ELEG 648
Vector Potentials

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Maxwell's Equation's are:

1. $\nabla \times \mathbf{E} = -j\omega\mathbf{B} = -j\omega\mu\mathbf{H}$
2. $\nabla \times \mathbf{H} = \mathbf{J} + j\omega\mathbf{D} = \mathbf{J} + j\omega\epsilon\mathbf{E}$
3. $\nabla \cdot \mathbf{D} = q$
4. $\nabla \cdot \mathbf{B} = 0$

How can we solve them? They are

- Partial differential equations.
- Coupled.
- Vector valued.
- Second order.

- Solving for the fields directly from Maxwell's Equations is extremely difficult.
- Therefore, we introduce a vector potential **A** which makes the solution simpler.
- In the quantum electrodynamic theory, this **A** is the primary variable of interest.

Auxiliary Potential Functions

Maxwell's equations:

$$\nabla \times \vec{E} = -j\omega\mu\vec{H} - \vec{M} \qquad \nabla \cdot \vec{D} = \rho_e$$

$$\nabla \times \vec{H} = j\omega\varepsilon\vec{E} + \vec{J} \qquad \nabla \cdot \vec{B} = \rho_m$$

For static fields with $\vec{M} = 0$:

$$\nabla \times \vec{E} = 0 \qquad \nabla \cdot \vec{D} = \rho_e$$

$$\nabla \times \vec{H} = \vec{J} \qquad \nabla \cdot \vec{B} = 0$$

For electrostatic fields:

$$\nabla \times \vec{E} = 0 \qquad \nabla \cdot \vec{D} = \rho_e$$

Vector identity: $\nabla \times \nabla f = 0$

Let $\vec{E} = -\nabla\phi$ $\quad$ ϕ: electric scalar potential

$$\nabla \cdot (\varepsilon\vec{E}) = -\nabla \cdot (\varepsilon\nabla\phi) = \rho_e$$

Auxiliary Potential Functions

If ε is a constant :

$$\nabla^2 \phi = -\frac{\rho_e}{\varepsilon}$$

Poisson's equation

In free space :

$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon} \iiint_V \frac{\rho(\vec{r}')}{R} dv' \qquad R = |\vec{r} - \vec{r}'|$$

For magnetostatic fields :

$$\nabla \times \vec{H} = \vec{J} \qquad \nabla \cdot \vec{B} = 0$$

Vector identity : $\nabla \cdot (\nabla \times \vec{a}) = 0$

Let $\vec{B} = \nabla \times \vec{A}$ $\quad \vec{A}$: magnetic vector potential

$$\vec{H} = \frac{\vec{B}}{\mu} = \frac{1}{\mu} \nabla \times \vec{A} \quad \Rightarrow \quad \nabla \times \left(\frac{1}{\mu} \nabla \times \vec{A} \right) = \vec{J}$$

Auxiliary Potential Functions

To uniquely determine $\vec{A}$: $\nabla \cdot \vec{A} = 0$ ⟸ Gauge condition

If μ is a constant:

$$\nabla \times (\nabla \times \vec{A}) = \mu \vec{J}$$

$$\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu \vec{J}$$

$$\nabla^2 \vec{A} = -\mu \vec{J}$$

Vector Poisson's equation

In free space: $\vec{A}(\vec{r}) = \dfrac{\mu}{4\pi} \iiint_V \dfrac{\vec{J}(\vec{r}\,')}{R} dv'$ $\quad R = |\vec{r} - \vec{r}\,'|$

Auxiliary Potential Functions

General case

$$\nabla \times \vec{E} = -j\omega\mu\vec{H} - \vec{M} \qquad \nabla \cdot \vec{D} = \rho_e$$

$$\nabla \times \vec{H} = j\omega\varepsilon\vec{E} + \vec{J} \qquad \nabla \cdot \vec{B} = \rho_m$$

