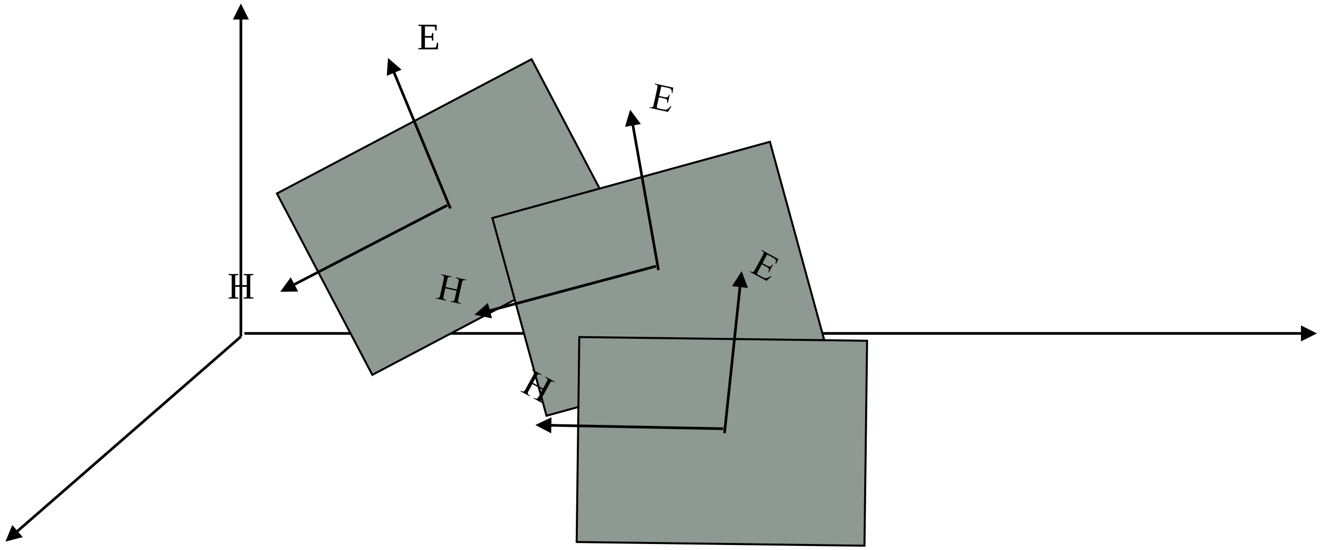
Bessel functions

Hankel functions

Wave Propagation and Polarization

TEM: Transverse Electromagnetic Waves

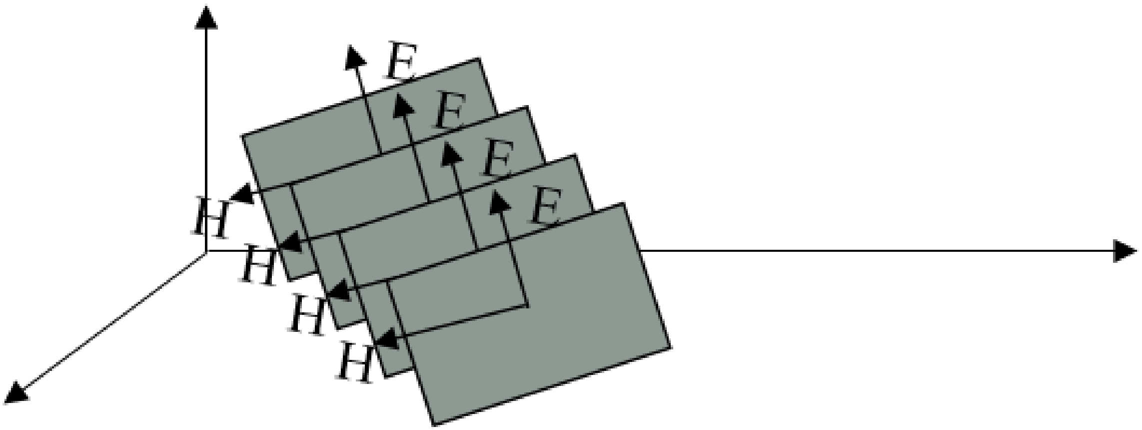
"A mode is a particular field configuration. For a given electromagnetic boundary value problem, many field configurations that satisfy the wave equation, Maxwell's equations, and boundary conditions usually exits. A TEM mode is one whole field intensities, both E and H, at every point in space are contained in a local plane, referred to as equiphase plane, that is independent of time"



