ELEG 413
Spring 2017
Lecture #2

Mark Mirotznik, Ph.D.
Professor
The University of Delaware
Tel: (302)831-4221
Email: mirotzni@ece.udel.edu

Maxwell's Equations in Differential Form

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} - \vec{M}$$

Faraday's Law

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}_c + \vec{J}_i$$

Ampere's Law

$$\nabla \cdot \vec{D} = \rho$$

Gauss's Law

$$\nabla \cdot \vec{B} = \rho_m$$

Gauss's Magnetic Law

Maxwell’s Equations in Differential Form: No Magnetic Sources

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Faraday’s Law

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}_c + \vec{J}_i$$

Ampere’s Law

$$\nabla \cdot \vec{D} = \rho$$

Gauss’s Law

$$\nabla \cdot \vec{B} = 0$$

Gauss’s Magnetic Law

Maxwell’s Equations Field Variables

$\vec{E} =$ **Electric Field, V/m**

$\vec{D} =$ **Electric Displacement, Q/m^2**

$\vec{H} =$ **Magnetic Field, A/m**

$\vec{B} =$ **Magnetic Flux Density, T**

Maxwell’s Equations Source Variables

$\vec{J}_c =$ **Conductive Current Density, A/m^2**

$\vec{J}_i =$ **Impressed Current Density, A/m^2**

$\vec{M} =$ **Magnetic Current Density, V/m^2**

$\rho =$ **Electric Charge Density, Q/m^3**

$\rho_m =$ **Magnetic Charge Density Wb/m^3**

CONSTITUTIVE RELATIONS

$$\vec{D} = \varepsilon \vec{E}$$

$\varepsilon = \varepsilon_r \varepsilon_o$=permittivity (F/m)

$\varepsilon_o = 8.854 \times 10^{-12}$ (F/m)

CONSTITUTIVE RELATIONS

$$\vec{B} = \mu \vec{H}$$

$\mu = \mu_o \mu_r$

μ_o=permeability of free space (H/m)

$\mu_o = 4\pi \times 10^{-7}$ (H/m)

CONSTITUTIVE RELATIONS

$$\vec{J}_c = \sigma \vec{E}$$

σ=conductivity (S/m)

Boundary Conditions

NOT A PEC

$$\hat{n}\times\left(\vec{E}^{(2)}-\vec{E}^{(1)}\right)=0$$

$$\hat{n}\times\left(\vec{H}^{(2)}-\vec{H}^{(1)}\right)=0$$

$$\hat{n}\cdot\left(\vec{B}^{(2)}-\vec{B}^{(1)}\right)=0$$

$$\hat{n}\cdot\left(\vec{D}^{(2)}-\vec{D}^{(1)}\right)=0$$

PEC

$$\hat{n}\times\vec{E}^{(2)}=0$$

$$\hat{n}\times\vec{H}^{(2)}=\vec{J}_s$$

$$\hat{n}\cdot\vec{B}^{(2)}=0$$

$$\hat{n}\cdot\vec{D}^{(2)}=\rho_s$$

THIS IS EVERYTHING

Constitutive Relations

$$\vec{J}_c = \sigma \vec{E}$$

$$\vec{D} = \varepsilon \vec{E}$$

$$\vec{B} = \mu \vec{H}$$

Boundary Conditions

NOT A PEC

$$\hat{n} \times \left(\vec{E}^{(2)} - \vec{E}^{(1)}\right) = 0$$

$$\hat{n} \times \left(\vec{H}^{(2)} - \vec{H}^{(1)}\right) = 0$$

$$\hat{n} \cdot \left(\vec{B}^{(2)} - \vec{B}^{(1)}\right) = 0$$

$$\hat{n} \cdot \left(\vec{D}^{(2)} - \vec{D}^{(1)}\right) = 0$$

PEC

$$\hat{n} \times \vec{E}^{(2)} = 0$$

$$\hat{n} \times \vec{H}^{(2)} = \vec{J}_s$$

$$\hat{n} \cdot \vec{B}^{(2)} = 0$$

$$\hat{n} \cdot \vec{D}^{(2)} = \rho_s$$

Maxwell's equations

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}_c + \vec{J}_i$$

$$\nabla \cdot \vec{D} = \rho$$

$$\nabla \cdot \vec{B} = 0$$

Example Problem

$$\varepsilon_1, \mu_o$$

$$\varepsilon_2, \mu_o$$

$$\vec{E}_1 = \cos(1000t - 2\pi\sqrt{\varepsilon_1}\, z)\vec{a}_x + R\cos(1000t + 2\pi\sqrt{\varepsilon_1}\, z)\vec{a}_x$$

$$\vec{E}_2 = T\cos(1000t - 2\pi\sqrt{\varepsilon_2}\, z)\vec{a}_x$$

FIND R and T on both sides!

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}_c + \vec{J}_i$$

$$\nabla \cdot \vec{D} = \rho$$

$$\nabla \cdot \vec{B} = 0$$

z=0

Example Problem

ε_1, μ_o

ε_2, μ_o

$$\vec{E}_1 = \cos(1000t - 2\pi\sqrt{\varepsilon_1}\ z)\vec{a}_x$$
$$+ R\cos(1000t + 2\pi\sqrt{\varepsilon_1}\ z)\vec{a}_x$$

$$\vec{E}_2 = T\cos(1000t - 2\pi\sqrt{\varepsilon_2}\ z)\vec{a}_x$$

FIND H on both sides!

$$\boxed{\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

z=0

Example Problem

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\varepsilon_1, \mu_o$$

$$\vec{E}_1 = \cos(1000t - 2\pi\sqrt{\varepsilon_1}\, z)\vec{a}_x + R\cos(1000t + 2\pi\sqrt{\varepsilon_1}\, z)\vec{a}_x$$

$$\frac{\partial \vec{B}_1}{\partial t} = -\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \begin{array}{l} \cos(1000t - 2\pi\sqrt{\varepsilon_1}\, z) \\ + R\cos(1000t + 2\pi\sqrt{\varepsilon_1}\, z) \end{array} & 0 & 0 \end{vmatrix}$$

$$\varepsilon_2, \mu_o$$

$$\vec{E}_2 = T\cos(1000t - 2\pi\sqrt{\varepsilon_2}\, z)\vec{a}_x$$

$$\frac{\partial \vec{B}_2}{\partial t} = -\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ T\cos(1000t - 2\pi\sqrt{\varepsilon_2}\, z) & 0 & 0 \end{vmatrix}$$

z=0

Example Problem

$$\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

ε_1,μ_o

$$\frac{\partial\vec{B}_1}{\partial t}=-\begin{vmatrix}\hat{a}_x & \hat{a}_y & \hat{a}_z\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ \begin{matrix}\cos(1000t-2\pi\sqrt{\varepsilon_1}\,z)\\ +R\cos(1000t+2\pi\sqrt{\varepsilon_1}\,z)\end{matrix} & 0 & 0\end{vmatrix}$$

$$\frac{\partial\vec{B}_1}{\partial t}=-\frac{\partial}{\partial z}\begin{bmatrix}\cos(1000t-2\pi\sqrt{\varepsilon_1}\,z)\\ +R\cos(1000t+2\pi\sqrt{\varepsilon_1}\,z)\end{bmatrix}\hat{a}_y$$

$$\frac{\partial\vec{B}_1}{\partial t}=-2\pi\sqrt{\varepsilon_1}\sin(1000t-2\pi\sqrt{\varepsilon_1}\,z)\hat{a}_y$$

$$+2\pi\sqrt{\varepsilon_1}\cdot R\sin(1000t+2\pi\sqrt{\varepsilon_1}\,z)\hat{a}_y$$