Field due to $\vec{J}$:

$$\nabla \times \vec{E}_e = -j\omega\mu\vec{H}_e$$

$$\nabla \times \vec{H}_e = j\omega\varepsilon\vec{E}_e + \vec{J}$$

$$\nabla \cdot \vec{D}_e = \rho_e$$

$$\nabla \cdot \vec{B}_e = 0$$

Field due to $\vec{M}$:

$$\nabla \times \vec{E}_m = -j\omega\mu\vec{H}_m - \vec{M}$$

$$\nabla \times \vec{H}_m = j\omega\varepsilon\vec{E}_m$$

$$\nabla \cdot \vec{D}_m = 0$$

$$\nabla \cdot \vec{B}_m = \rho_m$$

Total field:

$$\vec{E} = \vec{E}_e + \vec{E}_m \qquad \vec{D} = \vec{D}_e + \vec{D}_m$$

$$\vec{H} = \vec{H}_e + \vec{H}_m \qquad \vec{B} = \vec{B}_e + \vec{B}_m$$

Auxiliary Potential Functions

Consider the case of electric source :

Since $\nabla \cdot \vec{B}_e = 0$ $\qquad$ $\vec{B}_e = \nabla \times \vec{A}$

$$\vec{H}_e = \frac{\vec{B}_e}{\mu} = \frac{1}{\mu}\nabla \times \vec{A} \qquad \Rightarrow \qquad \nabla \times \vec{E}_e = -j\omega\mu\vec{H}_e = -j\omega\nabla \times \vec{A}$$

$$\nabla \times (\vec{E}_e + j\omega\vec{A}) = 0$$

From vector identity : $\nabla \times (-\nabla \phi_e) = 0$

$$\vec{E}_e + j\omega\vec{A} = -\nabla \phi_e \qquad \Rightarrow \qquad \vec{E}_e = -\nabla \phi_e - j\omega\vec{A}$$

$$\nabla \times \vec{H}_e = \nabla \times \left(\frac{1}{\mu}\nabla \times \vec{A}\right) = j\omega\varepsilon\vec{E}_e + \vec{J} = j\omega\varepsilon(-\nabla \phi_e - j\omega\vec{A}) + \vec{J}$$

$$\nabla(\nabla \cdot \vec{A}) - \nabla^2\vec{A} = -j\omega\mu\varepsilon\nabla \phi_e + k^2\vec{A} + \mu\vec{J}$$

Auxiliary Potential Functions

$$\nabla^2 \vec{A} + k^2 \vec{A} = -\mu \vec{J} + \nabla(\nabla \cdot \vec{A} + j\omega\mu\varepsilon\phi_e)$$

Choose $\nabla \cdot \vec{A} + j\omega\mu\varepsilon\phi_e = 0$ ⟹ $\nabla^2 \vec{A} + k^2 \vec{A} = -\mu \vec{J}$

↑ Lorentz gauge condition

$$\nabla^2 \vec{A} + k^2 \vec{A} = -\mu \vec{J}$$

$$\vec{E}_e = -j\omega\vec{A} + \frac{1}{j\omega\mu\varepsilon}\nabla(\nabla \cdot \vec{A})$$

$$\vec{H}_e = \frac{1}{\mu}\nabla \times \vec{A}$$

Auxiliary Potential Functions

Consider the case of magnetic source :

$$\nabla \cdot \vec{D}_m = 0 \quad \Rightarrow \quad \vec{D}_m = -\nabla \times \vec{F} \quad \Rightarrow \quad \vec{E}_m = -\frac{1}{\varepsilon}\nabla \times \vec{F}$$

$$\nabla \times \vec{H}_m = j\omega\varepsilon\vec{E}_m = -j\omega\nabla \times \vec{F} \quad \Rightarrow \quad \nabla \times (\vec{H}_m + j\omega\vec{F}) = 0$$

$$\vec{H}_m + j\omega\vec{F} = -\nabla\phi_m \quad \Rightarrow \quad \vec{H}_m = -\nabla\phi_m - j\omega\vec{F}$$

$$\nabla \times \vec{E}_m = \nabla \times \left(-\frac{1}{\varepsilon}\nabla \times \vec{F} \right) = -j\omega\mu\vec{H}_m - \vec{M}$$

$$= -j\omega\mu(-j\omega\vec{F} - \nabla\phi_m) - \vec{M}$$

$$\nabla \times (\nabla \times \vec{F}) = -j\omega\mu\varepsilon\nabla\phi_m + k^2\vec{F} + \varepsilon\vec{M}$$

$$\nabla(\nabla \cdot \vec{F}) - \nabla^2\vec{F} = -j\omega\mu\varepsilon\nabla\phi_m + k^2\vec{F} + \varepsilon\vec{M}$$

