ELEG 413
Lecture #3

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SUMMARY

Time Domain	Frequency Domain
$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} - \vec{M} \quad \nabla\times\vec{H} = \frac{\partial\vec{D}}{\partial t} + \vec{J}$ $\nabla\cdot\vec{D} = \rho \qquad \nabla\cdot\vec{B} = \rho_m$	$\nabla\times\tilde{E} = -j\omega\tilde{B} - \tilde{M} \quad \nabla\times\tilde{H} = j\omega\tilde{D} + \tilde{J}$ $\nabla\cdot\tilde{D} = \tilde{\rho} \qquad \nabla\cdot\tilde{B} = \tilde{\rho}_m$
$\hat{n}\times(\vec{E}_2 - \vec{E}_1) = 0 \quad \hat{n}\times(\vec{H}_2 - \vec{H}_1) = \vec{J}_s$ $\hat{n}\cdot(\vec{D}_2 - \vec{D}_1) = \rho_s \quad \hat{n}\cdot(\vec{B}_2 - \vec{B}_1) = 0$	$\hat{n}\times(\tilde{E}_2 - \tilde{E}_1) = 0 \quad \hat{n}\times(\tilde{H}_2 - \tilde{H}_1) = \tilde{J}_s$ $\hat{n}\cdot(\tilde{D}_2 - \tilde{D}_1) = \tilde{\rho}_s \quad \hat{n}\cdot(\tilde{B}_2 - \tilde{B}_1) = 0$ $Z_s = R_s + jX_s = (1+j)\sqrt{\frac{\omega\mu}{2\sigma}}$
$\vec{D} = \varepsilon\vec{E}$ $\vec{B} = \mu\vec{H}$ $\vec{J}_c = \sigma\vec{E}$	$\tilde{D} = \varepsilon\tilde{E}$ $\tilde{B} = \mu\tilde{H}$ $\tilde{J}_c = \sigma\tilde{E}$
$P_s = \oiint_s (\vec{E}\times\vec{H})\cdot ds$ $W_{m=}\left[\iiint_v [\frac{1}{2}\mu\lvert H\rvert^2]\,dv\right], \quad W_{e=}\iiint_v [\frac{1}{2}\varepsilon\lvert E\rvert^2]\,dv$ $P_i = \iiint_v [\vec{E}\cdot\vec{J}_i]\,dv = 0, \quad P_d = \iiint_v \sigma\lvert E\rvert^2\,dv = 0$	$P_s = \oiint_s (\tilde{E}\times\tilde{H}^*)\cdot ds$ $W_{m=}\left[\iiint_v [\frac{1}{4}\mu\lvert\tilde{H}\rvert^2]\,dv\right], \quad W_{e=}\iiint_v [\frac{1}{4}\varepsilon\lvert\tilde{E}\rvert^2]\,dv$ $P_i = \iiint_v [\frac{1}{2}\tilde{E}\cdot\tilde{J}_i^*]\,dv = 0, \quad P_d = \iiint_v \frac{1}{2}\sigma\lvert\tilde{E}\rvert^2\,dv = 0$

Example Problem

$$\varepsilon_1, \mu_o$$

$$\varepsilon_2, \mu_o$$

$$\vec{E}_1 = \cos(1000t - 2\pi\sqrt{\varepsilon_1}\, z)\vec{a}_x + R\cos(1000t + 2\pi\sqrt{\varepsilon_1}\, z)\vec{a}_x$$

$$\vec{E}_2 = T\cos(1000t - 2\pi\sqrt{\varepsilon_2}\, z)\vec{a}_x$$

FIND R and T on both sides!

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}_c + \vec{J}_i$$

$$\nabla \cdot \vec{D} = \rho$$

$$\nabla \cdot \vec{B} = 0$$

z=0

Example Problem

ε_1, μ_o

ε_2, μ_o

$$\vec{E}_1 = \cos(1000t - 2\pi\sqrt{\varepsilon_1}\ z)\vec{a}_x$$
$$+ R\cos(1000t + 2\pi\sqrt{\varepsilon_1}\ z)\vec{a}_x$$

$$\vec{E}_2 = T\cos(1000t - 2\pi\sqrt{\varepsilon_2}\ z)\vec{a}_x$$

FIND H on both sides!

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

z=0

Example Problem

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\varepsilon_1, \mu_o \qquad\qquad \varepsilon_2, \mu_o$$

$$\vec{E}_1 = \cos(1000t - 2\pi\sqrt{\varepsilon_1}\, z)\vec{a}_x + R\cos(1000t + 2\pi\sqrt{\varepsilon_1}\, z)\vec{a}_x$$

$$\vec{E}_2 = T\cos(1000t - 2\pi\sqrt{\varepsilon_2}\, z)\vec{a}_x$$

$$\frac{\partial \vec{B}_1}{\partial t} = -\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \begin{matrix}\cos(1000t - 2\pi\sqrt{\varepsilon_1}\, z) \\ + R\cos(1000t + 2\pi\sqrt{\varepsilon_1}\, z)\end{matrix} & 0 & 0 \end{vmatrix}$$

$$\frac{\partial \vec{B}_2}{\partial t} = -\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ T\cos(1000t - 2\pi\sqrt{\varepsilon_2}\, z) & 0 & 0 \end{vmatrix}$$

z=0

Example Problem

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

ε_1, μ_o

$$\frac{\partial \vec{B}_1}{\partial t} = -\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \begin{matrix} \cos(1000t - 2\pi\sqrt{\varepsilon_1}\, z) \\ + R\cos(1000t + 2\pi\sqrt{\varepsilon_1}\, z) \end{matrix} & 0 & 0 \end{vmatrix}$$

$$\frac{\partial \vec{B}_1}{\partial t} = -\frac{\partial}{\partial z}\begin{bmatrix} \cos(1000t - 2\pi\sqrt{\varepsilon_1}\, z) \\ + R\cos(1000t + 2\pi\sqrt{\varepsilon_1}\, z) \end{bmatrix}\hat{a}_y$$

$$\frac{\partial \vec{B}_1}{\partial t} = -2\pi\sqrt{\varepsilon_1}\sin(1000t - 2\pi\sqrt{\varepsilon_1}\, z)\hat{a}_y$$

$$+ 2\pi\sqrt{\varepsilon_1}\cdot R\sin(1000t + 2\pi\sqrt{\varepsilon_1}\, z)\hat{a}_y$$

ε_2, μ_o

$$\frac{\partial \vec{B}_2}{\partial t} = -\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ T\cos(1000t - 2\pi\sqrt{\varepsilon_2}\, z) & 0 & 0 \end{vmatrix}$$

$$\frac{\partial \vec{B}_2}{\partial t} = -\frac{\partial}{\partial z}\left(T\cos(1000t - 2\pi\sqrt{\varepsilon_2}\, z)\right)\hat{a}_y$$

$$\frac{\partial \vec{B}_2}{\partial t} = -T\cdot 2\pi\sqrt{\varepsilon_2}\sin(1000t - 2\pi\, z)\hat{a}_y$$

z=0

Example Problem

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

ε_1, μ_o

$$\frac{\partial \vec{B}_1}{\partial t} = -2\pi\sqrt{\varepsilon_1}\sin(1000t - 2\pi\sqrt{\varepsilon_1}\,z)\hat{a}_y$$

$$+2\pi\sqrt{\varepsilon_1}\cdot R\sin(1000t + 2\pi\sqrt{\varepsilon_1}\,z)\hat{a}_y$$

$$\vec{B}_1 = \int \begin{bmatrix} -2\pi\sqrt{\varepsilon_1}\sin(1000t - 2\pi\sqrt{\varepsilon_1} z)\hat{a}_y \\ +2\pi\sqrt{\varepsilon_1}\cdot R\sin(1000t + 2\pi\sqrt{\varepsilon_1}\,z)\hat{a}_y \end{bmatrix} dt$$

$$\vec{B}_1 = +2\pi\cdot\frac{\sqrt{\varepsilon_1}}{1000}\cos(1000t - 2\pi\sqrt{\varepsilon_1} z)\hat{a}_y$$

$$-2\pi\cdot\frac{\sqrt{\varepsilon_1}}{1000}R\cos(1000t + 2\pi\sqrt{\varepsilon_1}\,z)\hat{a}_y$$

