ELEG 413
Lecture #4

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SUMMARY

Time Domain	Frequency Domain
$\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}-\vec{M}$ $\quad \nabla\times\vec{H}=\frac{\partial\vec{D}}{\partial t}+\vec{J}$ $\nabla\cdot\vec{D}=\rho$ $\quad \nabla\cdot\vec{B}=\rho_m$	$\nabla\times\tilde{E}=-j\omega\tilde{B}-\tilde{M}$ $\quad \nabla\times\tilde{H}=j\omega\tilde{D}+\tilde{J}$ $\nabla\cdot\tilde{D}=\tilde{\rho}$ $\quad \nabla\cdot\tilde{B}=\tilde{\rho}_m$
$\hat{n}\times(\vec{E}_2-\vec{E}_1)=0$ $\quad \hat{n}\times(\vec{H}_2-\vec{H}_1)=\vec{J}_s$ $\hat{n}\cdot(\vec{D}_2-\vec{D}_1)=\rho_s$ $\quad \hat{n}\cdot(\vec{B}_2-\vec{B}_1)=0$	$\hat{n}\times(\tilde{E}_2-\tilde{E}_1)=0$ $\quad \hat{n}\times(\tilde{H}_2-\tilde{H}_1)=\tilde{J}_s$ $\hat{n}\cdot(\tilde{D}_2-\tilde{D}_1)=\tilde{\rho}_s$ $\quad \hat{n}\cdot(\tilde{B}_2-\tilde{B}_1)=0$ $Z_s=R_s+jX_s=(1+j)\sqrt{\frac{\omega\mu}{2\sigma}}$
$\vec{D}=\varepsilon\vec{E}$ $\vec{B}=\mu\vec{H}$ $\vec{J}_c=\sigma\vec{E}$	$\tilde{D}=\varepsilon\tilde{E}$ $\tilde{B}=\mu\tilde{H}$ $\tilde{J}_c=\sigma\tilde{E}$
$P_s=\oiint_s(\vec{E}\times\vec{H})\cdot ds$ $W_{m=}\left[\iiint_v[\frac{1}{2}\mu\|H\|^2]\,dv\right],\quad W_{e=}\iiint_v[\frac{1}{2}\varepsilon\|E\|^2]\,dv$ $P_i=\iiint_v[\vec{E}\cdot\vec{J}_i]\,dv=0,\quad P_d=\iiint_v\sigma\|E\|^2\,dv=0$	$P_s=\oiint_s(\tilde{E}\times\tilde{H}^*)\cdot ds$ $W_{m=}\left[\iiint_v[\frac{1}{4}\mu\|\tilde{H}\|^2]\,dv\right],\quad W_{e=}\iiint_v[\frac{1}{4}\varepsilon\|\tilde{E}\|^2]\,dv$ $P_i=\iiint_v[\frac{1}{2}\tilde{E}\cdot\tilde{J}_i^*]\,dv=0,\quad P_d=\iiint_v\frac{1}{2}\sigma\|\tilde{E}\|^2\,dv=0$

Classification of Materials

1. *Homogenous or Inhomogenous*
2. *Isotropic or Anisotropic*
3. *Linear or non-Linear*
4. *Dispersive or non-dispersive*

Electric Properties of Materials
Frequency Behavior (Complex Permittivity)

$$\nabla \times \tilde{E} = -j\omega \tilde{B}$$
$$\nabla \times \tilde{H} = j\omega \tilde{D} + \tilde{J}_i + \tilde{J}_c$$
$$\nabla \cdot \tilde{D} = \tilde{\rho}$$
$$\nabla \cdot \tilde{B} = 0$$

$$\tilde{D} = \varepsilon(\omega)\tilde{E}$$
$$\tilde{B} = \mu_o \tilde{H}$$
$$\tilde{J}_c = \sigma_s \tilde{E}$$

Dielectric constant

loss term

$$\varepsilon^*(\omega) = \varepsilon'(\omega) - j\varepsilon''(\omega)$$

is called the complex permittivity

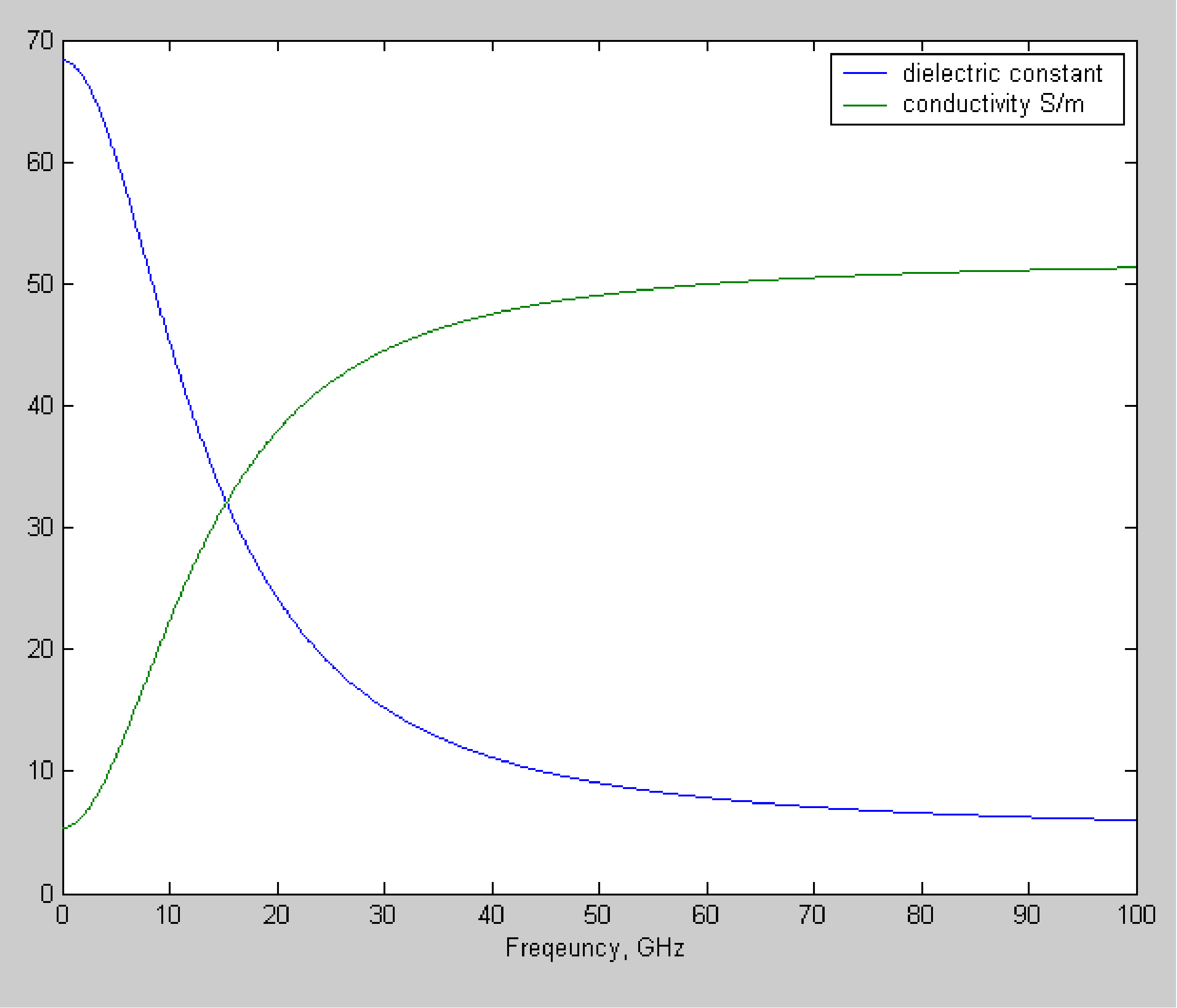
$$\nabla \times \tilde{E} = -j\omega \mu_o \tilde{H}$$
$$\nabla \times \tilde{H} = j\omega \varepsilon \tilde{E} + \tilde{J}_i + \sigma_s \tilde{E}$$
$$\nabla \cdot (\varepsilon \tilde{E}) = \tilde{\rho}$$
$$\nabla \cdot (\tilde{H}) = 0$$