Auxiliary Potential Functions

$$\nabla^2 \vec{F} + k^2 \vec{F} = -\varepsilon \vec{M} + \nabla(\nabla \cdot \vec{F} + j\omega\mu\varepsilon\phi_m)$$

Choose $\nabla \cdot \vec{F} = -j\omega\mu\varepsilon\phi_m$ ⇨ $\nabla^2 \vec{F} + k^2 \vec{F} = -\varepsilon \vec{M}$

⇧

Gauge condition

$$\nabla^2 \vec{F} + k^2 \vec{F} = -\varepsilon \vec{M}$$

$$\vec{E}_m = -\frac{1}{\varepsilon} \nabla \times \vec{F}$$

$$\vec{H}_m = -j\omega\vec{F} - \frac{j}{\omega\mu\varepsilon} \nabla(\nabla \cdot \vec{F})$$

Auxiliary Potential Functions

Summary:

Given: $\vec{J}, \vec{M}$

Solve: $\nabla^2 \vec{A} + k^2 \vec{A} = -\mu \vec{J}$

$$\nabla^2 \vec{F} + k^2 \vec{F} = -\varepsilon \vec{M}$$

Calculate:

$$\vec{E} = \vec{E}_e + \vec{E}_m = -j\omega \vec{A} - \frac{j}{\omega\mu\varepsilon} \nabla(\nabla \cdot \vec{A}) - \frac{1}{\varepsilon} \nabla \times \vec{F}$$

$$\vec{H} = \vec{H}_e + \vec{H}_m = \frac{1}{\mu} \nabla \times \vec{A} - j\omega \vec{F} - \frac{j}{\omega\mu\varepsilon} \nabla(\nabla \cdot \vec{F})$$

Auxiliary Potential Functions

Direct approach:

$$\nabla \times (\nabla \times \vec{E}_e) = -j\omega\mu \nabla \times \vec{H}_e = k^2 \vec{E}_e - j\omega\mu \vec{J}$$

$$\nabla(\nabla \cdot \vec{E}_e) - \nabla^2 \vec{E}_e = k_e^2 \vec{E}_e - j\omega\mu \vec{J}$$

$$\boxed{\nabla^2 \vec{E}_e + k^2 \vec{E}_e = j\omega\mu \vec{J} - \frac{1}{j\omega\varepsilon} \nabla(\nabla \cdot \vec{J})}$$

$$\nabla \times (\nabla \times \vec{H}_m) = -j\omega\varepsilon \nabla \times \vec{E}_m = k^2 \vec{H}_m + j\omega\varepsilon \vec{M}$$

$$\nabla(\nabla \cdot \vec{H}_m) - \nabla^2 \vec{H}_m = k^2 \vec{H}_m + j\omega\varepsilon \vec{M}$$

$$\boxed{\nabla^2 \vec{H}_m + k^2 \vec{H}_m = -j\omega\varepsilon \vec{M} - \frac{1}{j\omega\mu} \nabla(\nabla \cdot \vec{M})}$$

Compare two approaches.

Setting Up the Solution

We will concentrate on solving for the field due to a z-directed current, so our equation becomes

Scalar Wave Equation

$$\nabla^2 A_z + k^2 A_z = -\mu J_z$$

To solve this equation we note that it is

- Linear, and
- Spatially invariant.

How can we use this?