Wave Propagation and Polarization

Plane Waves

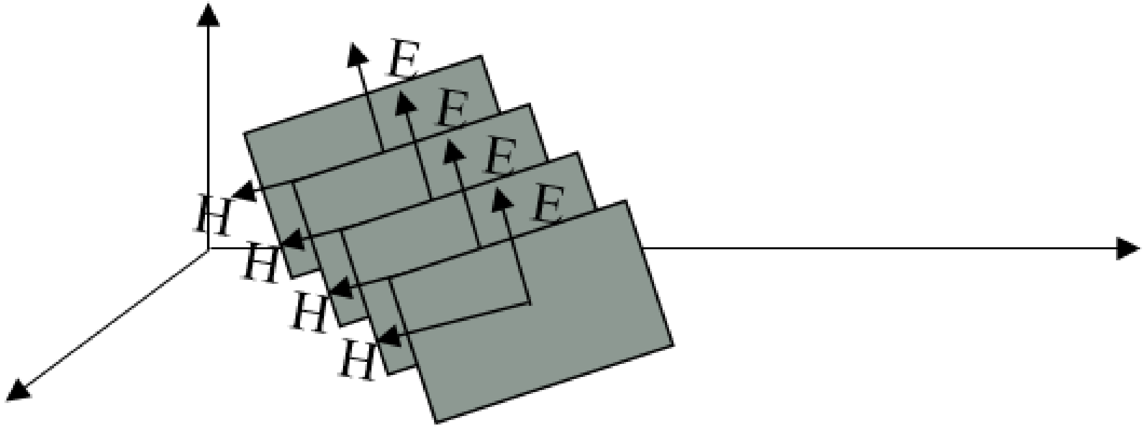
"If the space orientation of the planes for a TEM mode are the same (equiphase planes are parallel) then the fields form a plane wave.



Wave Propagation and Polarization

Uniform Plane Waves

"If in addition to having planar equiphases the field has equal amplitude (the amplitude of the field is the same over each plane) planar surfaces then it is called a uniform plane wave."



Uniform Plane Waves in Unbounded Lossless Medium
Principal Axis Propagation

$$\frac{\partial^2 E_x}{\partial x^2}+\frac{\partial^2 E_x}{\partial y^2}+\frac{\partial^2 E_x}{\partial z^2}+\beta^2 E_x=0$$

$$\frac{\partial^2 E_y}{\partial x^2}+\frac{\partial^2 E_y}{\partial y^2}+\frac{\partial^2 E_y}{\partial z^2}+\beta^2 E_y=0$$

$$\frac{\partial^2 E_z}{\partial x^2}+\frac{\partial^2 E_z}{\partial y^2}+\frac{\partial^2 E_z}{\partial z^2}+\beta^2 E_z=0$$

For uniform plane wave assume the solution is only a function of z and has only the x component of electric field.

$$\widetilde{E}(z)=\hat{a}_x E_x(z)=\hat{a}_x E_o^+ e^{-j\beta z}$$

Uniform Plane Waves in Unbounded Lossless Medium
Principal Axis Propagation

For uniform plane wave assume the solution is only a function of z and has only the x component of electric field.