ε_2,μ_o

$$\frac{\partial\vec{B}_2}{\partial t}=-\begin{vmatrix}\hat{a}_x & \hat{a}_y & \hat{a}_z\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ T\cos(1000t-2\pi\sqrt{\varepsilon_2}\,z) & 0 & 0\end{vmatrix}$$

$$\frac{\partial\vec{B}_2}{\partial t}=-\frac{\partial}{\partial z}\left(T\cos(1000t-2\pi\sqrt{\varepsilon_2}\,z)\right)\hat{a}_y$$

$$\frac{\partial\vec{B}_2}{\partial t}=-T\cdot 2\pi\sqrt{\varepsilon_2}\sin(1000t-2\pi\,z)\hat{a}_y$$

z=0

Example Problem

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

ε_1, μ_o

$$\frac{\partial \vec{B}_1}{\partial t} = -2\pi\sqrt{\varepsilon_1}\,\sin(1000t - 2\pi\sqrt{\varepsilon_1}\,z)\hat{a}_y$$

$$+\,2\pi\sqrt{\varepsilon_1}\cdot R\sin(1000t + 2\pi\sqrt{\varepsilon_1}\,z)\hat{a}_y$$

$$\vec{B}_1 = \int \begin{bmatrix} -\,2\pi\sqrt{\varepsilon_1}\,\sin(1000t - 2\pi\sqrt{\varepsilon_1}z)\hat{a}_y \\ +\,2\pi\sqrt{\varepsilon_1}\cdot R\sin(1000t + 2\pi\sqrt{\varepsilon_1}\,z)\hat{a}_y \end{bmatrix} dt$$

$$\vec{B}_1 = +2\pi\cdot\frac{\sqrt{\varepsilon_1}}{1000}\cos(1000t - 2\pi\sqrt{\varepsilon_1}z)\hat{a}_y$$

$$-\,2\pi\cdot\frac{\sqrt{\varepsilon_1}}{1000}R\cos(1000t + 2\pi\sqrt{\varepsilon_1}\,z)\hat{a}_y$$

ε_2, μ_o

$$\frac{\partial \vec{B}_2}{\partial t} = -T\cdot 2\pi\sin(1000t - 2\pi\sqrt{\varepsilon_2}\,z)\hat{a}_y$$

$$\vec{B}_2 = \int -T\cdot 2\pi\sin(1000t - 2\pi\sqrt{\varepsilon_2}z)\hat{a}_y dt$$

$$\vec{B}_2 = T\cdot 2\pi\frac{\sqrt{\varepsilon_2}}{1000}\cos(1000t - 2\pi\sqrt{\varepsilon_2}\,z)\hat{a}_y$$

z=0

Example Problem

$$\varepsilon_1, \mu_o$$

$$\varepsilon_2, \mu_o$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{E}_1 = \cos(1000t - 2\pi\sqrt{\varepsilon_1}\, z)\vec{a}_x$$

$$+ R\cos(1000t + 2\pi\sqrt{\varepsilon_1}\, z)\vec{a}_x$$

$$\vec{B}_1 = +2\pi \cdot \frac{\sqrt{\varepsilon_1}}{1000}\cos(1000t - 2\pi\sqrt{\varepsilon_1}\, z)\hat{a}_y$$

$$+ 2\pi \cdot \frac{\sqrt{\varepsilon_1}}{1000} R\cos(1000t + 2\pi\sqrt{\varepsilon_1}\, z)\hat{a}_y$$

$$\vec{E}_2 = T\cos(1000t - 2\pi\sqrt{\varepsilon_2}\, z)\vec{a}_x$$

$$\vec{B}_2 = T \cdot 2\pi\frac{\sqrt{\varepsilon_2}}{1000}\cos(1000t - 2\pi\sqrt{\varepsilon_2}\, z)\hat{a}_y$$

z=0

Example Problem

$$\boxed{\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

ε_1, μ_o

$$\vec{E}_1 = \cos(1000t - 2\pi\sqrt{\varepsilon_1}z)\vec{a}_x$$

$$+ R\cos(1000t + 2\pi\sqrt{\varepsilon_1}\,z)\vec{a}_x$$

$$\vec{H}_1 = +2\pi \cdot \frac{\sqrt{\varepsilon_1}}{1000}\cos(1000t - 2\pi\sqrt{\varepsilon_1}z)\hat{a}_y$$

$$-2\pi \cdot \frac{\sqrt{\varepsilon_1}}{1000}R\cos(1000t + 2\pi\sqrt{\varepsilon_1}\,z)\hat{a}_y$$

ε_2, μ_o

$$\vec{E}_2 = T\cos(1000t - 2\pi\sqrt{\varepsilon_2}\,z)\vec{a}_x$$

$$\vec{H}_2 = T \cdot 2\pi\frac{\sqrt{\varepsilon_2}}{1000}\cos(1000t - 2\pi\sqrt{\varepsilon_2}\,z)\hat{a}_y$$

z=0

Example Problem

ε_1, μ_o

ε_2, μ_o

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{E}_1 = \cos(1000t - 2\pi\sqrt{\varepsilon_1}z)\vec{a}_x$$

$$+ R\cos(1000t + 2\pi\sqrt{\varepsilon_1}\,z)\vec{a}_x$$

$$\vec{H}_1 = +2\pi \cdot \frac{\sqrt{\varepsilon_1}}{1000}\cos(1000t - 2\pi\sqrt{\varepsilon_1}z)\hat{a}_y$$

$$-2\pi \cdot \frac{\sqrt{\varepsilon_1}}{1000}R\cos(1000t + 2\pi\sqrt{\varepsilon_1}\,z)\hat{a}_y$$

$$\vec{E}_2 = T\cos(1000t - 2\pi\sqrt{\varepsilon_2}\,z)\vec{a}_x$$

$$\vec{H}_2 = T \cdot 2\pi\frac{\sqrt{\varepsilon_2}}{1000}\cos(1000t - 2\pi\sqrt{\varepsilon_2}\,z)\hat{a}_y$$

APPLY BC at Z=0 to H

$$\frac{2\pi}{\mu_o} \cdot \frac{\sqrt{\varepsilon_1}}{1000}\cos(1000t)\hat{a}_y - \frac{2\pi}{\mu_o} \cdot \frac{\sqrt{\varepsilon_1}}{1000}R\cos(1000t)\hat{a}_y = T\frac{2\pi}{\mu_o} \cdot \frac{\sqrt{\varepsilon_2}}{1000}\cos(1000t)\hat{a}_y$$

z=0

Example Problem

ε_1, μ_o

ε_2, μ_o

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{E}_1 = \cos(1000t - 2\pi\sqrt{\varepsilon_1}z)\vec{a}_x$$

$$+ R\cos(1000t + 2\pi\sqrt{\varepsilon_1}\,z)\vec{a}_x$$

$$\vec{E}_2 = T\cos(1000t - 2\pi\sqrt{\varepsilon_2}\,z)\vec{a}_x$$

$$\vec{H}_1 = +2\pi \cdot \frac{\sqrt{\varepsilon_1}}{1000}\cos(1000t - 2\pi\sqrt{\varepsilon_1}z)\hat{a}_y$$

$$-2\pi \cdot \frac{\sqrt{\varepsilon_1}}{1000}R\cos(1000t + 2\pi\sqrt{\varepsilon_1}\,z)\hat{a}_y$$

$$\vec{H}_2 = T \cdot 2\pi\frac{\sqrt{\varepsilon_2}}{1000}\cos(1000t - 2\pi\sqrt{\varepsilon_2}\,z)\hat{a}_y$$