Since the equation is linear, if

$$\begin{aligned}\nabla^2 A_{z1} + k^2 A_{z1} &= -\mu J_{z1} \\ \nabla^2 A_{z2} + k^2 A_{z2} &= -\mu J_{z2}\end{aligned}$$

then

$$\nabla^2 (aA_{z1} + bA_{z2}) + k^2 (aA_{z1} + bA_{z2}) = -\mu (aJ_{z1} + bJ_{z2})$$

- This linearity also applies to infinite sums and integrals.
- Suppose the ∇ symbol differentiates with respect to $\mathbf{r}$, and that ζ is a parameter.
- Suppose that for $\zeta_\mathrm{L} \leq \zeta \leq \zeta_\mathrm{U}$

$$\nabla^2 g(\mathbf{r}, \zeta) + k^2 g(\mathbf{r}, \zeta) = s(\mathbf{r}, \zeta).$$

If this is the case then suppose

$$\nabla^2 A_z + k^2 A_z = \mu \int_{\zeta_L}^{\zeta_U} J_z(\zeta) s(\mathbf{r}, \zeta)\, \mathrm{d}\zeta$$

we could then conclude that

$$A_z(\mathbf{r}) = \mu \int_{\zeta_L}^{\zeta_U} J_z(\zeta) g(\mathbf{r}, \zeta)\, \mathrm{d}\zeta$$

- This is called the principle of superposition.
- The function g is called a Green's function.
- The role of ζ is generally taken by $\mathbf{r}'$, the location of the source.

The Implication of Spatial Invariance

Because free space is invariant, if

$$\nabla^2 A_z(\mathbf{r}) + k^2 A_z(\mathbf{r}) = -\mu J_z(\mathbf{r})$$

then, if $\mathbf{r}'$ is any fixed location in space,

$$\nabla^2 A_z(\mathbf{r}-\mathbf{r}') + k^2 A_z(\mathbf{r}-\mathbf{r}') = -\mu J_z(\mathbf{r}-\mathbf{r}')$$

- Notice that spatial invariance is a less common occurrence than linearity.
- Maxwell's equations in free space have been shown linear with centuries of experimental evidence.
- Spatial invariance, on the other hand, does not exist in a waveguide!

The Delta Function in Multiple Dimensions

It helps to define a

Three-Dimensional Delta Function

$$\delta(\mathbf{r}) = \delta(x)\delta(y)\delta(z)$$

so that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\mathbf{r})\delta(\mathbf{r})\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = f(\mathbf{0})$$

Similarly, if we let $\mathbf{r}' = x'\mathbf{u}_x + y'\mathbf{u}_y + z'\mathbf{u}_z$, we have

Shifted Three-Dimensional Delta Function

$$\delta(\mathbf{r} - \mathbf{r}') = \delta(x - x')\delta(y - y')\delta(z - z')$$

Green's Function

The equation we wish to solve is

The Helmholtz Equation

$$\nabla^2 A_z + k^2 A_z = -\mu J_z$$

Now, we can write an

Impulsive Formulation of the Current

$$J_z(\mathbf{r}) = \iiint J_z(\mathbf{r}')\delta(\mathbf{r} - \mathbf{r}')\,\mathrm{d}V$$

This expression envisions the current as a superposition of shifted and weighted impulsive currents.

Green's Function

Suppose we can find a

Green's Function

$$\nabla^2 g(\mathbf{r}) + k^2 g(\mathbf{r}) = -\delta(\mathbf{r})$$

which solves the problem for a single impulse at the origin. Then, because of linearity and spatial invariance, the solution to the Helmholtz equation is written as a

Convolution

$$A_z(\mathbf{r}) = \iiint J_z(\mathbf{r}')g(\mathbf{r} - \mathbf{r}')\,\mathrm{d}V'$$

Finding the Green's Function

Consider the equation

$$\nabla^2 g(\mathbf{r}) + k^2 g(\mathbf{r}) = -\delta(\mathbf{r})$$

By symmetry, $g(\mathbf{r})$ is only a function of distance from the origin; that is

$$g(\mathbf{r}) = g(r)$$

where r is the radial coordinate in spherical coordinates. The equation thus reduces to

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}g}{\mathrm{d}r}\right) + k^2 g = -\delta(r)$$

Finding the Green's Function

Away from the origin, this reduces to

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}g}{\mathrm{d}r}\right)+r^2k^2g &= 0 \\ r^2\frac{\mathrm{d}^2g}{\mathrm{d}r^2}+2r\frac{\mathrm{d}g}{\mathrm{d}r}+r^2k^2g &= 0 \\ r\frac{\mathrm{d}^2g}{\mathrm{d}r^2}+2\frac{\mathrm{d}g}{\mathrm{d}r}+k^2rg &= 0 \\ \frac{\mathrm{d}^2(rg)}{\mathrm{d}r^2}+k^2(rg) &= 0 \end{aligned}$$