$$\nabla \times \tilde{E} = -j\omega \mu_o \tilde{H}$$
$$\nabla \times \tilde{H} = j\omega \varepsilon \tilde{E} + \tilde{J}_i + \sigma_s \tilde{E}$$
$$\nabla \cdot (\varepsilon \tilde{E}) = \tilde{\rho}$$
$$\nabla \cdot (\tilde{H}) = 0$$

$$\nabla \times \tilde{E} = -j\omega \mu_o \tilde{H}$$
$$\nabla \times \tilde{H} = j\omega \varepsilon^*(\omega) \tilde{E} + \tilde{J}_i$$
$$\nabla \cdot (\varepsilon^* \tilde{E}) = \tilde{\rho}$$
$$\nabla \cdot (\tilde{H}) = 0$$

$$\varepsilon^*(\omega) = \varepsilon'(\omega) - j\frac{\sigma_s}{\omega}$$
$$\varepsilon^*(\omega) = \varepsilon'(\omega) - j\varepsilon''(\omega)$$

Frequency Behavior of Sea Water



Wave Equation

Time Dependent Homogenous Wave Equation (E-Field)

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$$

$$\nabla \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t} + \sigma \vec{E} + \vec{J}$$

$$\nabla \cdot \vec{E} = \rho / \varepsilon$$

$$\nabla \cdot \vec{B} = 0$$

Take the curl of both of Faraday's Law

$$\nabla \times [\nabla \times \vec{E}] = \nabla \times (-\mu \frac{\partial \vec{H}}{\partial t})$$

$$\nabla \times \nabla \times \vec{E} = -\mu \frac{\partial}{\partial t} \nabla \times \vec{H}$$

Wave Equation

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$$\nabla \times \nabla \times \vec{E} = -\mu \frac{\partial}{\partial t} \nabla \times \vec{H}$$

$$\nabla \times \nabla \times \vec{E} = -\mu \frac{\partial}{\partial t}\left[\varepsilon \frac{\partial \vec{E}}{\partial t} + \sigma \vec{E} + \vec{J} \right]$$

$$\nabla \times \nabla \times \vec{E} = -\mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} - \mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu \frac{\partial \vec{J}}{\partial t}$$

Wave Equation

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Vector Identity

$$\nabla \times \nabla \times \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$

$$\nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = -\mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} - \mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu \frac{\partial \vec{J}}{\partial t}$$

Wave Equation

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$$\nabla\left(\frac{\rho}{\varepsilon}\right) - \nabla^2 \vec{E} = -\mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} - \mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu \frac{\partial \vec{J}}{\partial t}$$

Wave Equation

Time Dependent Homogenous Wave Equation (E-Field)

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$$

$$\nabla \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t} + \sigma \vec{E} + \vec{J}$$

$$\nabla \cdot \vec{E} = \rho / \varepsilon$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \nabla \times \vec{E} = -\mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} - \mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu \frac{\partial \vec{J}}{\partial t}$$

Vector Identity

$$\nabla \times \nabla \times \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$

$$\nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = -\mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} - \mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu \frac{\partial \vec{J}}{\partial t}$$

$$\nabla\left(\frac{\rho}{\varepsilon}\right) - \nabla^2 \vec{E} = -\mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} - \mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu \frac{\partial \vec{J}}{\partial t}$$

$$\nabla^2 \vec{E} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} - \mu\sigma \frac{\partial \vec{E}}{\partial t} = \mu \frac{\partial \vec{J}}{\partial t} + \frac{1}{\varepsilon} \nabla \rho$$

Wave Equation

Source-Free Time Dependent Homogenous Wave Equation (E-Field)

$$\nabla^2 \vec{E} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} - \mu\sigma \frac{\partial \vec{E}}{\partial t} = \mu \frac{\partial \vec{J}}{\partial t} + \frac{1}{\varepsilon} \nabla \rho$$

Source Free $\vec{J} = 0, \quad \rho = 0$

$$\nabla^2 \vec{E} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} - \mu\sigma \frac{\partial \vec{E}}{\partial t} = 0$$

Source-Free Lossless Time Dependent Homogenous Wave Equation (E-Field)

Lossless $\sigma = 0$

$$\nabla^2 \vec{E} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

Wave Equation

Source-Free Time Dependent Homogenous Wave Equation (H-Field)

$$\nabla^2 \vec{H} - \mu\varepsilon \frac{\partial^2 \vec{H}}{\partial t^2} - \mu\sigma \frac{\partial \vec{H}}{\partial t} = -\nabla \times \vec{J} + \frac{1}{\mu} \nabla \rho$$

Source Free $\vec{J} = 0, \quad \rho = 0$

$$\nabla^2 \vec{H} - \mu\varepsilon \frac{\partial^2 \vec{H}}{\partial t^2} - \mu\sigma \frac{\partial \vec{H}}{\partial t} = 0$$

Source Free and Lossless $\vec{J} = 0, \quad \rho = 0, \quad \sigma = 0$

$$\nabla^2 \vec{H} - \mu\varepsilon \frac{\partial^2 \vec{H}}{\partial t^2} = 0$$

Wave Equation: Time Harmonic

Time Domain

$$\nabla^2 \vec{E} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} - \mu\sigma \frac{\partial \vec{E}}{\partial t} = \mu \frac{\partial \vec{J}}{\partial t} + \frac{1}{\varepsilon} \nabla \rho$$

Source Free $\vec{J} = 0, \quad \rho = 0$

$$\nabla^2 \vec{E} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} - \mu\sigma \frac{\partial \vec{E}}{\partial t} = 0$$

Lossless $\sigma = 0$

$$\nabla^2 \vec{E} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

Frequency Domain

How do I convert these to their time harmonic forms?