APPLY BC at Z=0 to H

$$\sqrt{\varepsilon_1} - \sqrt{\varepsilon_1}R = \sqrt{\varepsilon_2}T$$

z=0

Example Problem

$$\varepsilon_1, \mu_o$$

$$\varepsilon_2, \mu_o$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{E}_1 = \cos(1000t - 2\pi\sqrt{\varepsilon_1}z)\vec{a}_x$$

$$+ R\cos(1000t + 2\pi\sqrt{\varepsilon_1}\,z)\vec{a}_x$$

$$\vec{E}_2 = T\cos(1000t - 2\pi\sqrt{\varepsilon_2}\,z)\vec{a}_x$$

$$\vec{H}_1 = +2\pi \cdot \frac{\sqrt{\varepsilon_1}}{1000}\cos(1000t - 2\pi\sqrt{\varepsilon_1}z)\hat{a}_y$$

$$- 2\pi \cdot \frac{\sqrt{\varepsilon_1}}{1000} R\cos(1000t + 2\pi\sqrt{\varepsilon_1}\,z)\hat{a}_y$$

$$\vec{H}_2 = T \cdot 2\pi \frac{\sqrt{\varepsilon_2}}{1000}\cos(1000t - 2\pi\sqrt{\varepsilon_2}\,z)\hat{a}_y$$

APPLY BC at Z=0 to E

$$\cos(1000t)\hat{a}_y - R\cos(1000t)\hat{a}_y = T\cos(1000t)\hat{a}_y$$

z=0

Example Problem

ε_1, μ_o

ε_2, μ_o

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{E}_1 = \cos(1000t - 2\pi\sqrt{\varepsilon_1}\,z)\vec{a}_x$$

$$+ R\cos(1000t + 2\pi\sqrt{\varepsilon_1}\,z)\vec{a}_x$$

$$\vec{H}_1 = +2\pi \cdot \frac{\sqrt{\varepsilon_1}}{1000}\cos(1000t - 2\pi\sqrt{\varepsilon_1}\,z)\hat{a}_y$$

$$-2\pi \cdot \frac{\sqrt{\varepsilon_1}}{1000} R\cos(1000t + 2\pi\sqrt{\varepsilon_1}\,z)\hat{a}_y$$

$$\vec{E}_2 = T\cos(1000t - 2\pi\sqrt{\varepsilon_2}\,z)\vec{a}_x$$

$$\vec{H}_2 = T \cdot 2\pi\frac{\sqrt{\varepsilon_2}}{1000}\cos(1000t - 2\pi\sqrt{\varepsilon_2}\,z)\hat{a}_y$$

APPLY BC at Z=0 to E

$$1 + R = T$$

z=0

Example Problem

$$1 - R = \frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}} T \qquad\qquad 1 + R = T$$

SOLVE

$$T = \frac{2}{\left(1 + \frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}\right)} \qquad\qquad R = \frac{\left(1 - \frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}\right)}{\left(1 + \frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}\right)}$$

TIME HARMONIC EM FIELDS

$$\vec{E}(x,y,z,t) = \vec{E}_o(x,y,z)\cos(\omega t + \phi(x,y,z))$$

Time varying Maxwell's equations work for fields with any time dependency. However, in engineering we often restrict the time dependency to sinusoidally varying signals. Why?

TIME HARMONIC EM FIELDS

$$\vec{E}(x,y,z,t) = \vec{E}_o(x,y,z)\cos(\omega t + \phi(x,y,z))$$

Time varying Maxwell's equations work for fields with any time dependency. However, in engineering we often restrict the time dependency to sinusoidally varying signals. Why?

Assume all sources have a sinusoidal time dependence and all materials properties are linear. Since Maxwell's equations are also linear WE ARE GUARENTEED that all electric and magnetic fields must also have the same sinusoidal time dependence. Only true for sinusoids!

TIME HARMONIC EM FIELDS

For sinusoidal (or time harmonic) sources we can mathematically write them the following way:

Euler's Formula

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

$$\vec{E}(x,y,z,t) = \vec{E}_o(x,y,z)\cos(\omega t + \phi(x,y,z))$$

$$\vec{E}(x,y,z,t) = \mathrm{Re}[\underbrace{\vec{E}_o(x,y,z)\cdot e^{j\phi(x,y,z)}}\, e^{j\omega t}]$$

Let $\tilde{E}(x,y,z) = \vec{E}_o(x,y,z)\cdot e^{j\phi(x,y,z)}$

$$\vec{E}(x,y,z,t) = \mathrm{Re}[\tilde{E}(x,y,z)e^{j\omega t}]$$

TIME HARMONIC EM FIELDS

For sinusoidal (or time harmonic) sources we can mathematically write them the following way:

$$\tilde{E}(x,y,z) = \vec{E}_o(x,y,z) \cdot e^{j\phi(x,y,z)}$$

$$\vec{E}(x,y,z,t) = \mathrm{Re}[\tilde{E}(x,y,z)e^{j\omega t}]$$

$\tilde{E}(x,y,z)$ is a complex function of space (phasor) called the time-harmonic electric field. All field values and sources can be represented by their time-harmonic form.

TIME HARMONIC EM FIELDS

This can be done for all of the field and source variables

$$\vec{E}(x,y,z,t) = \mathrm{Re}[\tilde{E}(x,y,z)e^{j\omega t}]$$

$$\vec{D}(x,y,z,t) = \mathrm{Re}[\tilde{D}(x,y,z)e^{j\omega t}]$$

$$\vec{H}(x,y,z,t) = \mathrm{Re}[\tilde{H}(x,y,z)e^{j\omega t}]$$

$$\vec{B}(x,y,z,t) = \mathrm{Re}[\tilde{B}(x,y,z)e^{j\omega t}]$$

$$\vec{J}(x,y,z,t) = \mathrm{Re}[\tilde{J}(x,y,z)e^{j\omega t}]$$

$$\vec{\rho}(x,y,z,t) = \mathrm{Re}[\tilde{\rho}(x,y,z)e^{j\omega t}]$$

TIME HARMONIC EM FIELDS

Lets see if you understand what a phasor represents

At some point in space you are given that the phasor for the electric field is given by:

$$\tilde{E}(0,1,-1) = (1-j)\hat{a}_x + 4\hat{a}_y + 7j\hat{a}_z$$

You are also told that you are fields are varying at 1.0 GHz

What does the electric field look like in time?

At some point in space you are given that the phasor for the electric field is given by:

$$\tilde{E}(0,1,-1) = (1-j)\hat{a}_x + 4\hat{a}_y + 7j\hat{a}_z$$

You are also told that you are fields are varying at 1.0 GHz

What does the electric field look like in time?