Therefore, for constants C_+ and C_- we have

$$g(r)=\frac{C_+}{r}e^{-jkr}+\frac{C_-}{r}e^{jkr}$$

Finding the Green's Function

It should be immediately obvious that

$$C_- = 0$$

Why? How do we find C_+? We

- Plug back into the original differential equation, and
- Integrate over a small sphere of radius ϵ (S_ϵ) including the origin. (Why do we do this?)

Thus,

$$\lim_{\epsilon \to 0} \left[\iiint\limits_{S_\epsilon} \nabla^2 g \, \mathrm{d}V + k^2 \iiint\limits_{S_\epsilon} g \, \mathrm{d}V = - \iiint\limits_{S_\epsilon} \delta(r) \mathrm{d}V \right]$$

Finding the Green's Function

We examine this term by term.

$$-\lim_{\epsilon\to 0}\iiint\limits_{S_\epsilon}\delta(r)\mathrm{d}V = -1$$

$$\lim_{\epsilon\to 0}\iiint\limits_{S_\epsilon}\frac{e^{-jkr}}{r}\,\mathrm{d}V = \lim_{\epsilon\to 0}\int\limits_0^{\epsilon}\int\limits_0^{\pi}\int\limits_0^{2\pi} re^{-jkr}\sin\theta\,\mathrm{d}\phi\,\mathrm{d}\theta\,\mathrm{d}r = 0$$

Finally, note that

$$\nabla g = \mathbf{u}_r\frac{\mathrm{d}g}{\mathrm{d}r} = -\mathbf{u}_r C_+\frac{jkre^{-jkr}+e^{-jkr}}{r^2}$$

Finding the Green's Function

Therefore

$$\lim_{\epsilon\to 0}\iiint_{S_\epsilon}\nabla^2 g\,\mathrm{d}V = \lim_{\epsilon\to 0}\iiint_{S_\epsilon}\nabla\cdot\nabla g\,\mathrm{d}V$$

$$= \lim_{\epsilon\to 0}\iint_{\partial S_\epsilon}\nabla g\cdot\mathrm{d}\mathbf{S} = \lim_{\epsilon\to 0}\int_0^{2\pi}\int_0^{\pi}\left[-\mathbf{u}_r C_+\frac{jk\epsilon e^{-jk\epsilon}+e^{-jk\epsilon}}{\epsilon^2}\right]\cdot\epsilon^2\mathbf{u}_r\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$$

$$= -4\pi C_+\lim_{\epsilon\to 0}\left[jk\epsilon e^{-jk\epsilon}+e^{-jk\epsilon}\right] = -4\pi C_+$$

So finally we have

$$-4\pi C_+ = -1$$

The Green's Function and the Magic Formula

We thus have

The Scalar Green's Function

$$g(r) = \frac{e^{-jkr}}{4\pi r} \quad \text{or} \quad g(\mathbf{r}) = \frac{e^{-jk|\mathbf{r}|}}{4\pi|\mathbf{r}|}$$

Therefore,

$$g(\mathbf{r} - \mathbf{r}') = \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r} - \mathbf{r}'|}$$

The magnetic potential is thus given by the

Magic Formula

$$A_z(\mathbf{r}) = \frac{\mu}{4\pi} \iiint J_z(\mathbf{r}') \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|} \mathrm{d}V'$$