Wave Equation: Time Harmonic

Time Domain

$$\nabla^2 \vec{E} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} - \mu\sigma \frac{\partial \vec{E}}{\partial t} = \mu \frac{\partial \vec{J}}{\partial t} + \frac{1}{\varepsilon} \nabla \rho$$

Source Free $\vec{J} = 0, \quad \rho = 0$

$$\nabla^2 \vec{E} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} - \mu\sigma \frac{\partial \vec{E}}{\partial t} = 0$$

Lossless $\sigma = 0$

$$\nabla^2 \vec{E} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

Frequency Domain

$$\nabla^2 \tilde{E} + \omega^2 \mu\varepsilon \tilde{E} - j\omega\mu\sigma \tilde{E} = \mu \tilde{J} + \frac{1}{\varepsilon} \nabla \tilde{\rho}$$

Source Free $\vec{J} = 0, \quad \rho = 0$

$$\nabla^2 \tilde{E} + \omega^2 \mu\varepsilon \tilde{E} - j\omega\mu\sigma \tilde{E} = 0$$

Lossless $\sigma = 0$

$$\nabla^2 \tilde{E} + \omega^2 \mu\varepsilon \tilde{E} = 0$$

“Helmholtz Equation”

General Solution Case: Time Harmonic Rectangular Coordinates

$$\nabla^2 \tilde{E} + \omega^2 \mu\varepsilon \tilde{E} = 0 \quad \Rightarrow \quad \nabla^2 \tilde{E} + \beta^2 \tilde{E} = 0$$

Wave Number $\beta = \omega\sqrt{\mu\varepsilon}$

$$\frac{\partial^2 \tilde{E}}{\partial x^2} + \frac{\partial^2 \tilde{E}}{\partial y^2} + \frac{\partial^2 \tilde{E}}{\partial z^2} + \beta^2 \tilde{E} = 0$$

General Solution Case: Time Harmonic Rectangular Coordinates

Wave Number $\beta = \omega\sqrt{\mu\varepsilon}$

$$\frac{\partial^2 \tilde{E}}{\partial x^2} + \frac{\partial^2 \tilde{E}}{\partial y^2} + \frac{\partial^2 \tilde{E}}{\partial z^2} + \beta^2 \tilde{E} = 0$$

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} + \beta^2 E_x = 0$$

$$\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_y}{\partial z^2} + \beta^2 E_y = 0$$

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} + \beta^2 E_z = 0$$

Separation of Variable Solutions

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} + \beta^2 E_x = 0$$

Assume Solution of the form:

$$E_x(x, y, z) = f(x)g(y)h(z)$$

Separation of Variable Solutions

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} + \beta^2 E_x = 0$$

Assume Solution of the form: $E_x(x, y, z) = f(x)g(y)h(z)$

$$f''(x)g(y)h(z) + f(x)g''(y)h(z) + f(x)g(y)h''(z) + \beta^2 f(x)g(y)h(z) = 0$$

Separation of Variable Solutions

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} + \beta^2 E_x = 0$$

Assume Solution of the form: $E_x(x, y, z) = f(x)g(y)h(z)$

$$f''(x)g(y)h(z) + f(x)g''(y)h(z) + f(x)g(y)h''(z) + \beta^2 f(x)g(y)h(z) = 0$$

$$\frac{1}{f(x)g(y)h(z)}[f''(x)g(y)h(z) + f(x)g''(y)h(z) + f(x)g(y)h''(z) + \beta^2 f(x)g(y)h(z)] = 0$$

Separation of Variable Solutions

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} + \beta^2 E_x = 0$$

Assume Solution of the form: $E_x(x, y, z) = f(x)g(y)h(z)$

$$f''(x)g(y)h(z) + f(x)g''(y)h(z) + f(x)g(y)h''(z) + \beta^2 f(x)g(y)h(z) = 0$$

⇓

$$\frac{1}{f(x)g(y)h(z)}[f''(x)g(y)h(z) + f(x)g''(y)h(z) + f(x)g(y)h''(z) + \beta^2 f(x)g(y)h(z)] = 0$$

⇓

$$\frac{f''(x)}{f(x)} + \frac{g''(y)}{g(y)} + \frac{h''(z)}{h(z)} + \beta^2 = 0$$

Separation of Variable Solutions

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} + \beta^2 E_x = 0$$

Assume Solution of the form: $E_x(x, y, z) = f(x)g(y)h(z)$

$$\underbrace{\frac{f''(x)}{f(x)}}_{\text{function of x}} + \underbrace{\frac{g''(y)}{g(y)}}_{\text{function of y}} + \underbrace{\frac{h''(z)}{h(z)}}_{\text{function of z}} + \overbrace{\beta^2}^{\text{constant}} = 0$$

Question: How can a function of x + a function of y + a function of z + constant equal zero for every value of x, y and z?

Separation of Variable Solutions

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} + \beta^2 E_x = 0$$

Assume Solution of the form: $E_x(x,y,z) = f(x)g(y)h(z)$

$$\underbrace{\frac{f''(x)}{f(x)}}_{\text{function of x}} + \underbrace{\frac{g''(y)}{g(y)}}_{\text{function of y}} + \underbrace{\frac{h''(z)}{h(z)}}_{\text{function of z}} + \overbrace{\beta^2}^{\text{constant}} = 0$$

constant

function of x

function of y

function of z

Answer: Each of the terms must also be equal to a constant!