At some point in space you are given that the phasor for the electric field is given by:

$$\tilde{E}(0,1,-1) = (1 - j)\hat{a}_x + 4\hat{a}_y + 7j\hat{a}_z$$

You are also told that you are fields are varying at 1.0 GHz

What does the electric field look like in time?

$$\vec{E}(0,1,-1,t) = \sqrt{1^2+1^2}\cos(2\pi\cdot 10^9 t + \tan^{-1}(\frac{-1}{1}))\hat{a}_x$$

$$+\sqrt{4^2}\cos(2\pi\cdot 10^9 t + \tan^{-1}(\frac{0}{4}))\hat{a}_y$$

$$+\sqrt{7^2}\cos(2\pi\cdot 10^9 t + \tan^{-1}(\frac{7}{0}))\hat{a}_z$$

PROPERTIES OF TIME HARMONIC FIELDS

Time derivative:

$$\frac{\partial}{\partial t}[\mathrm{Re}[\tilde{E}(x,y,z)e^{j\omega t}]] = j\omega[\mathrm{Re}[\tilde{E}(x,y,z)e^{j\omega t}]$$

Time integration:

$$\int[\mathrm{Re}[\tilde{E}(x,y,z)e^{j\omega t}]dt = \frac{1}{j\omega}[\mathrm{Re}[\tilde{E}(x,y,z)e^{j\omega t}]$$

TIME HARMONIC MAXWELL’S EQUATIONS

TIME DOMAIN

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}$$

$$\nabla \cdot \vec{D} = \rho$$

$$\nabla \cdot \vec{B} = \rho_m$$

FREQUENCY DOMAIN

??????

TIME HARMONIC MAXWELL'S EQUATIONS

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \Longrightarrow \quad \nabla \times \mathrm{Re}\left(\tilde{E}\, e^{j\omega t}\right) = -\frac{\partial}{\partial t} \mathrm{Re}\left(\tilde{B}\, e^{j\omega t}\right)$$

$$\vec{E}(x,y,z,t) = \mathrm{Re}[\tilde{E}(x,y,z)e^{j\omega t}]$$

$$\vec{D}(x,y,z,t) = \mathrm{Re}[\tilde{D}(x,y,z)e^{j\omega t}]$$

$$\vec{H}(x,y,z,t) = \mathrm{Re}[\tilde{H}(x,y,z)e^{j\omega t}]$$

$$\vec{B}(x,y,z,t) = \mathrm{Re}[\tilde{B}(x,y,z)e^{j\omega t}]$$

$$\vec{J}(x,y,z,t) = \mathrm{Re}[\tilde{J}(x,y,z)e^{j\omega t}]$$

$$\vec{\rho}(x,y,z,t) = \mathrm{Re}[\tilde{\rho}(x,y,z)e^{j\omega t}]$$

TIME HARMONIC MAXWELL’S EQUATIONS

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \mathrm{Re}\left(\tilde{E}\, e^{j\omega t}\right) = -\frac{\partial}{\partial t}\mathrm{Re}\left(\tilde{B}\, e^{j\omega t}\right)$$

$$\nabla \times \mathrm{Re}\left(\tilde{E}\, e^{j\omega t}\right) = -\frac{\partial}{\partial t}\mathrm{Re}\left(\tilde{B}\, e^{j\omega t}\right)$$

$$\Downarrow$$

$$\mathrm{Re}\left(\nabla \times \left(\tilde{E}\, e^{j\omega t}\right)\right) = -\mathrm{Re}\left(\frac{\partial}{\partial t}\left(\tilde{B}\, e^{j\omega t}\right)\right)$$

TIME HARMONIC MAXWELL’S EQUATIONS

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \nabla \times \mathrm{Re}\left(\tilde{E}\, e^{j\omega t}\right) = -\frac{\partial}{\partial t}\mathrm{Re}\left(\tilde{B}\, e^{j\omega t}\right)$$

$$\nabla \times \mathrm{Re}\left(\tilde{E}\, e^{j\omega t}\right) = -\frac{\partial}{\partial t}\mathrm{Re}\left(\tilde{B}\, e^{j\omega t}\right)$$

⬇

$$\mathrm{Re}\left(\nabla \times \left(\tilde{E}\, e^{j\omega t}\right)\right) = -\mathrm{Re}\left(\frac{\partial}{\partial t}\left(\tilde{B}\, e^{j\omega t}\right)\right)$$

⬇

$$\mathrm{Re}\left(\left(\nabla \times \tilde{E}\right) e^{j\omega t}\right) = -\mathrm{Re}\left(\tilde{B} \cdot \frac{\partial}{\partial t}\left(e^{j\omega t}\right)\right)$$

TIME HARMONIC MAXWELL’S EQUATIONS

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \nabla \times \mathrm{Re}\left(\tilde{E}\, e^{j\omega t}\right) = -\frac{\partial}{\partial t}\mathrm{Re}\left(\tilde{B}\, e^{j\omega t}\right)$$

$$\mathrm{Re}\left(\left(\nabla \times \tilde{E}\right) e^{j\omega t}\right) = -\mathrm{Re}\left(\tilde{B}\cdot\frac{\partial}{\partial t}\left(e^{j\omega t}\right)\right)$$

$$\Downarrow$$

$$\mathrm{Re}\left(\left(\nabla \times \tilde{E}\right) e^{j\omega t}\right) = -\mathrm{Re}\left(\tilde{B}\cdot j\omega e^{j\omega t}\right)$$

TIME HARMONIC MAXWELL’S EQUATIONS

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \nabla \times \mathrm{Re}\left(\tilde{E}\, e^{j\omega t}\right) = -\frac{\partial}{\partial t}\mathrm{Re}\left(\tilde{B}\, e^{j\omega t}\right)$$

$$\mathrm{Re}\left(\left(\nabla \times \tilde{E}\right) e^{j\omega t}\right) = -\mathrm{Re}\left(\tilde{B} \cdot \frac{\partial}{\partial t}\left(e^{j\omega t}\right)\right)$$

⬇

$$\cancel{\mathrm{Re}}\left(\left(\nabla \times \tilde{E}\right) e^{j\omega t}\right) = -\cancel{\mathrm{Re}}\left(\tilde{B} \cdot j\omega e^{j\omega t}\right)$$

⬇

$$\left(\nabla \times \tilde{E}\right) e^{j\omega t} = -\tilde{B} \cdot j\omega e^{j\omega t}$$

TIME HARMONIC MAXWELL’S EQUATIONS

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \mathrm{Re}\left(\tilde{E}\, e^{j\omega t}\right) = -\frac{\partial}{\partial t}\mathrm{Re}\left(\tilde{B}\, e^{j\omega t}\right)$$

$$\left(\nabla \times \tilde{E}\right)\cancel{e^{j\omega t}} = -\tilde{B}\cdot j\omega \cancel{e^{j\omega t}}$$

$$\Downarrow$$

$$\boxed{\nabla \times \tilde{E} = -j\omega\,\tilde{B}}$$

TIME HARMONIC MAXWELL'S EQUATIONS

$$\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

$$\nabla\times\vec{H}=\frac{\partial\vec{D}}{\partial t}+\vec{J}$$

$$\nabla\cdot\vec{D}=\rho$$

$$\nabla\cdot\vec{B}=\rho_m$$

$$\Longrightarrow$$

$$\nabla\times\mathrm{Re}\left(\tilde{E}\,e^{j\omega t}\right)=-\frac{\partial}{\partial t}\mathrm{Re}\left(\tilde{B}\,e^{j\omega t}\right)$$

$$\nabla\times\mathrm{Re}\left(\tilde{H}\,e^{j\omega t}\right)=\frac{\partial}{\partial t}\mathrm{Re}\left(\tilde{D}\,e^{j\omega t}\right)+\mathrm{Re}\left(\tilde{J}\,e^{j\omega t}\right)$$

$$\nabla\cdot\mathrm{Re}\left(\tilde{D}\,e^{j\omega t}\right)=\mathrm{Re}\left(\tilde{\rho}\,e^{j\omega t}\right)$$

$$\nabla\cdot\mathrm{Re}\left(\tilde{B}\,e^{j\omega t}\right)=\mathrm{Re}\left(\tilde{\rho}_m e^{j\omega t}\right)$$