The Solution to Maxwell's Equations

$$\vec{A}(r) = \frac{\mu}{4\pi} \iiint \vec{J}(r') \frac{e^{-jk|r-r'|}}{|r-r'|} dv'$$

$$\nabla^2 \vec{A} + k^2 \vec{A} = -\mu \vec{J}$$

$$\vec{E}_e = -j\omega \vec{A} + \frac{1}{j\omega\mu\varepsilon} \nabla (\nabla \cdot \vec{A})$$

$$\vec{H}_e = \frac{1}{\mu} \nabla \times \vec{A}$$

The Solution to Maxwell's Equations

If we have magnetic sources

$$\vec{F}(r) = \frac{\varepsilon}{4\pi} \iiint \vec{M}(r') \frac{e^{-jk|r-r'|}}{|r-r'|} dv'$$

$$\nabla^2 \vec{F} + k^2 \vec{F} = -\varepsilon \vec{M}$$

$$\vec{E}_m = -\frac{1}{\varepsilon} \nabla \times \vec{F}$$

$$\vec{H}_m = -j\omega \vec{F} - \frac{j}{\omega\mu\varepsilon} \nabla(\nabla \cdot \vec{F})$$

The Solution to Maxwell's Equations

If we have magnetic sources

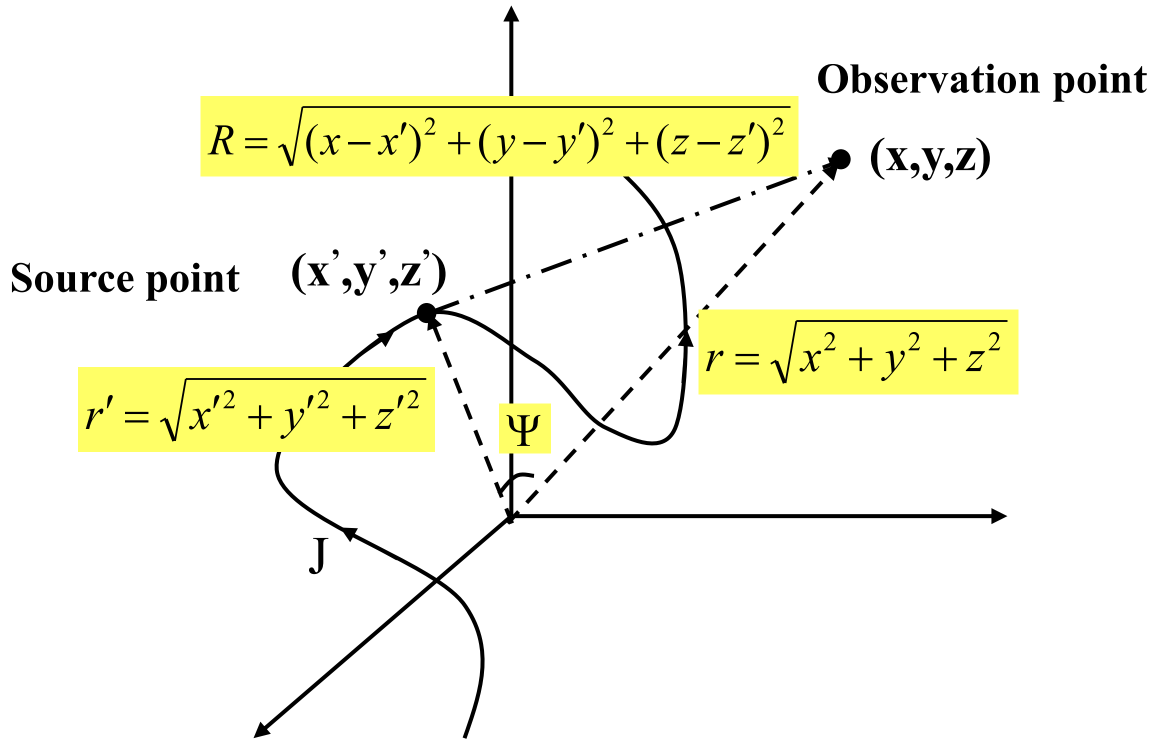
$$\vec{F}(r) = \frac{\varepsilon}{4\pi}\iiint \vec{M}(r')\frac{e^{-jk|r-r'|}}{|r-r'|}dv'$$