Separation of Variable Solutions

constant

$$\frac{f''(x)}{f(x)} + \frac{g''(y)}{g(y)} + \frac{h''(z)}{h(z)} + \overbrace{\beta^2}^{\text{constant}} = 0$$

function of x | **function of y** | **function of z**

(1) $$\frac{f''(x)}{f(x)} = -\beta_x{}^2$$

(2) $$\frac{g''(y)}{g(y)} = -\beta_y{}^2$$

(3) $$\frac{h''(z)}{h(z)} = -\beta_z{}^2$$

(4) $$\beta^2 = \beta_x{}^2 + \beta_y{}^2 + \beta_z{}^2$$

Separation of Variable Solutions

$$\underbrace{\frac{f''(x)}{f(x)}}_{\text{function of x}} + \underbrace{\frac{g''(y)}{g(y)}}_{\text{function of y}} + \underbrace{\frac{h''(z)}{h(z)}}_{\text{function of z}} + \overbrace{\beta^2}^{\text{constant}} = 0$$

(1) $$\frac{f''(x)}{f(x)} = -\beta_x^{\,2}$$

(2) $$\frac{g''(y)}{g(y)} = -\beta_y^{\,2}$$

(3) $$\frac{h''(z)}{h(z)} = -\beta_z^{\,2}$$

(4) $$\beta^2 = \beta_x^{\,2} + \beta_y^{\,2} + \beta_z^{\,2}$$

At this point we do not know what the new constants β_x, β_y and β_z are

Separation of Variable Solutions

$$(1) \quad \frac{f''(x)}{f(x)} = -\beta_x^{\,2} \quad \Rightarrow \quad f''(x) + \beta_x^{\,2} f(x) = 0$$

$$(2) \quad \frac{g''(y)}{g(y)} = -\beta_y^{\,2} \quad \Rightarrow \quad g''(y) + \beta_y^{\,2} g(y) = 0$$

$$(3) \quad \frac{h''(z)}{h(z)} = -\beta_z^{\,2} \quad \Rightarrow \quad h''(z) + \beta_z^{\,2} h(z) = 0$$

$$(4) \quad \beta^2 = \beta_x^{\,2} + \beta_y^{\,2} + \beta_z^{\,2}$$

Three ordinary differential equations instead of one partial differential equation! You should know how to solve these!

Separation of Variable Solutions

$$f''(x) + \beta_x^{\ 2} f(x) = 0$$

How do we solve this?

Separation of Variable Solutions

$$f''(x) + \beta_x^{\,2} f(x) = 0$$

Guess: $f(x) = A_1 e^{sx}$

$$A_1 s^2 e^{sx} + A_1 \beta_x^2 e^{sx} = 0$$

$$\Rightarrow \quad s^2 + \beta_x^2 = 0$$

$$\Rightarrow \quad s = \pm\sqrt{-\beta_x^2} = \pm j\beta_x$$

$$\Rightarrow \quad f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

Separation of Variable Solutions

$$f''(x) + \beta_x^{\,2} f(x) = 0$$

$$g''(y) + \beta_y^{\,2} g(y) = 0$$

$$h''(z) + \beta_z^{\,2} h(z) = 0$$

$$\Rightarrow$$

$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

$$g(y) = A_2 e^{-j\beta_y y} + B_2 e^{j\beta_y y}$$

$$h(z) = A_3 e^{-j\beta_z z} + B_3 e^{j\beta_z z}$$

Separation of Variable Solutions

$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

$$g(y) = A_2 e^{-j\beta_y y} + B_2 e^{j\beta_y y}$$

$$h(z) = A_3 e^{-j\beta_z z} + B_3 e^{j\beta_z z}$$

Lets look at these solutions a bit more carefully. What do they physically represent?

Separation of Variable Solutions

$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

$$g(y) = A_2 e^{-j\beta_y y} + B_2 e^{j\beta_y y}$$

$$h(z) = A_3 e^{-j\beta_z z} + B_3 e^{j\beta_z z}$$

Lets look at these solutions a bit more carefully. What do they physically represent?

Answer: It depends on these constants β_x, β_y and β_z

Separation of Variable Solutions

$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

Answer: It depends on these constants β_x, β_y and β_z

Case #1: β_x is a real number

$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

$$E(x,t) = \mathrm{Re}\left\{ f(x) \cdot e^{j\omega t} \right\}$$

$$E(x,t) = \mathrm{Re}\left\{ \left[A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x} \right] \cdot e^{j\omega t} \right\}$$

$$E(x,t) = \mathrm{Re}\left\{ A_1 e^{j(\omega t - \beta_x x)} + B_1 e^{j(\omega t + \beta_x x)} \right\}$$

$$E(x,t) = A_1 \cos(\omega t - \beta_x x) + B_1 \cos(\omega t + \beta_x x)$$

Separation of Variable Solutions

$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

Answer: It depends on these constants β_x, β_y and β_z

Case #1: β_x is a real number

$$E(x,t) = A_1 \cos(\omega t - \beta_x x) + B_1 \cos(\omega t + \beta_x x)$$

These solutions represent waves traveling in the x direction called “traveling waves”