$$\vec{E}(x,y,z,t)=\mathrm{Re}[\tilde{E}(x,y,z)e^{j\omega t}]$$

$$\vec{D}(x,y,z,t)=\mathrm{Re}[\tilde{D}(x,y,z)e^{j\omega t}]$$

$$\vec{H}(x,y,z,t)=\mathrm{Re}[\tilde{H}(x,y,z)e^{j\omega t}]$$

$$\vec{B}(x,y,z,t)=\mathrm{Re}[\tilde{B}(x,y,z)e^{j\omega t}]$$

$$\vec{J}(x,y,z,t)=\mathrm{Re}[\tilde{J}(x,y,z)e^{j\omega t}]$$

$$\vec{\rho}(x,y,z,t)=\mathrm{Re}[\tilde{\rho}(x,y,z)e^{j\omega t}]$$

TIME HARMONIC MAXWELL'S EQUATIONS

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}$$

$$\nabla \cdot \vec{D} = \rho$$

$$\nabla \cdot \vec{B} = \rho_m$$

$$\Longrightarrow$$

$$\nabla \times \mathrm{Re}\left(\tilde{E}\, e^{j\omega t}\right) = -\frac{\partial}{\partial t}\mathrm{Re}\left(\tilde{B}\, e^{j\omega t}\right)$$

$$\nabla \times \mathrm{Re}\left(\tilde{H}\, e^{j\omega t}\right) = \frac{\partial}{\partial t}\mathrm{Re}\left(\tilde{D}\, e^{j\omega t}\right) + \mathrm{Re}\left(\tilde{J}\, e^{j\omega t}\right)$$

$$\nabla \cdot \mathrm{Re}\left(\tilde{D}\, e^{j\omega t}\right) = \mathrm{Re}\left(\tilde{\rho}\, e^{j\omega t}\right)$$

$$\nabla \cdot \mathrm{Re}\left(\tilde{B}\, e^{j\omega t}\right) = \mathrm{Re}\left(\tilde{\rho}_m e^{j\omega t}\right)$$

Employing the derivative property results in the following set of equations:

$$\boxed{\begin{aligned} &\nabla \times \tilde{E} = -j\omega \tilde{B} \\ &\nabla \times \tilde{H} = j\omega \tilde{D} + \tilde{J} \\ &\nabla \cdot \tilde{D} = \tilde{\rho} \\ &\nabla \cdot \tilde{B} = \tilde{\rho}_m \end{aligned}}$$

TIME HARMONIC MAXWELL'S EQUATIONS

TIME DOMAIN

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}$$

$$\nabla \cdot \vec{D} = \rho$$

$$\nabla \cdot \vec{B} = \rho_m$$

FREQUENCY DOMAIN

$$\nabla \times \tilde{E} = -j\omega \tilde{B}$$

$$\nabla \times \tilde{H} = j\omega \tilde{D} + \tilde{J}$$

$$\nabla \cdot \tilde{D} = \tilde{\rho}$$

$$\nabla \cdot \tilde{B} = \tilde{\rho}_m$$

TIME HARMONIC EM FIELDS

BOUNDARY CONDITIONS AND CONSTITUTIVE PROPERTIES

The constitutive properties and boundary conditions are very similar for the time harmonic form:

Constitutive Properties

$$\boxed{\begin{aligned}\tilde{D} &= \varepsilon\,\tilde{E}\\ \tilde{B} &= \mu\,\tilde{H}\\ \tilde{J}_c &= \sigma\,\tilde{E}\end{aligned}}$$

General Boundary Conditions

$$\boxed{\begin{aligned}&\hat{n}\times(\tilde{E}_2-\tilde{E}_1)=0\\ &\hat{n}\times(\tilde{H}_2-\tilde{H}_1)=\tilde{J}_s\\ &\hat{n}\cdot(\tilde{D}_2-\tilde{D}_1)=\tilde{\rho}_s\\ &\hat{n}\cdot(\tilde{B}_2-\tilde{B}_1)=0\end{aligned}}$$

PEC Boundary Conditions

$$\boxed{\begin{aligned}&\hat{n}\times\tilde{E}_2=0\\ &\hat{n}\times\tilde{H}_2=\tilde{J}_s\\ &\hat{n}\cdot\tilde{D}_2=\tilde{\rho}_s\\ &\hat{n}\cdot\tilde{B}_2=0\end{aligned}}$$

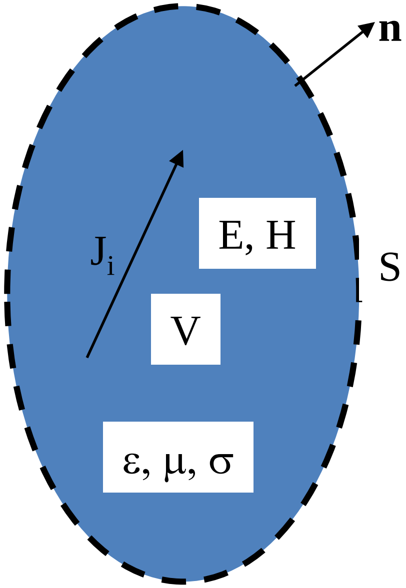
POWER and ENERGY

$$(eq1) \quad \nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$$

$$(eq2) \quad \nabla \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t} + \sigma \vec{E} + \vec{J}_i = \vec{J}_d + \vec{J}_c + \vec{J}_i$$

take

$$\vec{H} \cdot (eq1) - \vec{E} \cdot (eq2)$$



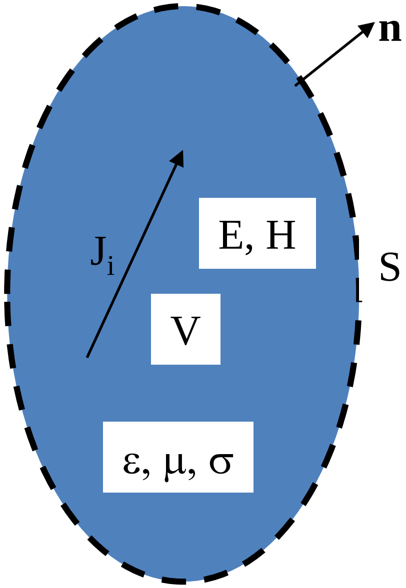
POWER and ENERGY

$$(eq1) \quad \nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$$

$$(eq2) \quad \nabla \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t} + \sigma \vec{E} + \vec{J}_i = \vec{J}_d + \vec{J}_c + \vec{J}_i$$

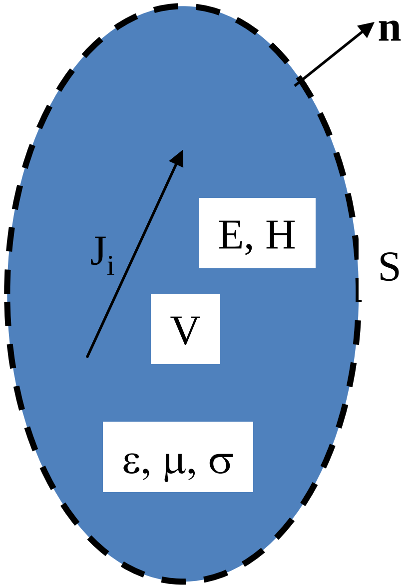
take

$$\vec{H} \cdot (eq1) - \vec{E} \cdot (eq2)$$



$$(eq3) \quad \vec{H} \cdot \nabla \times \vec{E} - \vec{E} \cdot \nabla \times \vec{H} = -\vec{H} \cdot \mu \frac{\partial \vec{H}}{\partial t} - \vec{E} \cdot (\vec{J}_d + \vec{J}_c + \vec{J}_i)$$