ε_2, μ_o

$$\frac{\partial \vec{B}_2}{\partial t} = -T\cdot 2\pi\sin(1000t - 2\pi\sqrt{\varepsilon_2}\,z)\hat{a}_y$$

$$\vec{B}_2 = \int -T\cdot 2\pi\sin(1000t - 2\pi\sqrt{\varepsilon_2} z)\hat{a}_y dt$$

$$\vec{B}_2 = T\cdot 2\pi\frac{\sqrt{\varepsilon_2}}{1000}\cos(1000t - 2\pi\sqrt{\varepsilon_2}\,z)\hat{a}_y$$

z=0

Example Problem

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

ε_1, μ_o

$$\vec{E}_1 = \cos(1000t - 2\pi\sqrt{\varepsilon_1}z)\vec{a}_x$$

$$+R\cos(1000t + 2\pi\sqrt{\varepsilon_1}\,z)\vec{a}_x$$

$$\vec{B}_1 = +2\pi\cdot\frac{\sqrt{\varepsilon_1}}{1000}\cos(1000t - 2\pi\sqrt{\varepsilon_1}z)\hat{a}_y$$

$$+2\pi\cdot\frac{\sqrt{\varepsilon_1}}{1000}R\cos(1000t + 2\pi\sqrt{\varepsilon_1}\,z)\hat{a}_y$$

ε_2, μ_o

$$\vec{E}_2 = T\cos(1000t - 2\pi\sqrt{\varepsilon_2}\,z)\vec{a}_x$$

$$\vec{B}_2 = T\cdot 2\pi\frac{\sqrt{\varepsilon_2}}{1000}\cos(1000t - 2\pi\sqrt{\varepsilon_2}\,z)\hat{a}_y$$

z=0

Example Problem

ε_1, μ_o

ε_2, μ_o

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{E}_1 = \cos(1000t - 2\pi\sqrt{\varepsilon_1}z)\vec{a}_x$$

$$+ R\cos(1000t + 2\pi\sqrt{\varepsilon_1}\,z)\vec{a}_x$$

$$\vec{H}_1 = +2\pi \cdot \frac{\sqrt{\varepsilon_1}}{1000}\cos(1000t - 2\pi\sqrt{\varepsilon_1}z)\hat{a}_y$$

$$-2\pi \cdot \frac{\sqrt{\varepsilon_1}}{1000}R\cos(1000t + 2\pi\sqrt{\varepsilon_1}\,z)\hat{a}_y$$

$$\vec{E}_2 = T\cos(1000t - 2\pi\sqrt{\varepsilon_2}\,z)\vec{a}_x$$

$$\vec{H}_2 = T \cdot 2\pi\frac{\sqrt{\varepsilon_2}}{1000}\cos(1000t - 2\pi\sqrt{\varepsilon_2}\,z)\hat{a}_y$$

z=0

Example Problem

$$\varepsilon_1, \mu_o$$

$$\varepsilon_2, \mu_o$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{E}_1 = \cos(1000t - 2\pi\sqrt{\varepsilon_1} z)\vec{a}_x$$

$$+ R\cos(1000t + 2\pi\sqrt{\varepsilon_1}\, z)\vec{a}_x$$

$$\vec{H}_1 = +2\pi \cdot \frac{\sqrt{\varepsilon_1}}{1000}\cos(1000t - 2\pi\sqrt{\varepsilon_1} z)\hat{a}_y$$

$$- 2\pi \cdot \frac{\sqrt{\varepsilon_1}}{1000} R\cos(1000t + 2\pi\sqrt{\varepsilon_1}\, z)\hat{a}_y$$

$$\vec{E}_2 = T\cos(1000t - 2\pi\sqrt{\varepsilon_2}\, z)\vec{a}_x$$

$$\vec{H}_2 = T \cdot 2\pi\frac{\sqrt{\varepsilon_2}}{1000}\cos(1000t - 2\pi\sqrt{\varepsilon_2}\, z)\hat{a}_y$$

APPLY BC at Z=0 to H

$$\frac{2\pi}{\mu_o} \cdot \frac{\sqrt{\varepsilon_1}}{1000}\cos(1000t)\hat{a}_y - \frac{2\pi}{\mu_o} \cdot \frac{\sqrt{\varepsilon_1}}{1000} R\cos(1000t)\hat{a}_y = T\frac{2\pi}{\mu_o} \cdot \frac{\sqrt{\varepsilon_2}}{1000}\cos(1000t)\hat{a}_y$$

z=0

Example Problem

$$\varepsilon_1, \mu_o$$

$$\varepsilon_2, \mu_o$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{E}_1 = \cos(1000t - 2\pi\sqrt{\varepsilon_1}\, z)\vec{a}_x + R\cos(1000t + 2\pi\sqrt{\varepsilon_1}\, z)\vec{a}_x$$

$$\vec{E}_2 = T\cos(1000t - 2\pi\sqrt{\varepsilon_2}\, z)\vec{a}_x$$

$$\vec{H}_1 = +2\pi\cdot\frac{\sqrt{\varepsilon_1}}{1000}\cos(1000t - 2\pi\sqrt{\varepsilon_1}\, z)\hat{a}_y - 2\pi\cdot\frac{\sqrt{\varepsilon_1}}{1000}R\cos(1000t + 2\pi\sqrt{\varepsilon_1}\, z)\hat{a}_y$$

$$\vec{H}_2 = T\cdot 2\pi\frac{\sqrt{\varepsilon_2}}{1000}\cos(1000t - 2\pi\sqrt{\varepsilon_2}\, z)\hat{a}_y$$

APPLY BC at Z=0 to H

$$\sqrt{\varepsilon_1} - \sqrt{\varepsilon_1}R = \sqrt{\varepsilon_2}T$$

z=0

Example Problem

$$\varepsilon_1, \mu_o \qquad \varepsilon_2, \mu_o$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{E}_1 = \cos(1000t - 2\pi\sqrt{\varepsilon_1}z)\vec{a}_x$$

$$+ R\cos(1000t + 2\pi\sqrt{\varepsilon_1}\, z)\vec{a}_x$$

$$\vec{E}_2 = T\cos(1000t - 2\pi\sqrt{\varepsilon_2}\, z)\vec{a}_x$$

$$\vec{H}_1 = +2\pi \cdot \frac{\sqrt{\varepsilon_1}}{1000}\cos(1000t - 2\pi\sqrt{\varepsilon_1}z)\hat{a}_y$$

$$-2\pi \cdot \frac{\sqrt{\varepsilon_1}}{1000} R\cos(1000t + 2\pi\sqrt{\varepsilon_1}\, z)\hat{a}_y$$

$$\vec{H}_2 = T \cdot 2\pi \frac{\sqrt{\varepsilon_2}}{1000}\cos(1000t - 2\pi\sqrt{\varepsilon_2}\, z)\hat{a}_y$$

APPLY BC at Z=0 to E

$$\cos(1000t)\hat{a}_y - R\cos(1000t)\hat{a}_y = T\cos(1000t)\hat{a}_y$$

z=0

Example Problem

ε_1, μ_o

ε_2, μ_o

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{E}_1 = \cos(1000t - 2\pi\sqrt{\varepsilon_1}z)\vec{a}_x$$

$$+ R\cos(1000t + 2\pi\sqrt{\varepsilon_1}\, z)\vec{a}_x$$

$$\vec{H}_1 = +2\pi \cdot \frac{\sqrt{\varepsilon_1}}{1000}\cos(1000t - 2\pi\sqrt{\varepsilon_1}z)\hat{a}_y$$

$$-2\pi \cdot \frac{\sqrt{\varepsilon_1}}{1000}R\cos(1000t + 2\pi\sqrt{\varepsilon_1}\, z)\hat{a}_y$$

$$\vec{E}_2 = T\cos(1000t - 2\pi\sqrt{\varepsilon_2}\, z)\vec{a}_x$$

$$\vec{H}_2 = T \cdot 2\pi\frac{\sqrt{\varepsilon_2}}{1000}\cos(1000t - 2\pi\sqrt{\varepsilon_2}\, z)\hat{a}_y$$

APPLY BC at Z=0 to E

$$1 + R = T$$

z=0

Example Problem

$$1-R=\frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}T \qquad 1+R=T$$

SOLVE

$$\boxed{T=\frac{2}{\left(1+\frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}\right)}} \qquad \boxed{R=\frac{\left(1-\frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}\right)}{\left(1+\frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}\right)}}$$