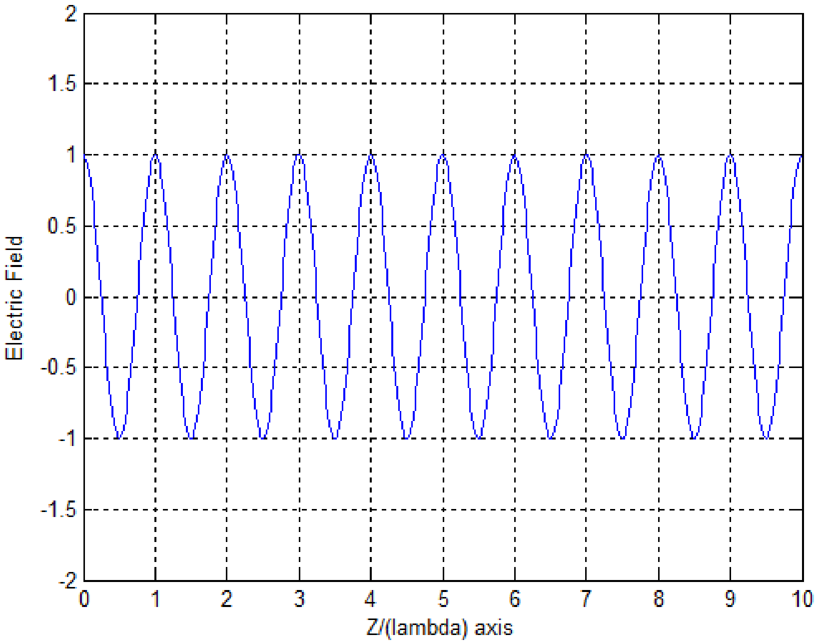
Separation of Variable Solutions

$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

Answer: It depends on these constants β_x, β_y and β_z

Case #1: β_x is a real number

$$E(x,t) = \cos(\omega t - \beta_x x)$$



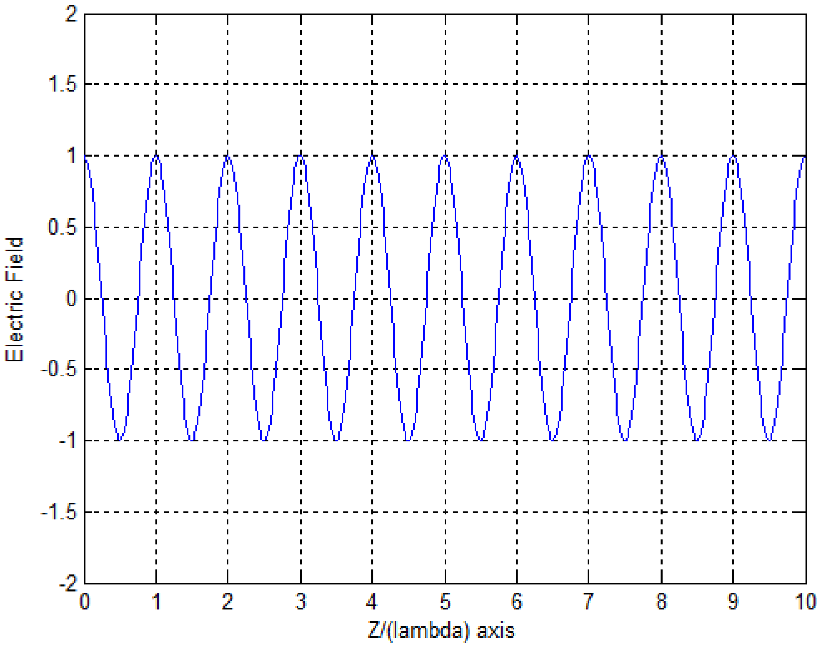
Separation of Variable Solutions

$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

Answer: It depends on these constants β_x, β_y and β_z

Case #1: β_x is a real number

$$E(x,t) = B_1 \cos(\omega t + \beta_x x)$$



Separation of Variable Solutions

$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

Answer: It depends on these constants β_x, β_y and β_z

Case #1: β_x is a real number

$$E(x,t) = \cos(\omega t - \beta_x x) + \cos(\omega t + \beta_x x)$$

?????

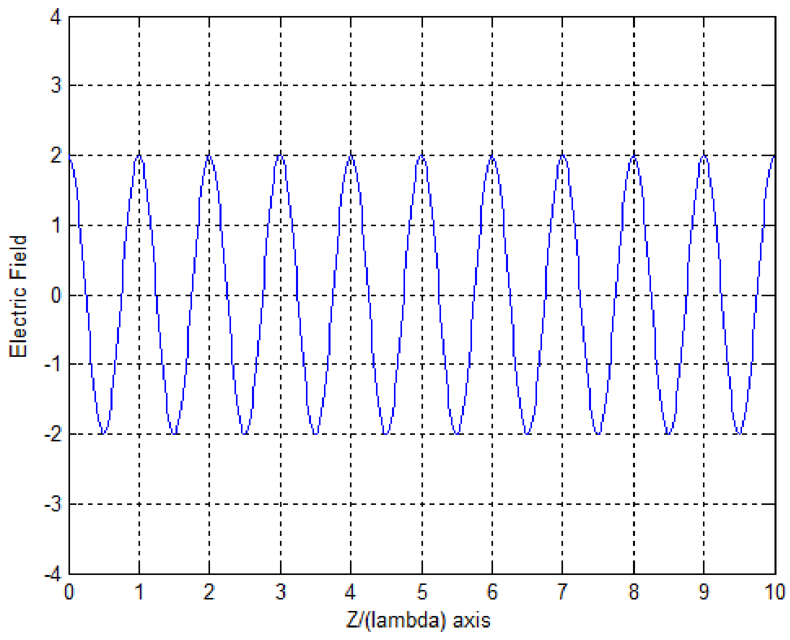
Separation of Variable Solutions

$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

Answer: It depends on these constants β_x, β_y and β_z

Case #1: β_x is a real number

$$E(x,t) = \cos(\omega t - \beta_x x) + \cos(\omega t + \beta_x x)$$



Separation of Variable Solutions

$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

Answer: It depends on these constants β_x, β_y and β_z

Case #2: β_x is a purely imaginary number

Separation of Variable Solutions

$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

Answer: It depends on these constants β_x, β_y and β_z

Case #2: β_x is a purely imaginary number

Let $\beta_x = j\alpha$ where α is purely real

Separation of Variable Solutions

$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

Answer: It depends on these constants β_x, β_y and β_z

Case #2: β_x is a purely imaginary number

Let $\beta_x = j\alpha$ where α is purely real

$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

$$f(x) = A_1 e^{-j \cdot j\alpha x} + B_1 e^{j \cdot j\alpha x}$$

$$f(x) = A_1 e^{\alpha x} + B_1 e^{-\alpha x}$$

Separation of Variable Solutions

$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

Answer: It depends on these constants β_x, β_y and β_z

Case #2: β_x is a purely imaginary number

Let $\beta_x = j\alpha$ where α is purely real

$$f(x) = A_1 e^{\alpha x} + B_1 e^{-\alpha x}$$

$$E(x,t) = \text{Re}\left\{ f(x) \cdot e^{j\omega t} \right\}$$

$$E(x,t) = \text{Re}\left\{ \left[A_1 e^{\alpha x} + B_1 e^{-\alpha x} \right] \cdot e^{j\omega t} \right\}$$

$$E(x,t) = \text{Re}\left\{ A_1 e^{\alpha x} e^{j\omega t} + B_1 e^{-\alpha x} e^{j\omega t} \right\}$$

$$E(x,t) = A_1 e^{\alpha x} \,\text{Re}\left\{ e^{j\omega t} \right\} + B_1 e^{-\alpha x} \,\text{Re}\left\{ e^{j\omega t} \right\}$$

Separation of Variable Solutions

$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

Answer: It depends on these constants β_x, β_y and β_z

Case #2: β_x is a purely imaginary number

Let $\beta_x = j\alpha$ where α is purely real

$$f(x) = A_1 e^{\alpha x} + B_1 e^{-\alpha x}$$

$$E(x,t) = A_1 e^{\alpha x}\,\mathrm{Re}\left\{e^{j\omega t}\right\} + B_1 e^{-\alpha x}\,\mathrm{Re}\left\{e^{j\omega t}\right\}$$

$$E(x,t) = A_1 e^{\alpha x}\cos(\omega t) + B_1 e^{-\alpha x}\cos(\omega t)$$

Separation of Variable Solutions

$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

Answer: It depends on these constants β_x, β_y and β_z

Case #2: β_x is a purely imaginary number

Let $\beta_x = j\alpha$ where α is purely real

$$f(x) = A_1 e^{\alpha x} + B_1 e^{-\alpha x}$$

$$E(x,t) = A_1 e^{\alpha x} \cos(\omega t) + B_1 e^{-\alpha x} \cos(\omega t)$$

These solutions represent waves that do not travel but instead either grow or attenuate in space. These solutions are called “evanescent waves”