POWER and ENERGY



$$(eq1) \quad \nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$$

$$(eq2) \quad \nabla \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t} + \sigma \vec{E} + \vec{J}_i = \vec{J}_d + \vec{J}_c + \vec{J}_i$$

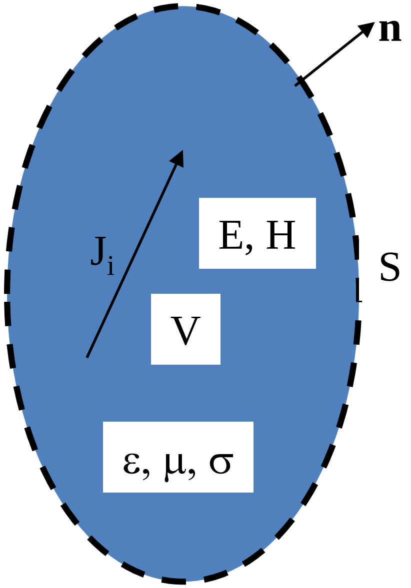
take $\vec{H} \cdot (eq1) - \vec{E} \cdot (eq2)$

$$(eq3) \quad \vec{H} \cdot \nabla \times \vec{E} - \vec{E} \cdot \nabla \times \vec{H} = -\vec{H} \cdot \mu \frac{\partial \vec{H}}{\partial t} - \vec{E} \cdot (\vec{J}_d + \vec{J}_c + \vec{J}_i)$$

Using the vector identity $\nabla \cdot (A \times B) = B \cdot (\nabla \times A) - A \cdot (\nabla \times B)$

$$(eq4) \quad \nabla \cdot \left(\vec{E} \times \vec{H} \right) = -\vec{H} \cdot \mu \frac{\partial \vec{H}}{\partial t} - \vec{E} \cdot \varepsilon \frac{\partial \vec{E}}{\partial t} - \vec{E} \cdot (\vec{J}_c + \vec{J}_i)$$

POWER and ENERGY



$$(eq1) \quad \nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$$

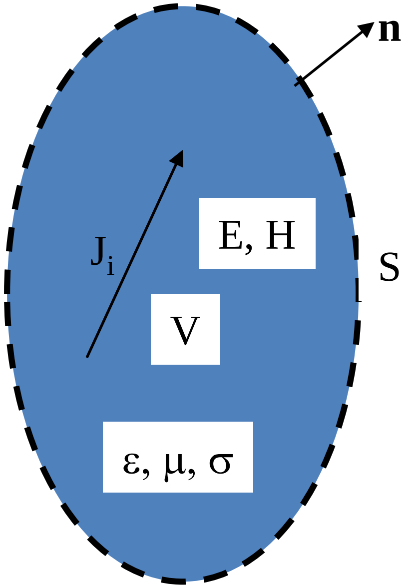
$$(eq2) \quad \nabla \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t} + \sigma \vec{E} + \vec{J}_i = \vec{J}_d + \vec{J}_c + \vec{J}_i$$

Using the vector identity $\nabla \cdot (A \times B) = B \cdot (\nabla \times A) - A \cdot (\nabla \times B)$

$$(eq4) \quad \nabla \cdot \left(\vec{E} \times \vec{H}\right) = -\vec{H} \cdot \mu \frac{\partial \vec{H}}{\partial t} - \vec{E} \cdot \varepsilon \frac{\partial \vec{E}}{\partial t} - \vec{E} \cdot (\vec{J}_c + \vec{J}_i)$$

$$(eq5) \quad \iiint \nabla \cdot \left(\vec{E} \times \vec{H}\right) dv = -\iiint \vec{H} \cdot \mu \frac{\partial \vec{H}}{\partial t} dv - \iiint \vec{E} \cdot \varepsilon \frac{\partial \vec{E}}{\partial t} dv - \iiint \vec{E} \cdot \sigma \vec{E} dv - \iiint \vec{E} \cdot \vec{J}_i dv$$

POWER and ENERGY



$$(eq1)\quad \nabla\times\vec{E} = -\mu\frac{\partial\vec{H}}{\partial t}$$

$$(eq2)\quad \nabla\times\vec{H} = \varepsilon\frac{\partial\vec{E}}{\partial t} + \sigma\vec{E} + \vec{J}_i = \vec{J}_d + \vec{J}_c + \vec{J}_i$$

$$(eq4)\quad \nabla\cdot\left(\vec{E}\times\vec{H}\right) = -\vec{H}\cdot\mu\frac{\partial\vec{H}}{\partial t} - \vec{E}\cdot\varepsilon\frac{\partial\vec{E}}{\partial t} - \vec{E}\cdot(\vec{J}_c + \vec{J}_i)$$

$$(eq5)\quad \iiint\nabla\cdot\left(\vec{E}\times\vec{H}\right)dv = -\iiint\vec{H}\cdot\mu\frac{\partial\vec{H}}{\partial t}dv - \iiint\vec{E}\cdot\varepsilon\frac{\partial\vec{E}}{\partial t}dv - \iiint\vec{E}\cdot\sigma\vec{E}dv - \iiint\vec{E}\cdot\vec{J}_i dv$$

Use divergence theorem

$$(eq6)\quad \oiint\vec{E}\times\vec{H}\cdot d\vec{s} = -\iiint\mu\vec{H}\cdot\frac{\partial\vec{H}}{\partial t}dv - \iiint\varepsilon\,\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}dv - \iiint\sigma\,\vec{E}\cdot\vec{E}dv - \iiint\vec{E}\cdot\vec{J}_i dv$$

POWER and ENERGY (continued)

$$\boxed{(eq6)\quad \oint\!\!\!\!\iint_s (\vec{E}\times\vec{H})\cdot ds + \iiint_v [\mu\,\vec{H}\cdot\frac{\partial\vec{H}}{\partial t} + \varepsilon\,\vec{E}\cdot\frac{\partial\vec{E}}{\partial t} + \sigma\,\vec{E}\cdot\vec{E} + \vec{E}\cdot\vec{J}_i)]\,dv = 0}$$

$$\mu\,\vec{H}\cdot\frac{\partial\vec{H}}{\partial t} = \frac{\partial}{\partial t}\left[\frac{1}{2}\mu\left|H\right|^2\right]$$

$$\varepsilon\,\vec{E}\cdot\frac{\partial\vec{E}}{\partial t} = \frac{\partial}{\partial t}\left[\frac{1}{2}\varepsilon\left|E\right|^2\right]$$

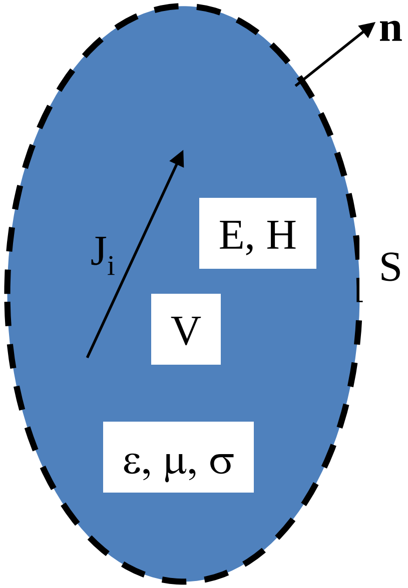
$$\sigma\,\vec{E}\cdot\vec{E} = \sigma\left|E\right|^2$$

$$\boxed{(eq7)\quad \oint\!\!\!\!\iint_s (\vec{E}\times\vec{H})\cdot ds + \iiint_v [\frac{\partial}{\partial t}\left[\frac{1}{2}\mu\left|\vec{H}\right|^2\right] + \frac{\partial}{\partial t}\left[\frac{1}{2}\varepsilon\left|\vec{E}\right|^2\right] + \sigma\left|\vec{E}\right|^2 + \vec{E}\cdot\vec{J}_i)]\,dv = 0}$$

POWER and ENERGY (continued)

$$(eq7)\quad \oiint_s (\vec{E}\times\vec{H})\cdot d\vec{s} + \iiint_v [\frac{\partial}{\partial t}\left[\frac{1}{2}\mu\left|\vec{H}\right|^2\right] + \frac{\partial}{\partial t}\left[\frac{1}{2}\varepsilon\left|\vec{E}\right|^2\right] + \sigma\left|\vec{E}\right|^2 + \vec{E}\cdot\vec{J}_i)]\,dv = 0$$

What do the terms represent physically?