$$\tilde{E}(z) = \hat{a}_x E_x(z) = \hat{a}_x E_o^+ e^{-j\beta z}$$

$$\tilde{H} = -\frac{1}{j\omega\mu}\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$

$$\tilde{H} = -\frac{1}{j\omega\mu_1}\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_o e^{-j\beta z} & 0 & 0 \end{vmatrix} = -\hat{a}_y \frac{1}{j\omega\mu_1}\frac{\partial}{\partial z}(E_o e^{-j\beta z}) = \hat{a}_y \frac{\beta}{\omega\mu} E_o e^{-j\beta z}$$

$$\tilde{H} = \hat{a}_y \frac{\beta}{\omega\mu} E_o e^{-j\beta z} = \hat{a}_y \frac{\omega\sqrt{\mu\varepsilon}}{\omega\mu} E_o e^{-j\beta z} = \hat{a}_y \frac{1}{\sqrt{\mu/\varepsilon}} E_o e^{-j\beta z}$$

Uniform Plane Waves in Unbounded Lossless Medium
Principal Axis Propagation

For uniform plane wave assume the solution is only a function of z and has only the x component of electric field.

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What observations can we make about the relationship between E and H for uniform plane waves?

Uniform Plane Waves in Unbounded Lossless Medium
Principal Axis Propagation

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$$\tilde{H}(z) = \hat{a}_y \frac{1}{\sqrt{\mu/\varepsilon}} E_o e^{-j\beta z}$$

What observations can we make about the relationship between E and H for uniform plane waves?

(1) Both E and H are traveling in the same direction (+z) with the same phase (or wave number) β

(2) Both E and H are polarized in directions that are orthogonal (i.e. 90 degrees) from the direction of propagation and they are orthogonal to each other.

(3) H has a magnitude that is smaller than E by $\frac{1}{\sqrt{\mu/\varepsilon}}$ or $\frac{1}{\eta}$

Uniform Plane Waves in Unbounded Lossless Medium
Principal Axis Propagation

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How do I figure out which way H is polarized if I know E and the direction of propagation?

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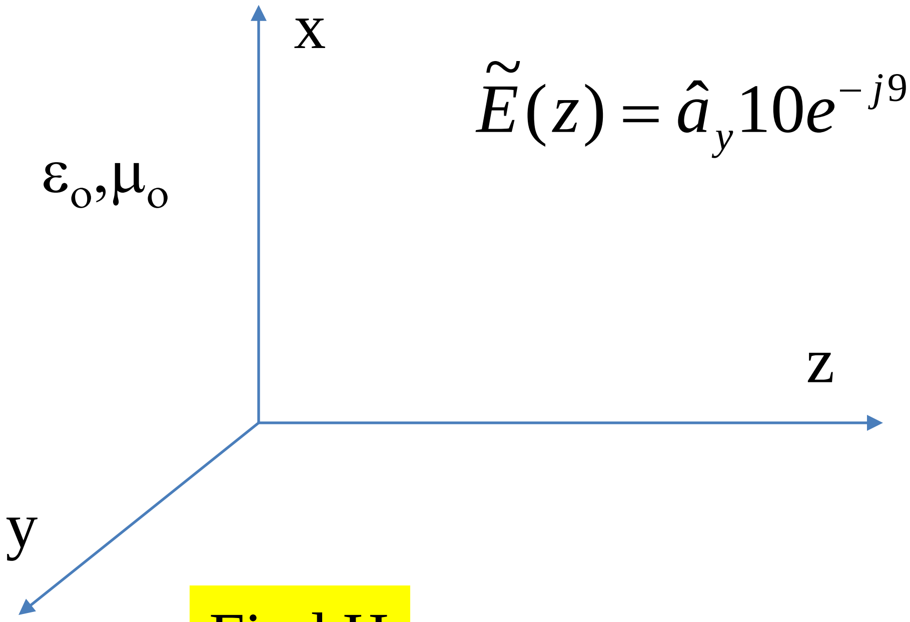
How do I figure out which way H is polarized if I know E and the direction of propagation? Which direction is the energy moving?

$$\hat{S} = \tilde{E} \times \tilde{H}$$

The Poynting vector S must be in the direction of propagation (e.g. +z for this example). Thus E cross H using the **right hand rule** must point in the direction of energy or propagation.