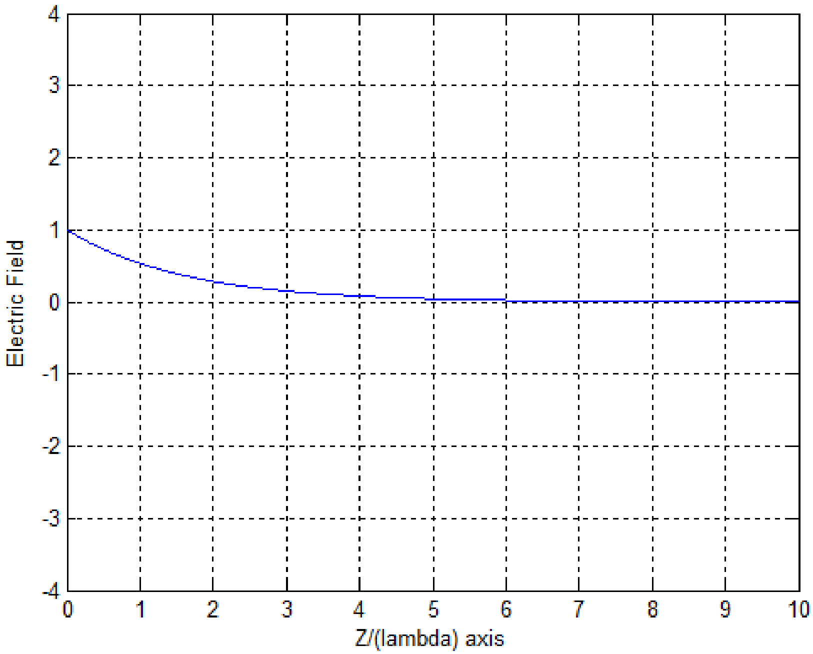
Separation of Variable Solutions

$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

Answer: It depends on these constants β_x, β_y and β_z

Case #2: β_x is a purely imaginary number

$$E(x,t) = e^{-\alpha x}\cos(\omega t)$$



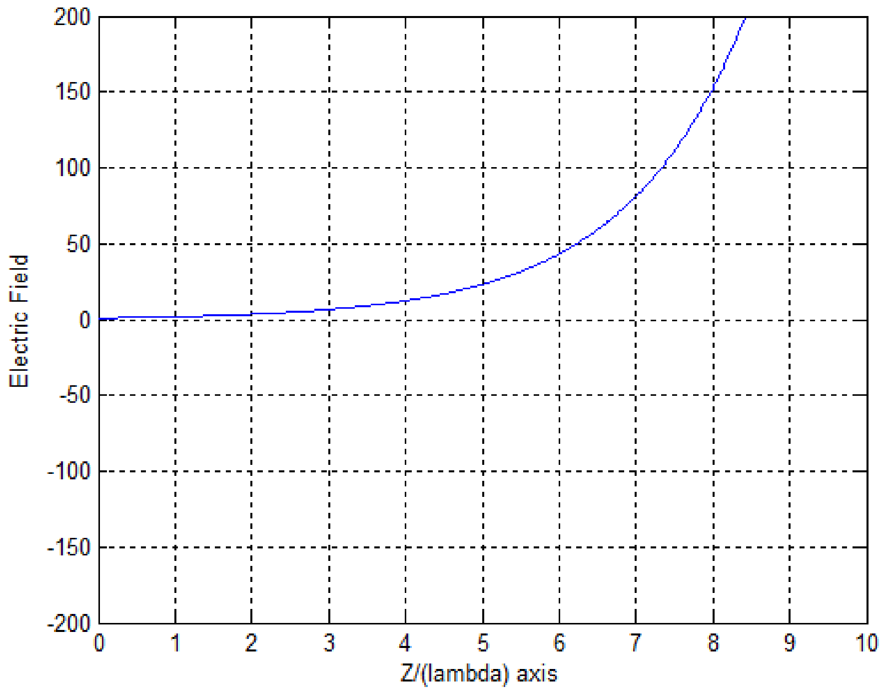
Separation of Variable Solutions

$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

Answer: It depends on these constants β_x, β_y and β_z

Case #2: β_x is a purely imaginary number

$$E(x,t) = e^{\alpha x}\cos(\omega t)$$



Separation of Variable Solutions

$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

Answer: It depends on these constants β_x, β_y and β_z

Case #3: β_x is a complex number

Let $\beta_x = \pm\gamma \pm j\alpha$ where α and γ are purely real

Separation of Variable Solutions

$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

Answer: It depends on these constants β_x, β_y and β_z

Case #3: β_x is a complex number

Let $\beta_x = \pm\gamma \pm j\alpha$ where α and γ are purely real

$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

$$f(x) = A_1 e^{-j\cdot(\pm\gamma + j\alpha)x} + B_1 e^{j(\pm\gamma - j\alpha)x}$$

$$f(x) = A_1 e^{\alpha x} e^{-j\gamma x} + A_1 e^{-\alpha x} e^{-j\gamma x} + B_1 e^{-\alpha x} e^{j\gamma x} + B_1 e^{\alpha x} e^{j\gamma x}$$

Separation of Variable Solutions

$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

Answer: It depends on these constants β_x, β_y and β_z

Case #3: β_x is a complex number

Let $\beta_x = \gamma \pm j\alpha$ where α and γ are purely real

$$f(x) = A_1 e^{\alpha x} e^{-j\gamma x} + A_1 e^{-\alpha x} e^{-j\gamma x} + B_1 e^{-\alpha x} e^{j\gamma x} + B_1 e^{\alpha x} e^{j\gamma x}$$

$$E(x,t) = \mathrm{Re}\left\{ f(x) \cdot e^{j\omega t} \right\}$$

$$E(x,t) = \mathrm{Re}\left\{ \left[A_1 e^{\alpha x} e^{-j\gamma x} + B_1 e^{-\alpha x} e^{j\gamma x} + A_1 e^{-\alpha x} e^{-j\gamma x} + B_1 e^{\alpha x} e^{j\gamma x} \right] \cdot e^{j\omega t} \right\}$$

$$E(x,t) = A_1 e^{\alpha x} \mathrm{Re}\left\{ e^{j(\omega t - \gamma x)} \right\} + A_1 e^{-\alpha x} \mathrm{Re}\left\{ e^{j(\omega t - \gamma x)} \right\} + B_1 e^{-\alpha x} \mathrm{Re}\left\{ e^{j(\omega t + \gamma x)} \right\} + B_1 e^{\alpha x} \mathrm{Re}\left\{ e^{j(\omega t + \gamma x)} \right\}$$