Example Problem

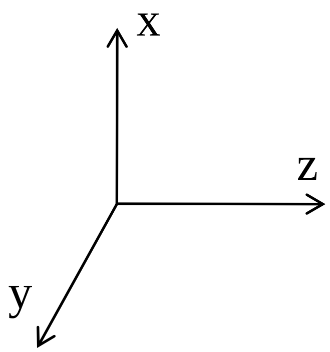
Two half spaces

ε_1, μ_1

ε_2, μ_2

$$\vec{E}_1 = E_o \cos(\omega t - \beta_1 z)\vec{a}_x + \Gamma E_o \cos(\omega t + \beta_1 z)\vec{a}_x$$

$$\vec{E}_2 = T E_o \cos(\omega t - \beta_2 z)\vec{a}_x$$



Problem: Using Maxwell's equations + constituent equations + boundary condition find expressions for β_1, β_2, Γ and T

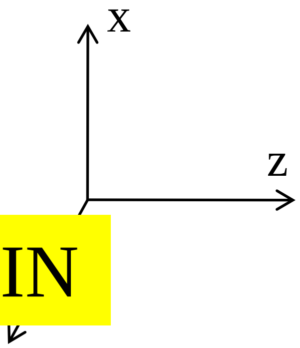
Example Problem

Two half spaces

$$\varepsilon_1,\ \mu_1 \qquad\qquad \varepsilon_2,\ \mu_2$$

$$\vec{E}_1 = E_o \cos(\omega t - \beta_1 z)\vec{a}_x + \Gamma E_o \cos(\omega t + \beta_1 z)\vec{a}_x$$

$$\vec{E}_2 = TE_o \cos(\omega t - \beta_2 z)\vec{a}_x$$

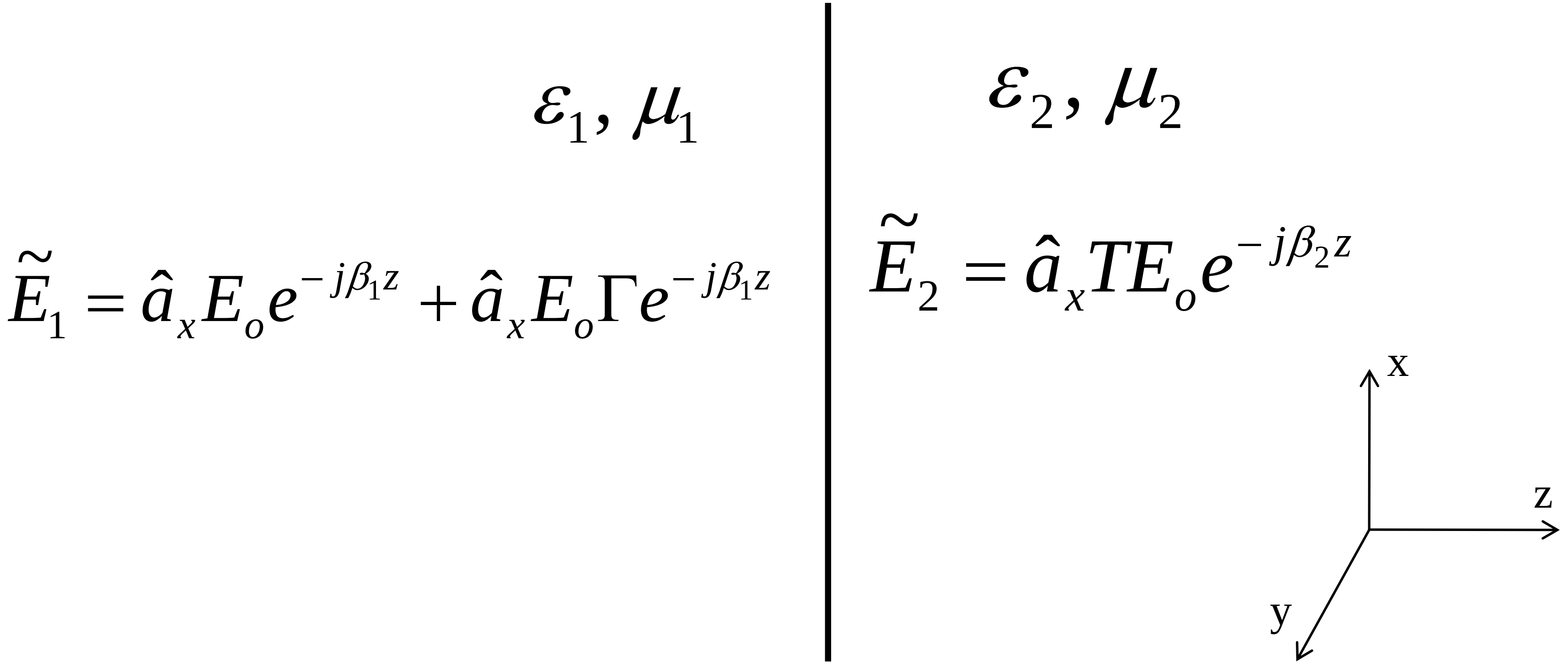


CONVERT TO FREQUNCY DOMAIN

Problem: Using Maxwell's equations + constituent equations + boundary condition find expressions for β_1, β_2, Γ and T

Example Problem

Two lossless half spaces



ε_1, μ_1

ε_2, μ_2

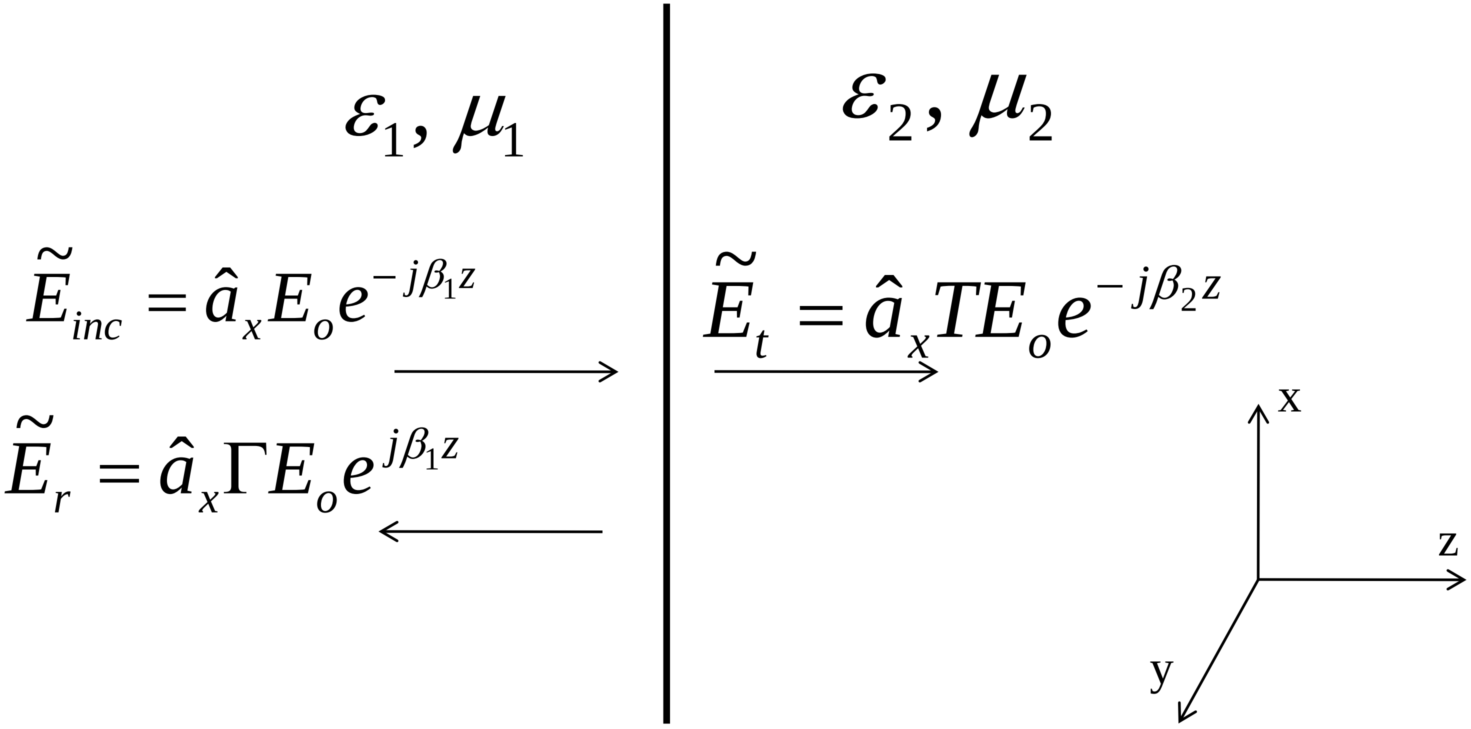
$$\tilde{E}_1 = \hat{a}_x E_o e^{-j\beta_1 z} + \hat{a}_x E_o \Gamma e^{-j\beta_1 z}$$

$$\tilde{E}_2 = \hat{a}_x T E_o e^{-j\beta_2 z}$$

Problem: Using Maxwell's equations + constituent equations + boundary condition find expressions for β_1, β_2, Γ and T