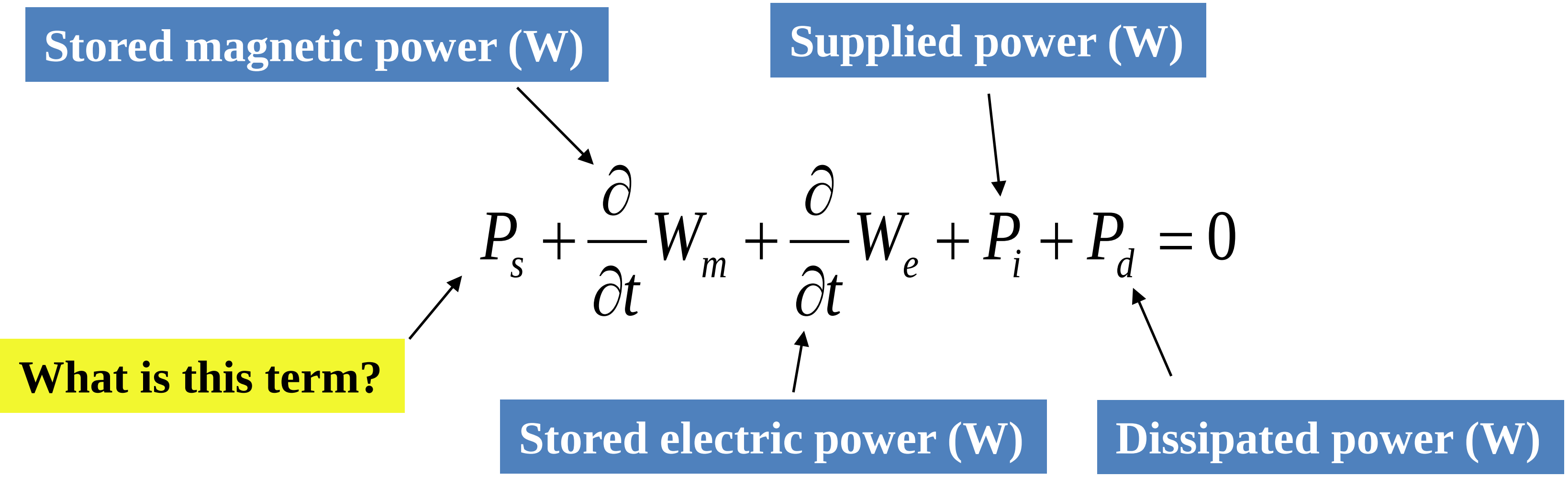
POWER and ENERGY (continued)

$$(eq7)\quad \oiint_s (\vec{E}\times\vec{H})\cdot d\vec{s} + \iiint_v [\frac{\partial}{\partial t}\left[\frac{1}{2}\mu\left|\vec{H}\right|^2\right] + \frac{\partial}{\partial t}\left[\frac{1}{2}\varepsilon\left|\vec{E}\right|^2\right] + \sigma\left|\vec{E}\right|^2 + \vec{E}\cdot\vec{J}_i)]\,dv = 0$$

$$P_s = \oiint_s (\vec{E}\times\vec{H})\cdot ds$$

$$W_{m=}\left[\iiint_v [\frac{1}{2}\mu\left|H\right|^2]\,dv\right],\quad W_{e=}\iiint_v [\frac{1}{2}\varepsilon\left|E\right|^2]\,dv$$

$$P_i = \iiint_v [\vec{E}\cdot\vec{J}_i]\,dv = 0,\quad P_d = \iiint_v \sigma\left|E\right|^2 dv = 0$$



Stored magnetic power (W)

Supplied power (W)

$$P_s + \frac{\partial}{\partial t}W_m + \frac{\partial}{\partial t}W_e + P_i + P_d = 0$$

What is this term?

Stored electric power (W)

Dissipated power (W)

POWER and ENERGY (continued)

$$P_s = \oiint_s (\vec{E} \times \vec{H}) \cdot ds$$

$$W_{m=}\left[\iiint_v [\frac{1}{2}\mu |H|^2]\, dv\right], \quad W_{e=}\iiint_v [\frac{1}{2}\varepsilon |E|^2]\, dv$$

$$P_i = \iiint_v [\vec{E} \cdot \vec{J}_i]\, dv = 0, \quad P_d = \iiint_v \sigma |E|^2\, dv = 0$$

Stored magnetic power (W)

Supplied power (W)

$$P_s + \frac{\partial}{\partial t} W_m + \frac{\partial}{\partial t} W_e + P_i + P_d = 0$$

What is this term?

Stored electric power (W)

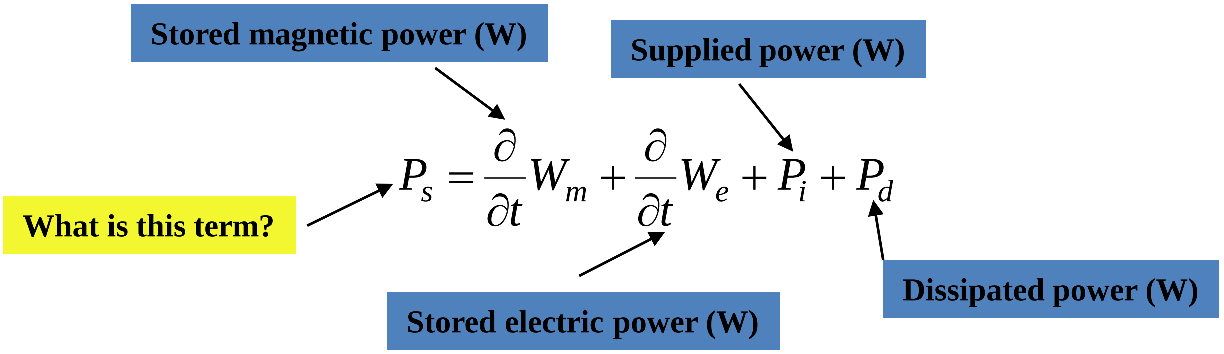
Dissipated power (W)

P_s = power exiting the volume through radiation

$$\boxed{\vec{S} = \vec{E} \times \vec{H}}$$ W/m^2 Poynting vector

POWER and ENERGY (continued)

$$\boxed{\begin{aligned}&P_s = \oiint_s (\vec{E}\times\vec{H})\cdot ds\\ &W_{m=}\left[\iiint_v [\frac{1}{2}\mu|H|^2]\,dv\right], \quad W_{e=}\iiint_v [\frac{1}{2}\varepsilon|E|^2]\,dv\\ &P_i = \iiint_v [\vec{E}\cdot\vec{J}_i]\,dv = 0, \quad P_d = \iiint_v \sigma|E|^2\,dv = 0\end{aligned}}$$