$$\nabla^2\vec{F} + k^2\vec{F} = -\varepsilon\vec{M}$$

$$\vec{E}_m = -\frac{1}{\varepsilon}\nabla\times\vec{F}$$

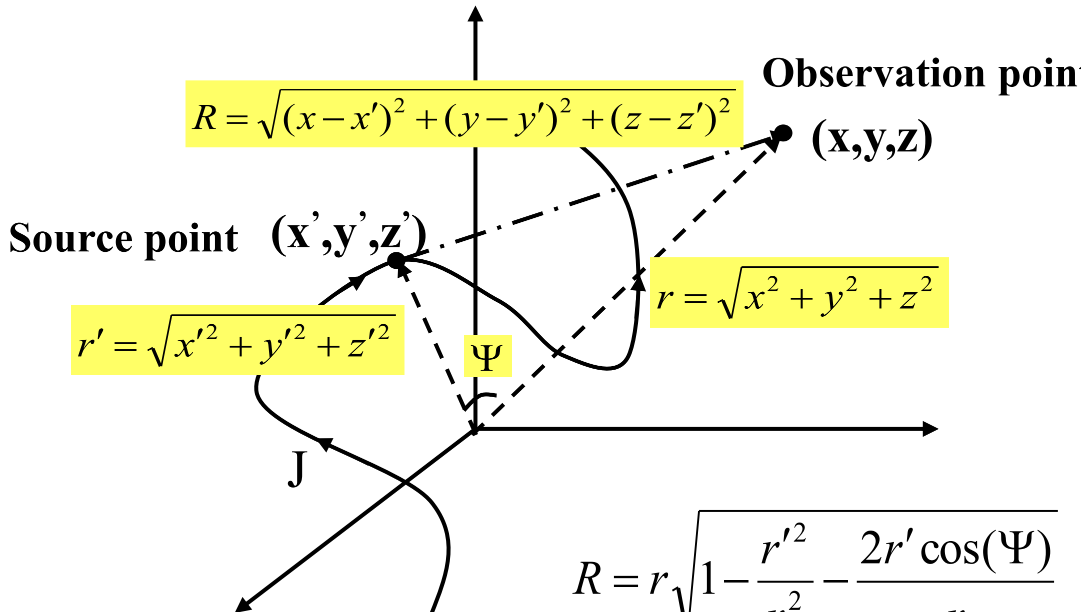
$$\vec{H}_m = -j\omega\vec{F} - \frac{j}{\omega\mu\varepsilon}\nabla(\nabla\cdot\vec{F})$$

Far Field Approximation



$$
\begin{aligned}
R &= \sqrt{(x-x')^2+(y-y')^2+(z-z')^2} \\
&= \sqrt{(x^2+y^2+z^2)-(x'^2+y'^2+z'^2)-2(xx'+yy'+zz')} \\
&= \sqrt{r^2-r'^2-2rr'\cos(\Psi)} \\
&= r\sqrt{1-\frac{r'^2}{r^2}-\frac{2r'\cos(\Psi)}{r}}
\end{aligned}
$$

Far Field Approximation



$$R = r\sqrt{1-\frac{r'^2}{r^2}-\frac{2r'\cos(\Psi)}{r}}$$

In the far field r>>r'

$$R = r\sqrt{1-\frac{r'^2}{r^2}-\frac{2r'\cos(\Psi)}{r}}$$

$$= r\cdot\left(1-\frac{r'^2}{2r^2}-\frac{2r'\cos(\Psi)}{2r}+\frac{r'^4}{8r^4}+\frac{4r'^2\cos^2(\Psi)}{8r^2}-\ldots.\right)$$

$$\approx r\cdot\left(1-\frac{r'\cos(\Psi)}{r}\right)\approx r-r'\cos(\Psi)$$

Far-Field Approximation (1)

$$\vec{A} = \frac{\mu}{4\pi} \iiint_V \vec{J}(\vec{r}\,') \frac{e^{-jkR}}{R} dv' \qquad \vec{F} = \frac{\varepsilon}{4\pi} \iiint_V \vec{M}(\vec{r}\,') \frac{e^{-jkR}}{R} dv'$$

In the far-field zone:

$$R = \sqrt{r^2 + r'^2 - 2rr'\cos\psi} \approx r - r'\cos\psi$$

$$= \quad r - \mathbf{u}_r \cdot \mathbf{r}'$$

We can use this approximation to R in the exponent, but need not in the denominator. Why? This gives rise to the