Example Problem

Two lossless half spaces



ε_1, μ_1

ε_2, μ_2

$$\tilde{E}_{inc} = \hat{a}_x E_o e^{-j\beta_1 z}$$

$$\tilde{E}_r = \hat{a}_x \Gamma E_o e^{j\beta_1 z}$$

$$\tilde{E}_t = \hat{a}_x T E_o e^{-j\beta_2 z}$$

Problem: Using Maxwell’s equations + constituent equations + boundary condition find expressions for β_1, β_2, Γ and T

(1) Start by finding H using $\nabla\times\tilde{E} = -j\omega\mu\tilde{H} \Rightarrow \tilde{H} = -\frac{1}{j\omega\mu}\nabla\times\tilde{E}$

$$\tilde{H} = -\frac{1}{j\omega\mu}\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$

$$\tilde{H}_{inc} = -\frac{1}{j\omega\mu_1}\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_o e^{-j\beta_1 z} & 0 & 0 \end{vmatrix} = -\hat{a}_y \frac{1}{j\omega\mu_1}\frac{\partial}{\partial z}(E_o e^{-j\beta_1 z}) = \hat{a}_y \frac{\beta_1}{\omega\mu_1} E_o e^{-j\beta_1 z}$$

(1) Start by finding H using $\nabla\times\tilde{E} = -j\omega\mu\tilde{H} \Rightarrow \tilde{H} = -\frac{1}{j\omega\mu}\nabla\times\tilde{E}$

$$\tilde{H}_r = -\frac{1}{j\omega\mu_1}\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_o\Gamma e^{j\beta_1 z} & 0 & 0 \end{vmatrix} = -\hat{a}_y\,\frac{\beta_1}{\omega\mu_1}E_o\Gamma e^{j\beta_1 z}$$

(1) Start by finding H using $\nabla \times \tilde{E} = -j\omega\mu\tilde{H} \Rightarrow \tilde{H} = -\frac{1}{j\omega\mu}\nabla \times \tilde{E}$

$$\tilde{H}_t = -\frac{1}{j\omega\mu_2}\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_o T e^{-j\beta_2 z} & 0 & 0 \end{vmatrix} = \hat{a}_y \frac{\beta_2}{\omega\mu_2} E_o T e^{j\beta_2 z}$$

(2) See if we can find β using $\nabla \times \tilde{H} = j\omega\tilde{D}$

$$\nabla \times \tilde{H} = j\omega\varepsilon\,\tilde{E} \Rightarrow \tilde{E} = \frac{1}{j\omega\varepsilon}\nabla \times \tilde{H}$$

$$\tilde{E} = \frac{1}{j\omega\varepsilon}\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix}$$

$$\tilde{E}_{inc} = \hat{a}_x E_o e^{-j\beta_1 z} = \frac{1}{j\omega\varepsilon_1}\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & \frac{\beta_1}{\omega\mu_1}E_o e^{-j\beta_1 z} & 0 \end{vmatrix} = -\hat{a}_x \frac{1}{j\omega\varepsilon_1}\frac{\partial}{\partial z}(\frac{\beta_1}{\omega\mu_1}E_o e^{-j\beta_1 z})$$

$$= \hat{a}_x \frac{\beta_1{}^2}{\omega^2\mu_1\varepsilon_1}E_o e^{-j\beta_1 z}$$

(2) See if we can find β using

$$\nabla \times \tilde{H} = j\omega \tilde{D}$$

$$\nabla \times \tilde{H} = j\omega\varepsilon \tilde{E} \Rightarrow \tilde{E} = \frac{1}{j\omega\varepsilon} \nabla \times \tilde{H}$$

$$\tilde{E}_{inc} = \hat{a}_x E_o e^{-j\beta_1 z} = \frac{1}{j\omega\varepsilon_1} \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & \frac{\beta_1}{\omega\mu_1} E_o e^{-j\beta_1 z} & 0 \end{vmatrix} = -\hat{a}_x \frac{1}{j\omega\varepsilon_1} \frac{\partial}{\partial z} \left(\frac{\beta_1}{\omega\mu_1} E_o e^{-j\beta_1 z} \right)$$

$$= \hat{a}_x \frac{\beta_1^{\,2}}{\omega^2 \mu_1 \varepsilon_1} E_o e^{-j\beta_1 z}$$

$$\Rightarrow \hat{a}_x E_o e^{-j\beta_1 z} = \hat{a}_x \frac{\beta_1^{\,2}}{\omega^2 \mu_1 \varepsilon_1} E_o e^{-j\beta_1 z}$$

$$\Rightarrow \frac{\beta_1^{\,2}}{\omega^2 \mu_1 \varepsilon_1} = 1$$

$$\Rightarrow \beta_1 = \omega\sqrt{\mu_1 \varepsilon_1}$$

Similarly: $\beta_2 = \omega\sqrt{\mu_2 \varepsilon_2}$

Combine the two last results:

$$\widetilde{H}_{inc} = \hat{a}_y \frac{\beta_1}{\omega\mu_1} E_o e^{-j\beta_1 z}$$

$$\widetilde{H}_r = -\hat{a}_y \frac{\beta_1}{\omega\mu_1} E_o \Gamma e^{j\beta_1 z}$$

$$\widetilde{H}_t = \hat{a}_y \frac{\beta_2}{\omega\mu_2} E_o T e^{j\beta_2 z}$$

$$\beta_1 = \omega\sqrt{\mu_1\varepsilon_1}$$

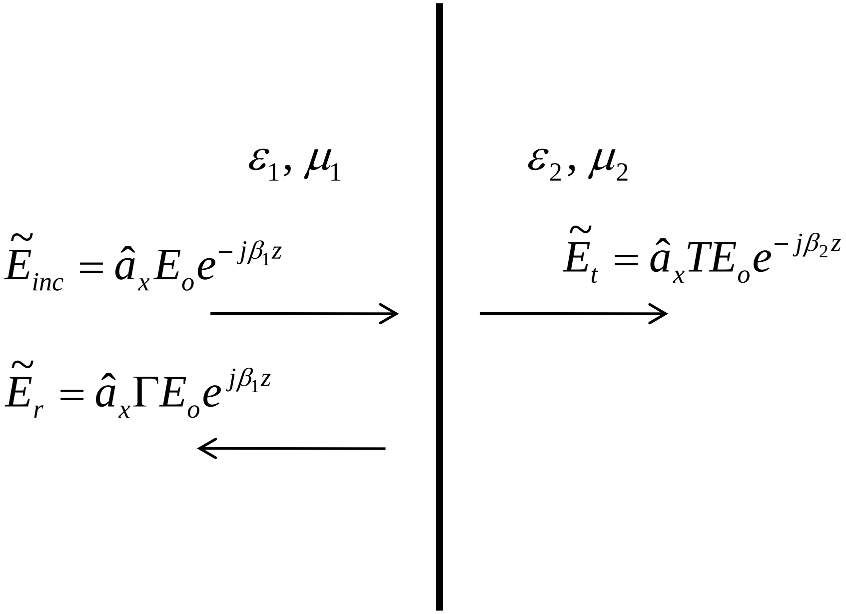
$$\beta_2 = \omega\sqrt{\mu_2\varepsilon_2}$$

$$\widetilde{H}_{inc} = \hat{a}_y \frac{\beta_1}{\omega\mu_1} E_o e^{-j\beta_1 z} = \hat{a}_y \frac{\omega\sqrt{\mu_1\varepsilon_1}}{\omega\mu_1} E_o e^{-j\beta_1 z} = \hat{a}_y \sqrt{\frac{\varepsilon_1}{\mu_1}} E_o e^{-j\beta_1 z}$$

$$\widetilde{H}_r = -\hat{a}_y \frac{\beta_1}{\omega\mu_1} E_o \Gamma e^{j\beta_1 z} = -\hat{a}_y \sqrt{\frac{\varepsilon_1}{\mu_1}} E_o \Gamma e^{j\beta_1 z}$$

$$\widetilde{H}_t = \hat{a}_y \frac{\beta_2}{\omega\mu_2} E_o T e^{j\beta_2 z} = \hat{a}_y \sqrt{\frac{\varepsilon_2}{\mu_2}} E_o T e^{-j\beta_2 z}$$