Separation of Variable Solutions

$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

Answer: It depends on these constants β_x, β_y and β_z

Case #3: β_x is a complex number

Let $\beta_x = \gamma \pm j\alpha$ where α and γ are purely real

$$f(x) = A_1 e^{\alpha x} e^{-j\gamma x} + A_1 e^{-\alpha x} e^{-j\gamma x} + B_1 e^{-\alpha x} e^{j\gamma x} + B_1 e^{\alpha x} e^{j\gamma x}$$

$$E(x,t) = A_1 e^{\alpha x} \operatorname{Re}\left\{e^{j(\omega t - \gamma x)}\right\} + A_1 e^{-\alpha x} \operatorname{Re}\left\{e^{j(\omega t - \gamma x)}\right\} + B_1 e^{-\alpha x} \operatorname{Re}\left\{e^{j(\omega t + \gamma x)}\right\} + B_1 e^{\alpha x} \operatorname{Re}\left\{e^{j(\omega t + \gamma x)}\right\}$$

$$E(x,t) = A_1 e^{\alpha x} \cos(\omega t - \gamma x) + A_1 e^{-\alpha x} \cos(\omega t - \gamma x)$$
$$+ B_1 e^{\alpha x} \cos(\omega t + \gamma x) + B_1 e^{-\alpha x} \cos(\omega t + \gamma x)$$

Separation of Variable Solutions

$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

Answer: It depends on these constants β_x, β_y and β_z

Case #3: β_x is a complex number

Let $\beta_x = \gamma \pm j\alpha$ where α and γ are purely real

$$E(x,t) = A_1 e^{\alpha x}\cos(\omega t - \gamma x) + A_1 e^{-\alpha x}\cos(\omega t - \gamma x)$$
$$+ B_1 e^{\alpha x}\cos(\omega t + \gamma x) + B_1 e^{-\alpha x}\cos(\omega t + \gamma x)$$

What do these solutions look like physically?

Separation of Variable Solutions

$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

Answer: It depends on these constants β_x, β_y and β_z

Case #3: β_x is a complex number

Let $\beta_x = \gamma \pm j\alpha$ where α and γ are purely real

$$E(x,t) = A_1 e^{\alpha x} \cos(\omega t - \gamma x) + A_1 e^{-\alpha x} \cos(\omega t - \gamma x)$$
$$+ B_1 e^{\alpha x} \cos(\omega t + \gamma x) + B_1 e^{-\alpha x} \cos(\omega t + \gamma x)$$

These solutions represent waves that travel and either grow or attenuate as they travel.

Separation of Variable Solutions

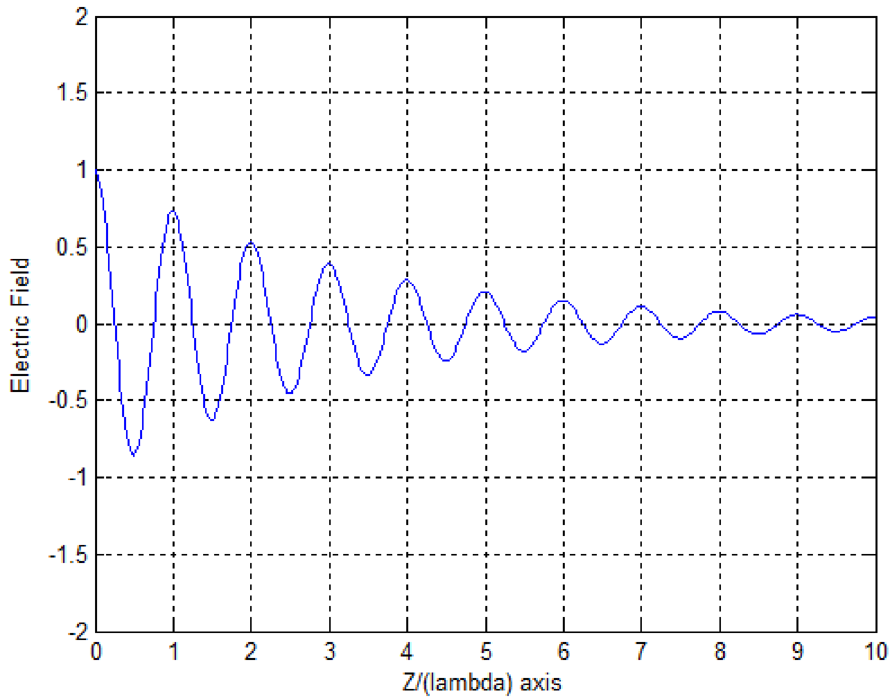
$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

Answer: It depends on these constants β_x, β_y and β_z

Case #3: β_x is a complex number

Let $\beta_x = \gamma \pm j\alpha$ where α and γ are purely real

$$E(x,t) = e^{-\alpha x}\cos(\omega t - \gamma x)$$



Separation of Variable Solutions

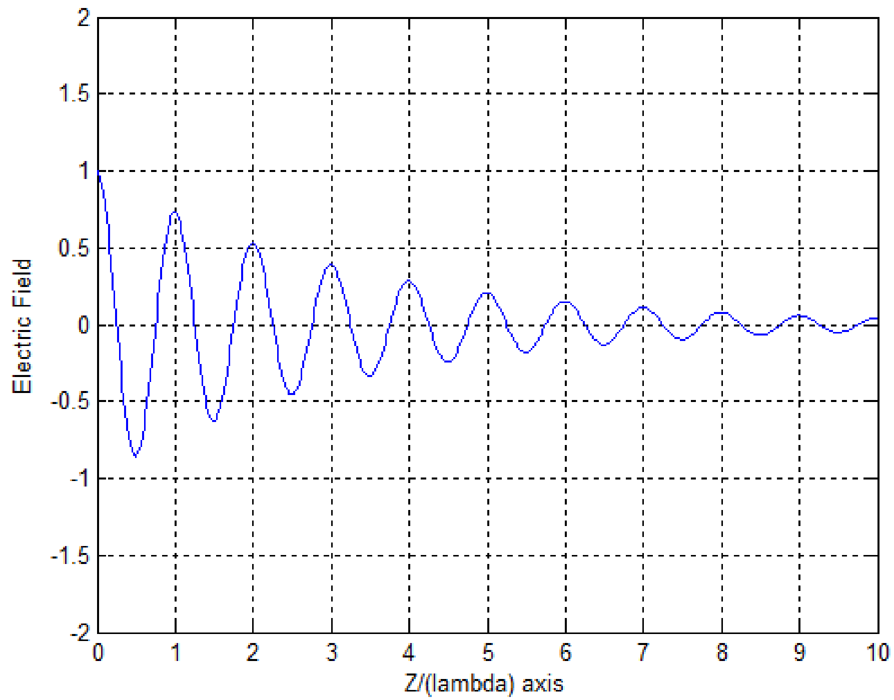
$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