Uniform Plane Waves in Unbounded Lossless Medium
Principal Axis Propagation



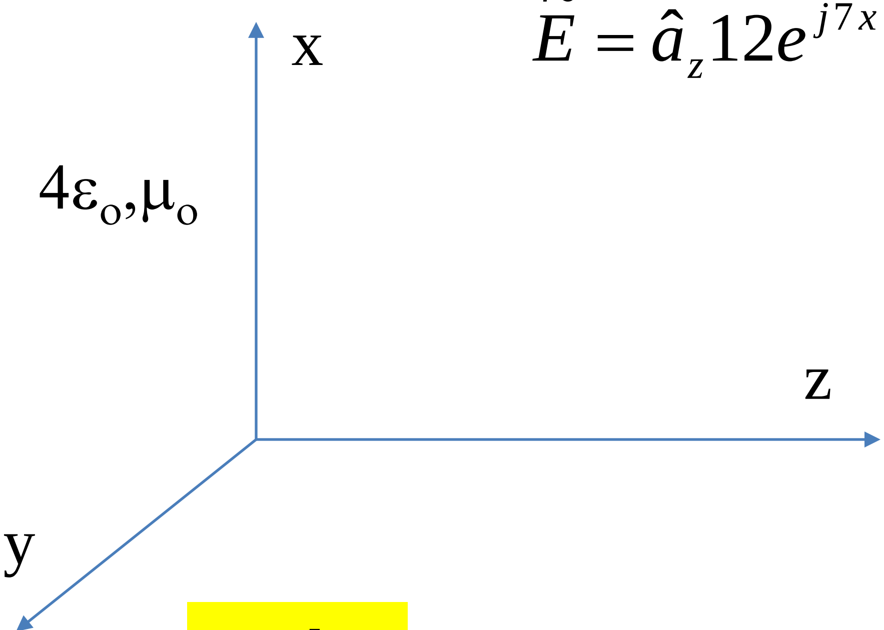
$$\tilde{E}(z) = \hat{a}_y 10 e^{-j9z}$$