Lets Review:



$\varepsilon_1,\, \mu_1$

$\varepsilon_2,\, \mu_2$

$\tilde{E}_{inc} = \hat{a}_x E_o e^{-j\beta_1 z}$

$\tilde{E}_t = \hat{a}_x T E_o e^{-j\beta_2 z}$

$\tilde{E}_r = \hat{a}_x \Gamma E_o e^{j\beta_1 z}$

$$\beta_1 = \omega\sqrt{\mu_1\varepsilon_1}$$

$$\beta_2 = \omega\sqrt{\mu_2\varepsilon_2}$$

$$\tilde{H}_{inc} = \hat{a}_y \sqrt{\frac{\varepsilon_1}{\mu_1}} E_o e^{-j\beta_1 z}$$

$$\tilde{H}_r = -\hat{a}_y \sqrt{\frac{\varepsilon_1}{\mu_1}} E_o \Gamma e^{j\beta_1 z}$$

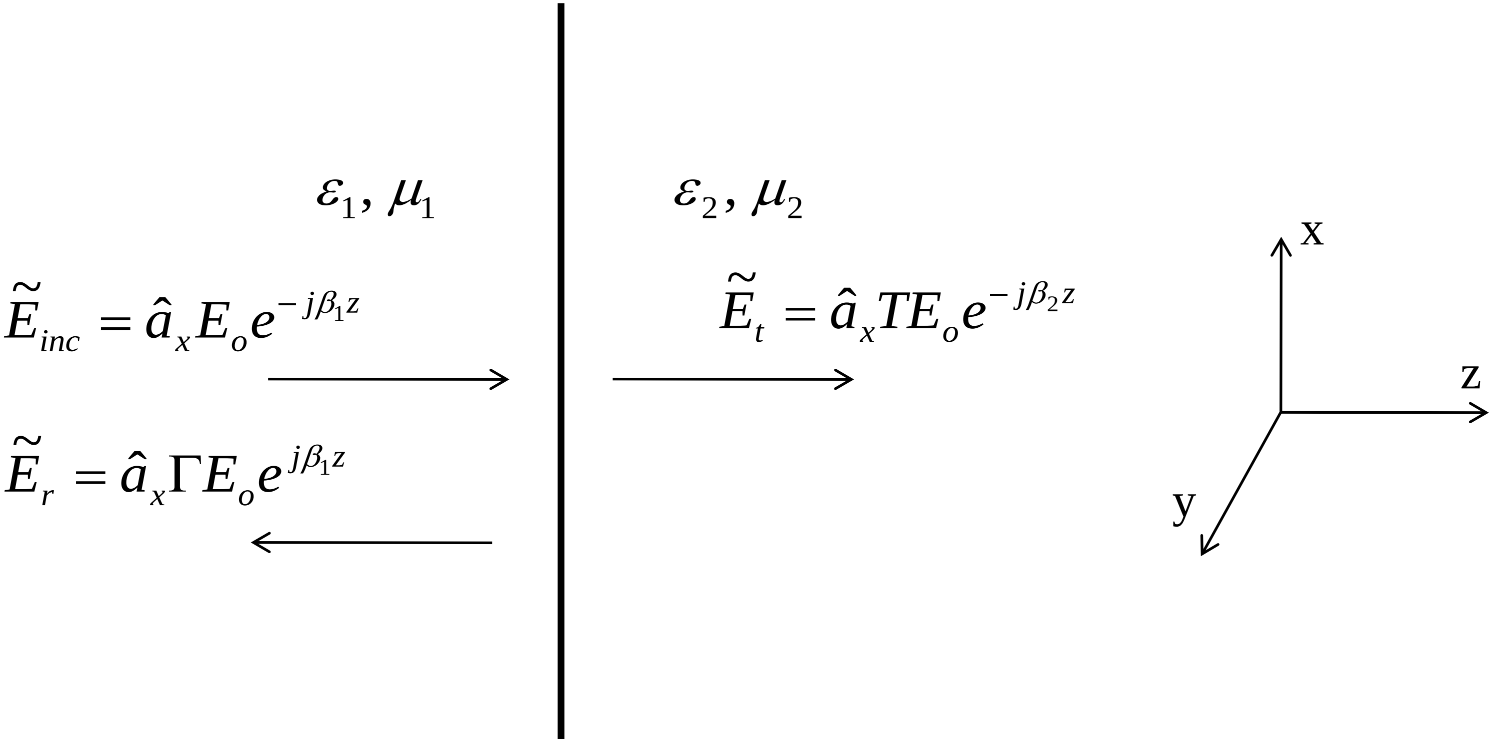
$$\tilde{H}_t = \hat{a}_y \sqrt{\frac{\varepsilon_2}{\mu_2}} E_o T e^{-j\beta_2 z}$$

$$\tilde{E}_{inc} = \hat{a}_x E_o e^{-j\beta_1 z}$$

$$\tilde{E}_r = \hat{a}_x E_o \Gamma e^{j\beta_1 z}$$

$$\tilde{E}_t = \hat{a}_x E_o T e^{-j\beta_2 z}$$

Now Apply Boundary Conditions:



ε_1, μ_1

ε_2, μ_2

$$\tilde{E}_{inc} = \hat{a}_x E_o e^{-j\beta_1 z}$$

$$\tilde{E}_r = \hat{a}_x \Gamma E_o e^{j\beta_1 z}$$

$$\tilde{E}_t = \hat{a}_x T E_o e^{-j\beta_2 z}$$

$$\hat{n} \times \left(\tilde{E}_2 - \tilde{E}_1\right) = 0$$

$$\hat{n} \times \left(\tilde{H}_2 - \tilde{H}_1\right) = 0$$

at z=0:

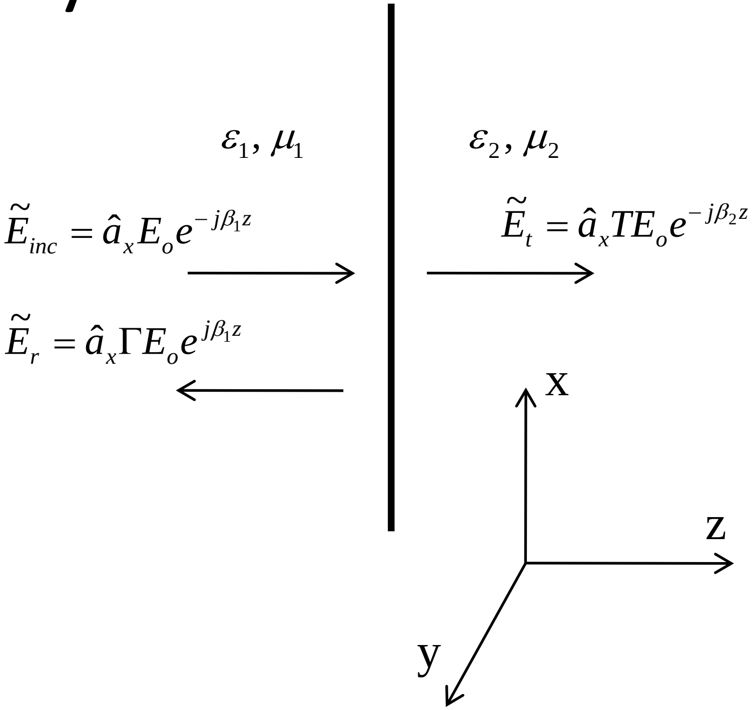
$$\tilde{E}_{inc} + \tilde{E}_r = \tilde{E}_t$$

$$\tilde{H}_{inc} + \tilde{H}_r = \tilde{H}_t$$

Now Apply Boundary Conditions:

at z=0: $\tilde{E}_{inc} + \tilde{E}_r = \tilde{E}_t$

$\tilde{H}_{inc} + \tilde{H}_r = \tilde{H}_t$



$\varepsilon_1,\ \mu_1$

$\varepsilon_2,\ \mu_2$

$\tilde{E}_{inc} = \hat{a}_x E_o e^{-j\beta_1 z}$

$\tilde{E}_r = \hat{a}_x \Gamma E_o e^{j\beta_1 z}$

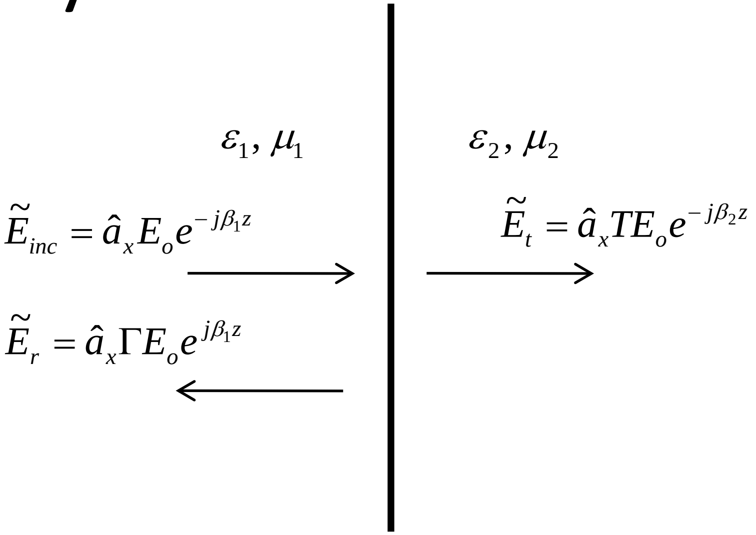
$\tilde{E}_t = \hat{a}_x T E_o e^{-j\beta_2 z}$

$$(1)\qquad \hat{a}_x E_o e^{-j\beta_1 0} + \hat{a}_x E_o \Gamma e^{j\beta_1 0} = \hat{a}_x E_o T e^{-j\beta_2 0}$$

$$(2)\qquad \hat{a}_y \sqrt{\frac{\varepsilon_1}{\mu_1}} E_o e^{-j\beta_1 0} - \hat{a}_y \sqrt{\frac{\varepsilon_1}{\mu_1}} E_o \Gamma e^{j\beta_1 0} = \hat{a}_y \sqrt{\frac{\varepsilon_2}{\mu_2}} E_o T e^{-j\beta_2 0}$$

Now Apply Boundary Conditions:

at z=0: $$\tilde{E}_{inc} + \tilde{E}_r = \tilde{E}_t$$
$$\tilde{H}_{inc} + \tilde{H}_r = \tilde{H}_t$$



ε_1, μ_1

ε_2, μ_2

$\tilde{E}_{inc} = \hat{a}_x E_o e^{-j\beta_1 z}$

$\tilde{E}_t = \hat{a}_x T E_o e^{-j\beta_2 z}$

$\tilde{E}_r = \hat{a}_x \Gamma E_o e^{j\beta_1 z}$

$$(1) \qquad \hat{a}_x E_o e^{-j\beta_1 0} + \hat{a}_x E_o \Gamma e^{j\beta_1 0} = \hat{a}_x E_o T e^{-j\beta_2 0}$$

$$(2) \qquad \hat{a}_y \sqrt{\frac{\varepsilon_1}{\mu_1}} E_o e^{-j\beta_1 0} - \hat{a}_y \sqrt{\frac{\varepsilon_1}{\mu_1}} E_o \Gamma e^{j\beta_1 0} = \hat{a}_y \sqrt{\frac{\varepsilon_2}{\mu_2}} E_o T e^{-j\beta_2 0}$$

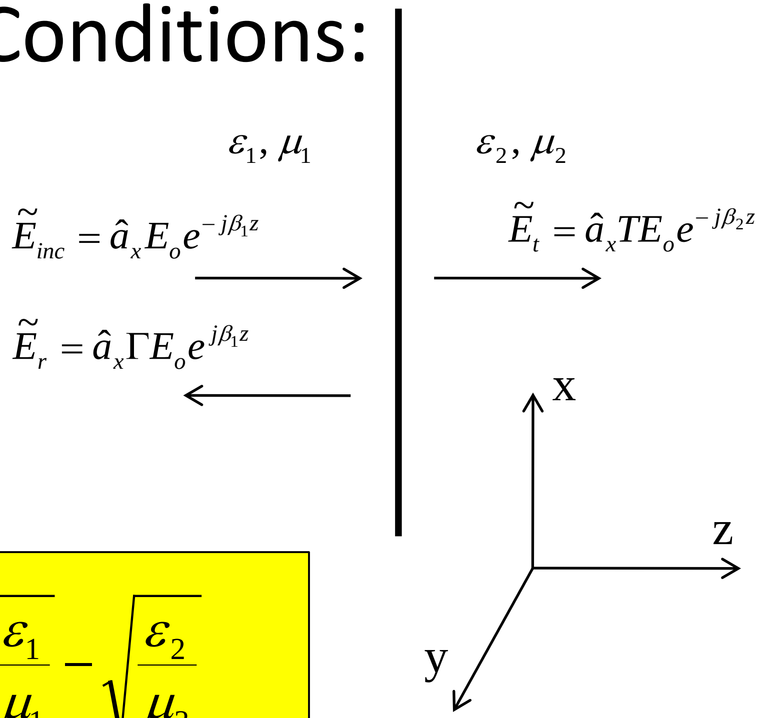
results in
2 equations +
2 unknowns

$$1 + \Gamma = T$$
$$\sqrt{\frac{\varepsilon_1}{\mu_1}} - \sqrt{\frac{\varepsilon_1}{\mu_1}}\Gamma = \sqrt{\frac{\varepsilon_2}{\mu_2}} T$$

Now Apply Boundary Conditions:

$$1+\Gamma = T$$

$$\sqrt{\frac{\varepsilon_1}{\mu_1}} - \sqrt{\frac{\varepsilon_1}{\mu_1}}\Gamma = \sqrt{\frac{\varepsilon_2}{\mu_2}}T$$



$\varepsilon_1,\ \mu_1$

$\varepsilon_2,\ \mu_2$

$\tilde{E}_{inc} = \hat{a}_x E_o e^{-j\beta_1 z}$

$\tilde{E}_t = \hat{a}_x T E_o e^{-j\beta_2 z}$

$\tilde{E}_r = \hat{a}_x \Gamma E_o e^{j\beta_1 z}$

Solution:

$$\Gamma = \frac{\sqrt{\frac{\varepsilon_1}{\mu_1}} - \sqrt{\frac{\varepsilon_2}{\mu_3}}}{\sqrt{\frac{\varepsilon_1}{\mu_1}} + \sqrt{\frac{\varepsilon_2}{\mu_2}}}$$

$$T = \frac{2\sqrt{\frac{\varepsilon_1}{\mu_1}}}{\sqrt{\frac{\varepsilon_1}{\mu_1}} + \sqrt{\frac{\varepsilon_2}{\mu_2}}}$$

Classification of Materials

1. *Homogenous or Inhomogenous:* If the material properties are independent of spatial location then the material is homogenous, otherwise it is called inhomogenous

$$\varepsilon(x, y, z) \Rightarrow \mathit{Inhomogenous}$$

Classification of Materials

2. *Isotropic or Anisotropic:* If the material properties are independent of the polarization of the applied field then the material is isotropic, otherwise it is called anisotropic.

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \Rightarrow \vec{D} = \overset{=}{\varepsilon}\,\vec{E} \qquad anisotropic$$

Classification of Materials

3. *Linear or non-Linear:* If the material properties are independent on the magnitude and phase of the electric and magnetic fields, otherwise it is called non-linear