Answer: It depends on these constants β_x, β_y and β_z

Case #3: β_x is a complex number

Let $\beta_x = \gamma \pm j\alpha$ where α and γ are purely real

$$E(x,t) = e^{-\alpha x}\cos(\omega t + \gamma x)$$



Separation of Variable Solutions

$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

Answer: It depends on these constants β_x, β_y and β_z

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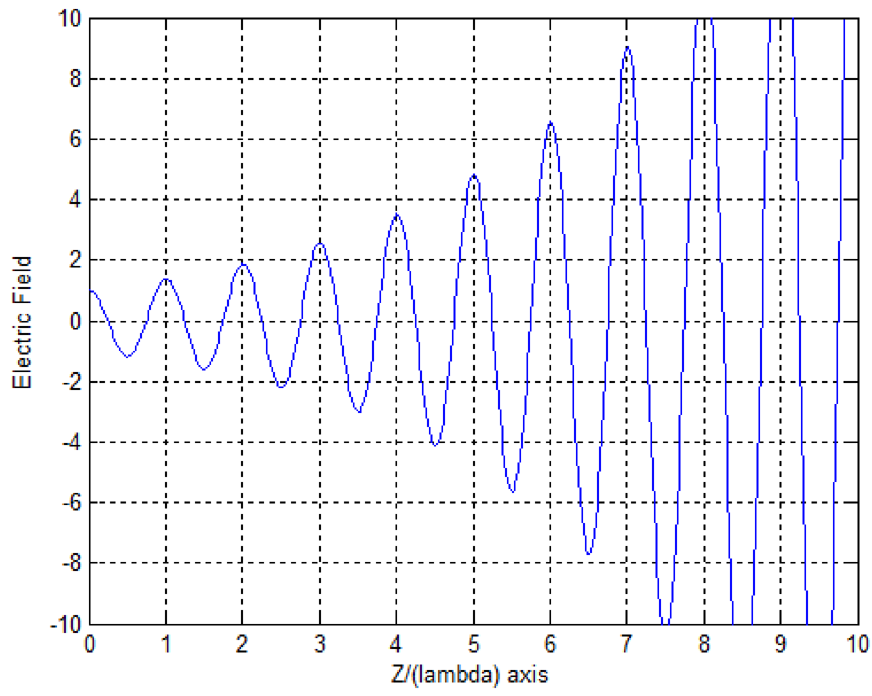
Let $\beta_x = \gamma \pm j\alpha$ where α and γ are purely real

$$E(x,t) = e^{\alpha x}\cos(\omega t - \gamma x)$$



Separation of Variable Solutions

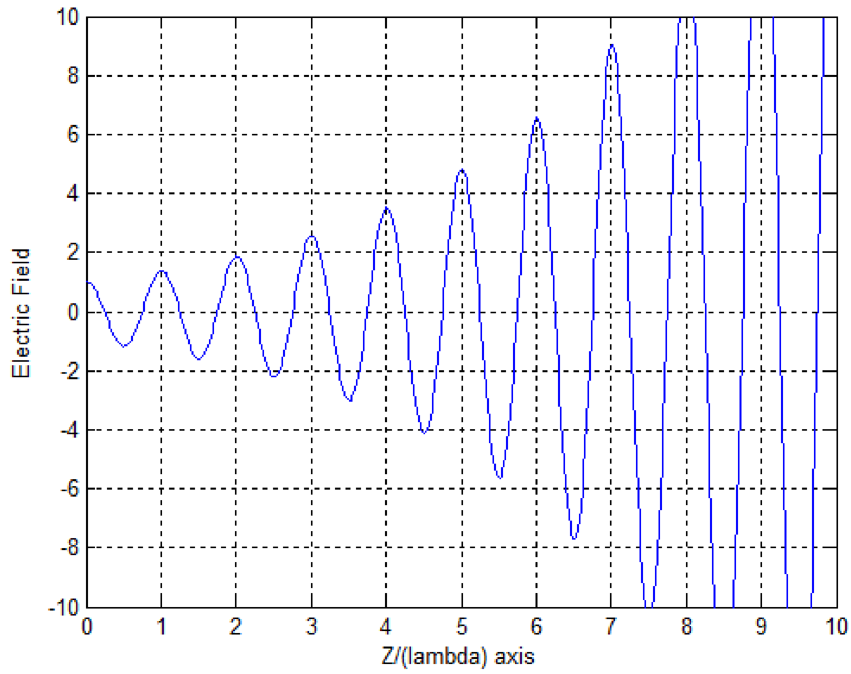
$$f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$$

Answer: It depends on these constants β_x, β_y and β_z

Case #3: β_x is a complex number

Let $\beta_x = \gamma \pm j\alpha$ where α and γ are purely real

$$E(x,t) = e^{\alpha x}\cos(\omega t + \gamma x)$$



Separation of Variable Solutions

β purely real	β purely imaginary	β complex
Traveling and standing waves $f(x) = A_1 e^{-j\beta_x x} + B_1 e^{j\beta_x x}$ $E(x,t) = A_1 \cos(\omega t - \beta_x x)$ $+ B_1 \cos(\omega t + \beta_x x)$	**Evanescent waves** $f(x) = A_1 e^{\alpha x} + B_1 e^{-\alpha x}$ $E(x,t) = A_1 e^{\alpha x} \cos(\omega t)$ $+ B_1 e^{-\alpha x} \cos(\omega t)$	**Exponentially modulated traveling wave** $f(x) = A_1 e^{\alpha x} e^{-j\gamma x} + A_1 e^{-\alpha x} e^{-j\gamma x}$ $+ B_1 e^{-\alpha x} e^{j\gamma x} + B_1 e^{\alpha x} e^{j\gamma x}$ $E(x,t) = A_1 e^{\alpha x} \cos(\omega t - \gamma x) + A_1 e^{-\alpha x} \cos(\omega t - \gamma x)$ $+ B_1 e^{\alpha x} \cos(\omega t + \gamma x) + B_1 e^{-\alpha x} \cos(\omega t + \gamma x)$