Find H

Uniform Plane Waves in Unbounded Lossless Medium
Principal Axis Propagation

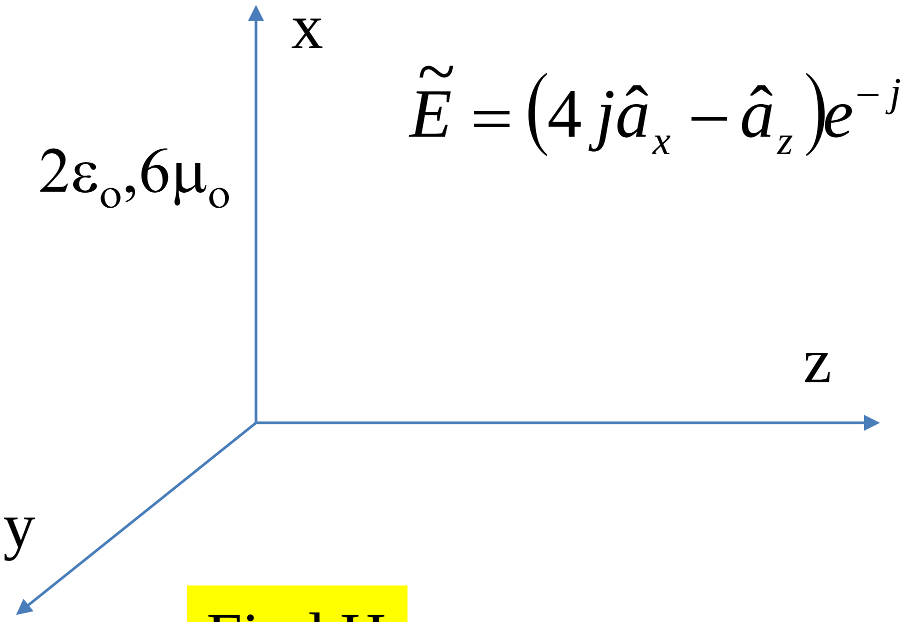
$$\tilde{E} = \hat{a}_z 12 e^{j7x}$$

$4\varepsilon_o, \mu_o$



Find H

Uniform Plane Waves in Unbounded Lossless Medium
Principal Axis Propagation



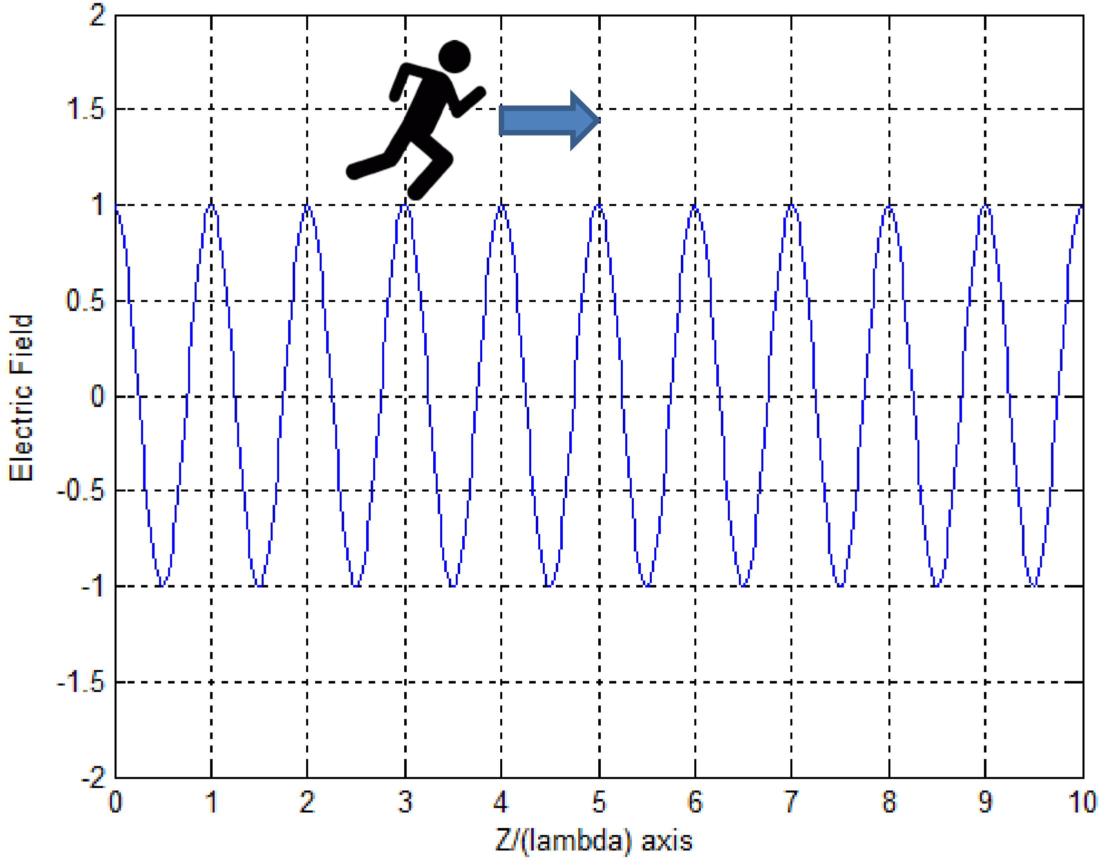
$2\varepsilon_o, 6\mu_o$

$$\tilde{E} = \left(4j\hat{a}_x - \hat{a}_z\right)e^{-j5y}$$

Find H

Phase Velocity

How fast do I have to run to keep at the same phase point?