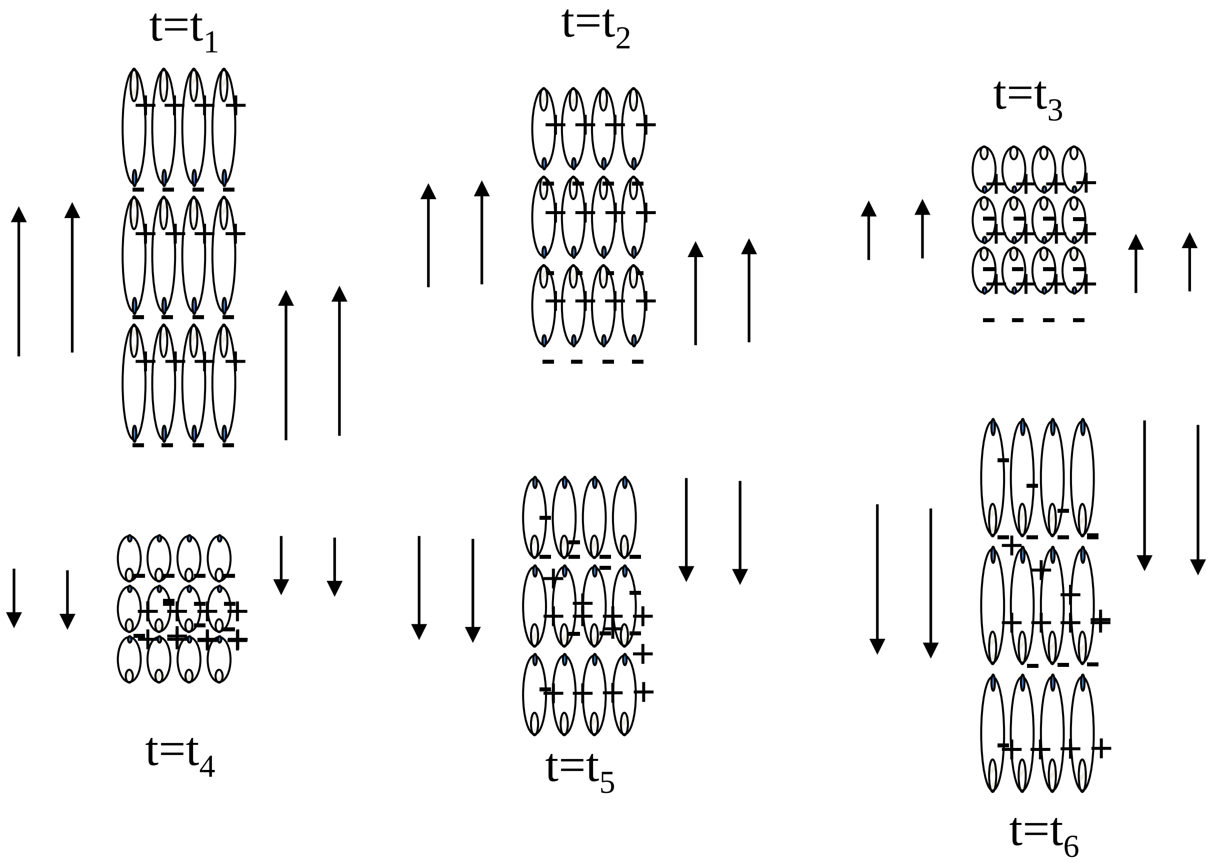
$$\vec{D} = \varepsilon_o \vec{E} + \chi_o \vec{E} + \chi_1 \vec{E}^2 + \chi_3 \vec{E}^3 + \ldots$$

Classification of Materials

4. *Dispersive or non-dispersive:* If the material properties are independent of frequency then the material is non-dispersive, otherwise it is called dispersive

Classification of Materials

4. Dispersive or non-dispersive: If the material properties are independent of frequency then the material is non-dispersive, otherwise it is called dispersive



A material's atoms or molecules attempt to keep up with a changing electric field. This results in two things: (1) friction causes energy loss via heat and (2) the dynamic response of the molecules will be a function of the frequency of the applied field (i.e. frequency dependent material properties)

Electric Properties of Materials
Frequency Behavior (Complex Permittivity)

$$
\begin{aligned}
&\nabla \times \tilde{E} = -j\omega \tilde{B} \\
&\nabla \times \tilde{H} = j\omega \tilde{D} + \tilde{J}_i + \tilde{J}_c \\
&\nabla \cdot \tilde{D} = \tilde{\rho} \\
&\nabla \cdot \tilde{B} = 0
\end{aligned}
$$

$$
\begin{aligned}
&\tilde{D} = \varepsilon(\omega)\,\tilde{E} \\
&\tilde{B} = \mu_o \tilde{H} \\
&\tilde{J}_c = \sigma_s\,\tilde{E}
\end{aligned}
$$

Dielectric constant

loss term

$$\varepsilon^*(\omega) = \varepsilon'(\omega) - j\varepsilon''(\omega)$$

is called the complex permittivity

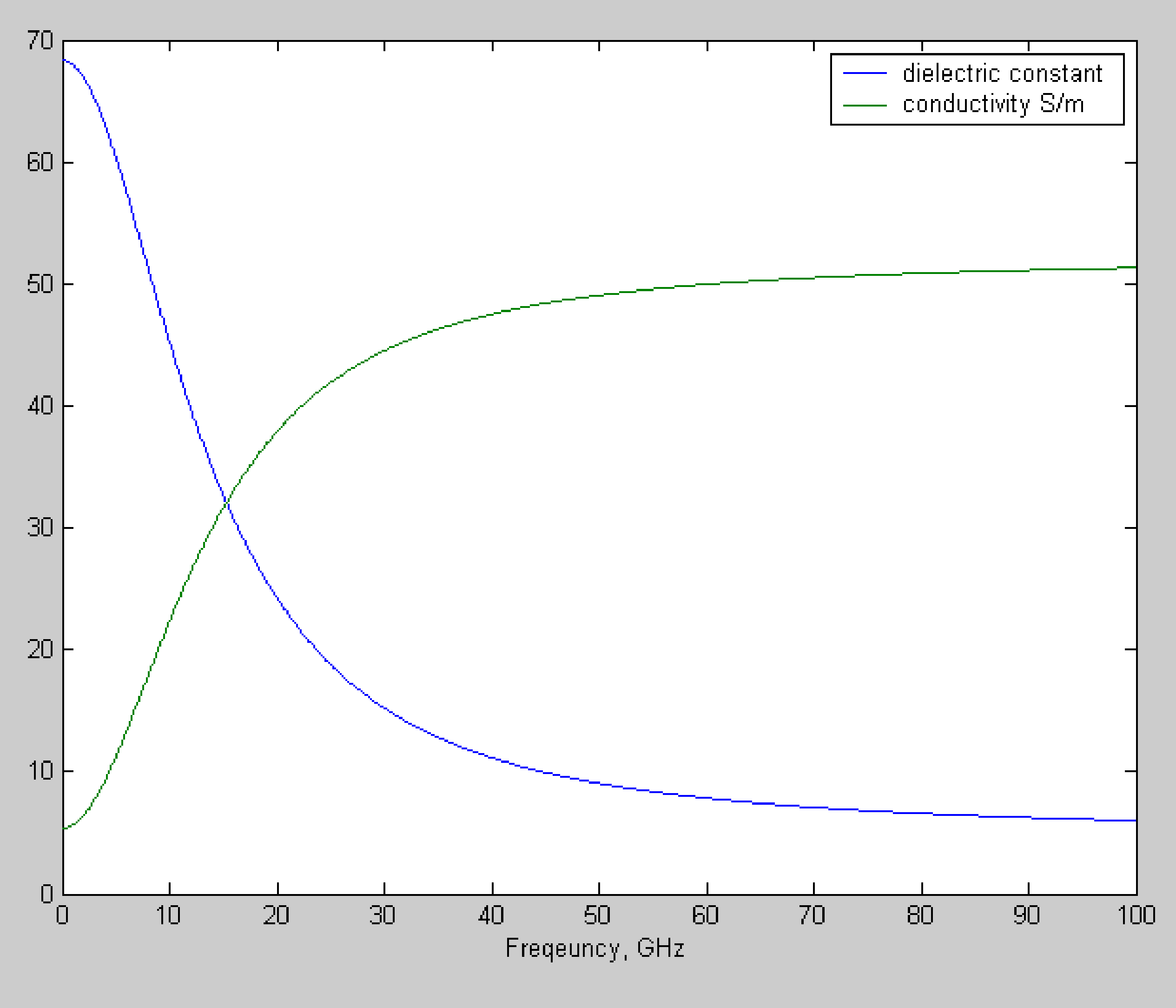
$$
\begin{aligned}
&\nabla \times \tilde{E} = -j\omega\,\mu_o \tilde{H} \\
&\nabla \times \tilde{H} = j\omega\,\varepsilon\,\tilde{E} + \tilde{J}_i + \sigma_s\,\tilde{E} \\
&\nabla \cdot (\varepsilon\,\tilde{E}) = \tilde{\rho} \\
&\nabla \cdot (\tilde{H}) = 0
\end{aligned}
$$

$$
\begin{aligned}
&\nabla \times \tilde{E} = -j\omega\,\mu_o \tilde{H} \\
&\nabla \times \tilde{H} = j\omega\varepsilon\tilde{E} + \tilde{J}_i + \sigma_s\,\tilde{E} \\
&\nabla \cdot (\varepsilon\,\tilde{E}) = \tilde{\rho} \\
&\nabla \cdot (\tilde{H}) = 0
\end{aligned}
$$

$$
\begin{aligned}
&\nabla \times \tilde{E} = -j\omega\,\mu_o \tilde{H} \\
&\nabla \times \tilde{H} = j\omega\,\varepsilon^*(\omega)\,\tilde{E} + \tilde{J}_i \\
&\nabla \cdot (\varepsilon^*\,\tilde{E}) = \tilde{\rho} \\
&\nabla \cdot (\tilde{H}) = 0
\end{aligned}
$$

$$
\begin{aligned}
&\varepsilon^*(\omega) = \varepsilon'(\omega) - j\frac{\sigma_s}{\omega} \\
&\varepsilon^*(\omega) = \varepsilon'(\omega) - j\varepsilon''(\omega)
\end{aligned}
$$