Stored magnetic power (W)

Supplied power (W)

What is this term?

$$P_s = \frac{\partial}{\partial t}W_m + \frac{\partial}{\partial t}W_e + P_i + P_d$$

Stored electric power (W)

Dissipated power (W)

P_s = power exiting the volume through radiation

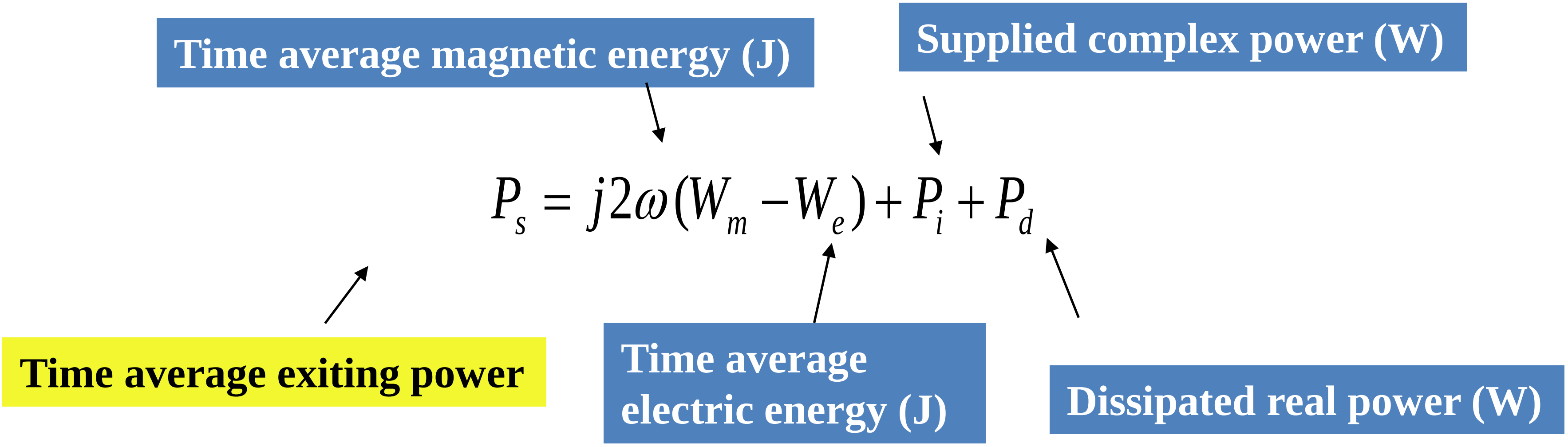
$\boxed{\vec{S} = \vec{E}\times\vec{H}}$ W/m^2 Poynting vector

POWER and ENERGY: TIME HARMONIC

$$P_s = \oiint_s (\tilde{E}\times\tilde{H}^*)\cdot ds$$

$$W_{m=}\left[\iiint_v [\frac{1}{4}\mu|\tilde{H}|^2]\,dv\right],\quad W_{e=}\iiint_v [\frac{1}{4}\varepsilon|\tilde{E}|^2]\,dv$$

$$P_i = \iiint_v [\frac{1}{2}\tilde{E}\cdot\tilde{J}_i^*]\,dv = 0,\quad P_d = \iiint_v \frac{1}{2}\sigma|\tilde{E}|^2\,dv = 0$$



Time average magnetic energy (J)

Supplied complex power (W)

$$P_s = j2\omega(W_m - W_e) + P_i + P_d$$

Time average exiting power

Time average electric energy (J)

Dissipated real power (W)

CONTINUITY OF CURRENT LAW

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}$$

$$\nabla \cdot \vec{D} = \rho$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \cdot (\nabla \times \vec{H}) = \nabla \cdot [\frac{\partial \vec{D}}{\partial t} + \vec{J}] = \frac{\partial}{\partial t}[\nabla \cdot \vec{D}] + \nabla \cdot \vec{J}$$

vector identity $\nabla \cdot (\nabla \times \vec{A}) = 0$

$$0 = \frac{\partial}{\partial t}[\nabla \cdot \vec{D}] + \nabla \cdot \vec{J}$$

$$0 = \frac{\partial}{\partial t}[\rho] + \nabla \cdot \vec{J}$$

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$$

time harmonic

$$\nabla \cdot \vec{J} = -j\omega\rho$$

SUMMARY

Time Domain	Frequency Domain
$\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}-\vec{M}$ $\quad\nabla\times\vec{H}=\frac{\partial\vec{D}}{\partial t}+\vec{J}$ $\nabla\cdot\vec{D}=\rho$ $\quad\nabla\cdot\vec{B}=\rho_m$	$\nabla\times\tilde{E}=-j\omega\tilde{B}-\tilde{M}$ $\quad\nabla\times\tilde{H}=j\omega\tilde{D}+\tilde{J}$ $\nabla\cdot\tilde{D}=\tilde{\rho}$ $\quad\nabla\cdot\tilde{B}=\tilde{\rho}_m$
$\hat{n}\times(\vec{E}_2-\vec{E}_1)=0$ $\quad\hat{n}\times(\vec{H}_2-\vec{H}_1)=\vec{J}_s$ $\hat{n}\cdot(\vec{D}_2-\vec{D}_1)=\rho_s$ $\quad\hat{n}\cdot(\vec{B}_2-\vec{B}_1)=0$	$\hat{n}\times(\tilde{E}_2-\tilde{E}_1)=0$ $\quad\hat{n}\times(\tilde{H}_2-\tilde{H}_1)=\tilde{J}_s$ $\hat{n}\cdot(\tilde{D}_2-\tilde{D}_1)=\tilde{\rho}_s$ $\quad\hat{n}\cdot(\tilde{B}_2-\tilde{B}_1)=0$ $Z_s=R_s+jX_s=(1+j)\sqrt{\frac{\omega\mu}{2\sigma}}$
$\vec{D}=\varepsilon\vec{E}$ $\vec{B}=\mu\vec{H}$ $\vec{J}_c=\sigma\vec{E}$	$\tilde{D}=\varepsilon\tilde{E}$ $\tilde{B}=\mu\tilde{H}$ $\tilde{J}_c=\sigma\tilde{E}$
$P_s=\oiint_s(\vec{E}\times\vec{H})\cdot ds$ $W_{m=}\left[\iiint_v[\frac{1}{2}\mu\lvert H\rvert^2]\,dv\right],\quad W_{e=}\iiint_v[\frac{1}{2}\varepsilon\lvert E\rvert^2]\,dv$ $P_i=\iiint_v[\vec{E}\cdot\vec{J}_i]\,dv=0,\quad P_d=\iiint_v\sigma\lvert E\rvert^2\,dv=0$	$P_s=\oiint_s(\tilde{E}\times\tilde{H}^*)\cdot ds$ $W_{m=}\left[\iiint_v[\frac{1}{4}\mu\lvert\tilde{H}\rvert^2]\,dv\right],\quad W_{e=}\iiint_v[\frac{1}{4}\varepsilon\lvert\tilde{E}\rvert^2]\,dv$ $P_i=\iiint_v[\frac{1}{2}\tilde{E}\cdot\tilde{J}_i^*]\,dv=0,\quad P_d=\iiint_v\frac{1}{2}\sigma\lvert\tilde{E}\rvert^2\,dv=0$