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Convert to time domain

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Convert to time domain

$$E(z,t) = \mathrm{Re}\left\{\tilde{E}(z) \cdot e^{j\omega t}\right\}$$

$$E(z,t) = \mathrm{Re}\left\{E_o e^{-j\beta z} \cdot e^{j\omega t}\right\}\hat{a}_x$$

$$E(z,t) = E_o \cos(\omega t - \beta z)\hat{a}_x$$

Phase Velocity

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Convert to time domain

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Which term is the phase?

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Uniform Plane Waves in Unbounded Lossless Medium
Principal Axis Propagation

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We want to know how fast I must run to keep at
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$$\phi = \omega t - \beta z$$

We want to know how fast I must run to keep at the same phase point (i.e. constant phase)

$$\phi = \omega t - \beta z = \phi_o \quad \textit{constant}$$

$$\frac{d}{dt}\phi_o = \frac{d}{dt}\left[\omega t - \beta z\right] = 0$$

Phase Velocity

How fast do I have to run to keep at the same phase point?

$$E(z,t) = E_o \cos(\omega t - \beta z)\hat{a}_x$$

$$\phi = \omega t - \beta z$$

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$$\frac{d}{dt}\left[\omega t - \beta z\right] = \omega \underbrace{\frac{dt}{dt}}_{=1} - \beta \underbrace{\frac{dz}{dt}}_{= v_p} = 0$$

Phase Velocity

How fast do I have to run to keep at the same phase point?

$$E(z,t) = E_o \cos(\omega t - \beta z)\hat{a}_x$$

$$\phi = \omega t - \beta z$$

$$\frac{d}{dt}\left[\omega t - \beta z\right] = \omega \underbrace{\frac{dt}{dt}}_{=1} - \beta \underbrace{\frac{dz}{dt}}_{= v_\mathrm{p}} = 0$$

$$\omega - \beta v_p = 0 \quad \Rightarrow \quad \boxed{v_p = \frac{\omega}{\beta}}$$

True in general

Phase Velocity

How fast do I have to run to keep at the same phase point?

$$E(z,t) = E_o \cos(\omega t - \beta z)\hat{a}_x$$

$$\boxed{v_p = \frac{\omega}{\beta}} \Rightarrow \text{True in general}$$

$$\boxed{v_p = \frac{\omega}{\omega\sqrt{\varepsilon\mu}} = \frac{1}{\sqrt{\varepsilon\mu}} = c} \Rightarrow \text{True for uniform plane waves}$$

Energy and Power for Uniform Plane Waves

$$\tilde{E}(z) = \hat{a}_x E_o e^{-j\beta z}$$

$$\tilde{H}(z) = \hat{a}_y \frac{1}{\eta} E_o e^{-j\beta z}$$

$$w_e^{ave} = \frac{1}{4}\varepsilon\left|\tilde{E}\right|^2 = \frac{1}{4}\varepsilon\left|E_o e^{-j\beta z}\right|^2 = \frac{1}{4}\varepsilon\left|E_o\right|^2$$

$$w_m^{ave} = \frac{1}{4}\mu\left|\tilde{H}\right|^2 = \frac{1}{4}\mu\left|\frac{1}{\eta}E_o e^{-j\beta z}\right|^2 = \frac{\mu}{4\eta^2}\left|E_o\right|^2 = \frac{1}{4}\varepsilon\left|E_o\right|^2$$

$$\tilde{S}^{ave} = \frac{1}{2}\tilde{E}\times\tilde{H}^* = \frac{1}{2}\left[E_o e^{-j\beta z}\hat{a}_x\right]\times\left[\frac{1}{\eta}E_o e^{-j\beta z}\hat{a}_y\right]^*$$

$$= \frac{1}{2}E_o e^{-j\beta z}\cdot\frac{1}{\eta}E_o e^{j\beta z}\left[\hat{a}_x\right]\times\left[\hat{a}_y\right] = \frac{1}{2\eta}\left|E_o\right|^2\hat{a}_z$$