Frequency Behavior of Sea Water



Wave Equation

Time Dependent Homogenous Wave Equation (E-Field)

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$$

$$\nabla \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t} + \sigma \vec{E} + \vec{J}$$

$$\nabla \cdot \vec{E} = {}^{\rho}\!/_{\varepsilon}$$

$$\nabla \cdot \vec{B} = 0$$

Take the curl of both of Faraday's Law

$$\nabla \times [\nabla \times \vec{E}] = \nabla \times (-\mu \frac{\partial \vec{H}}{\partial t})$$

$$\nabla \times \nabla \times \vec{E} = -\mu \frac{\partial}{\partial t} \nabla \times \vec{H}$$

Wave Equation

Time Dependent Homogenous Wave Equation (E-Field)

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$$

$$\nabla \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t} + \sigma \vec{E} + \vec{J}$$

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Take the curl of both of Faraday's Law

$$\nabla \times [\nabla \times \vec{E}] = \nabla \times (-\mu \frac{\partial \vec{H}}{\partial t})$$

$$\nabla \times \nabla \times \vec{E} = -\mu \frac{\partial}{\partial t} \nabla \times \vec{H}$$

$$\nabla \times \nabla \times \vec{E} = -\mu \frac{\partial}{\partial t} \left[\varepsilon \frac{\partial \vec{E}}{\partial t} + \sigma \vec{E} + \vec{J} \right]$$

$$\nabla \times \nabla \times \vec{E} = -\mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} - \mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu \frac{\partial \vec{J}}{\partial t}$$

Wave Equation

Time Dependent Homogenous Wave Equation (E-Field)

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$$

$$\nabla \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t} + \sigma \vec{E} + \vec{J}$$

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Vector Identity

$$\nabla \times \nabla \times \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$

$$\nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = -\mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} - \mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu \frac{\partial \vec{J}}{\partial t}$$

Wave Equation

Time Dependent Homogenous Wave Equation (E-Field)

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$$

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Wave Equation

Time Dependent Homogenous Wave Equation (E-Field)

$$\nabla\times\vec{E} = -\mu\frac{\partial\vec{H}}{\partial t}$$

$$\nabla\times\vec{H} = \varepsilon\frac{\partial\vec{E}}{\partial t} + \sigma\vec{E} + \vec{J}$$

$$\nabla\cdot\vec{E} = \rho/\varepsilon$$

$$\nabla\cdot\vec{B} = 0$$

$$\nabla\times\nabla\times\vec{E} = -\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} - \mu\sigma\frac{\partial\vec{E}}{\partial t} - \mu\frac{\partial\vec{J}}{\partial t}$$

Vector Identity

$$\nabla\times\nabla\times\vec{A} = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A}$$

$$\nabla(\nabla\cdot\vec{E}) - \nabla^2\vec{E} = -\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} - \mu\sigma\frac{\partial\vec{E}}{\partial t} - \mu\frac{\partial\vec{J}}{\partial t}$$

$$\nabla\left(\frac{\rho}{\varepsilon}\right) - \nabla^2\vec{E} = -\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} - \mu\sigma\frac{\partial\vec{E}}{\partial t} - \mu\frac{\partial\vec{J}}{\partial t}$$

Wave Equation

Time Dependent Homogenous Wave Equation (E-Field)

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$$

$$\nabla \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t} + \sigma \vec{E} + \vec{J}$$

$$\nabla \cdot \vec{E} = {}^{\rho}\!/\!_{\varepsilon}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \nabla \times \vec{E} = -\mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} - \mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu \frac{\partial \vec{J}}{\partial t}$$

Vector Identity

$$\nabla \times \nabla \times \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$

$$\nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = -\mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} - \mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu \frac{\partial \vec{J}}{\partial t}$$

$$\nabla\left(\frac{\rho}{\varepsilon}\right) - \nabla^2 \vec{E} = -\mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} - \mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu \frac{\partial \vec{J}}{\partial t}$$

$$\boxed{\nabla^2 \vec{E} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} - \mu\sigma \frac{\partial \vec{E}}{\partial t} = \mu \frac{\partial \vec{J}}{\partial t} + \frac{1}{\varepsilon} \nabla \rho}$$

Wave Equation

Source-Free Time Dependent Homogenous Wave Equation (E-Field)

$$\boxed{\nabla^2\vec{E}-\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}-\mu\sigma\frac{\partial\vec{E}}{\partial t}=\mu\frac{\partial\vec{J}}{\partial t}+\frac{1}{\varepsilon}\nabla\rho}$$

Source Free $\vec{J}=0, \quad \rho=0$

$$\boxed{\nabla^2\vec{E}-\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}-\mu\sigma\frac{\partial\vec{E}}{\partial t}=0}$$

Source-Free Lossless Time Dependent Homogenous Wave Equation (E-Field)

Lossless $\sigma=0$

$$\boxed{\nabla^2\vec{E}-\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}=0}$$

Wave Equation

Source-Free Time Dependent Homogenous Wave Equation (H-Field)

$$\nabla^2 \vec{H} - \mu\varepsilon \frac{\partial^2 \vec{H}}{\partial t^2} - \mu\sigma \frac{\partial \vec{H}}{\partial t} = -\nabla \times \vec{J} + \frac{1}{\mu} \nabla \rho$$

Source Free $\vec{J} = 0, \quad \rho = 0$

$$\nabla^2 \vec{H} - \mu\varepsilon \frac{\partial^2 \vec{H}}{\partial t^2} - \mu\sigma \frac{\partial \vec{H}}{\partial t} = 0$$

Source Free and Lossless $\vec{J} = 0, \quad \rho = 0, \quad \sigma = 0$

$$\nabla^2 \vec{H} - \mu\varepsilon \frac{\partial^2 \vec{H}}{\partial t^2} = 0$$

Wave Equation: Time Harmonic

Time Domain

$$\nabla^2 \vec{E} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} - \mu\sigma \frac{\partial \vec{E}}{\partial t} = \mu \frac{\partial \vec{J}}{\partial t} + \frac{1}{\varepsilon} \nabla \rho$$

Source Free $\vec{J} = 0, \quad \rho = 0$

⬇

$$\nabla^2 \vec{E} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} - \mu\sigma \frac{\partial \vec{E}}{\partial t} = 0$$

Lossless $\sigma = 0$

⬇

$$\nabla^2 \vec{E} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

Frequency Domain

How do I convert these to their time harmonic forms?

Wave Equation: Time Harmonic

Time Domain

$$\nabla^2\vec{E} - \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} - \mu\sigma\frac{\partial\vec{E}}{\partial t} = \mu\frac{\partial\vec{J}}{\partial t} + \frac{1}{\varepsilon}\nabla\rho$$

Source Free $\vec{J} = 0, \quad \rho = 0$

$$\nabla^2\vec{E} - \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} - \mu\sigma\frac{\partial\vec{E}}{\partial t} = 0$$

Lossless $\sigma = 0$

$$\nabla^2\vec{E} - \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} = 0$$

Frequency Domain

$$\nabla^2\tilde{E} + \omega^2\mu\varepsilon\tilde{E} - j\omega\mu\sigma\tilde{E} = \mu\tilde{J} + \frac{1}{\varepsilon}\nabla\tilde{\rho}$$

Source Free $\vec{J} = 0, \quad \rho = 0$

$$\nabla^2\tilde{E} + \omega^2\mu\varepsilon\tilde{E} - j\omega\mu\sigma\tilde{E} = 0$$

Lossless $\sigma = 0$

$$\nabla^2\tilde{E} + \omega^2\mu\varepsilon\tilde{E} = 0$$

“Helmholtz Equation”