Far Field Magnetic Vector Potential

$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi}\frac{e^{-jkr}}{r}\iiint \mathbf{J}(\mathbf{r}')e^{jk\mathbf{u}_r\cdot\mathbf{r}'}dV'$$

Note:

- The integrand is a function of only θ and ϕ, not r. This allows us to talk about antenna patterns.
- The radial dependence is a
  - Decay at the rate r^{-1} (Why?), and
  - Wave travel in the radial direction (e^{-jkr})

Far-Field Approximation (1)

$$\vec{A} = \frac{\mu}{4\pi} \iiint_V \vec{J}(\vec{r}') \frac{e^{-jkR}}{R} dv' \qquad \vec{F} = \frac{\varepsilon}{4\pi} \iiint_V \vec{M}(\vec{r}') \frac{e^{-jkR}}{R} dv'$$

In the far-field zone:

$$R = \sqrt{r^2 + r'^2 - 2rr'\cos\psi} \approx r - r'\cos\psi$$

$$\vec{A} = \frac{\mu}{4\pi} \iiint_V \vec{J} \frac{e^{-jk(r - r'\cos\psi)}}{r} dv' = \frac{\mu}{4\pi r} e^{-jkr} \iiint_V \vec{J} e^{jkr'\cos\psi} dv'$$

$$= \frac{\mu}{4\pi r} e^{-jkr} \vec{N} \qquad \vec{N} = \iiint_V \vec{J} e^{jkr'\cos\psi} dv'$$

$$\vec{F} = \frac{\varepsilon}{4\pi r} e^{-jkr} \vec{L} \qquad \vec{L} = \iiint_V \vec{M} e^{jkr'\cos\psi} dv'$$

We can write the expression from the previous slide as

$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi}\frac{e^{-jkr}}{r}\mathbf{N}(\theta,\phi) = \mu f(r)\mathbf{N}(\theta,\phi)$$

Now

$$\mathbf{H}(\mathbf{r}) = \frac{1}{\mu}\nabla\times\mathbf{A} = f(r)\nabla\times\mathbf{N}(\theta,\phi) + \mathbf{N}(\theta,\phi)\times\nabla f(r)$$

and

$$\nabla\times\mathbf{N}(\theta,\phi) \propto r^{-1} + O(r^{-2})$$

Since we are ignoring terms that decay faster than r^{-1},

$$\mathbf{H}(\mathbf{r}) = \mathbf{N}(\theta,\phi)\times\nabla f(r)$$

$$
\begin{aligned}
\nabla f(r) &= \mathbf{u}_r \frac{\partial f(r)}{\partial r} \\
&= \mathbf{u}_r \frac{-jkre^{-jkr} - e^{-jkr}}{4\pi r^2} \\
&\approx -\mathbf{u}_r \frac{jke^{-jkr}}{4\pi r} = -jkf(r)\mathbf{u}_r
\end{aligned}
$$

Thus we have

The Far Magnetic Field

$$
\mathbf{H}(\mathbf{r}) = jk\frac{e^{-jkr}}{4\pi r}\mathbf{u}_r \times \iiint \mathbf{J}(\mathbf{r}')e^{jk\mathbf{u}_r\cdot\mathbf{r}'}dV' = \frac{jk}{\mu}\mathbf{u}_r \times \mathbf{A}(\mathbf{r})
$$

The Electric Field

The far electric field can be found with the same technique. Recall

$$\begin{aligned}\mathbf{E}(\mathbf{r}) &= -j\omega\left[\mathbf{A}(\mathbf{r}) + \frac{1}{k^2}\nabla\nabla\cdot\mathbf{A}(\mathbf{r})\right] \\ &= -j\omega\mu\left\{f(r)\mathbf{N}(\theta,\phi) + \frac{1}{k^2}\nabla\nabla\cdot[f(r)\mathbf{N}(\theta,\phi)]\right\}\end{aligned}$$

Now

$$\begin{aligned}\nabla\cdot[f(r)\mathbf{N}(\theta,\phi)] &= \nabla f(r)\cdot\mathbf{N}(\theta,\phi) + f(r)\nabla\cdot\mathbf{N}(\theta,\phi) \\ &\approx \nabla f(r)\cdot\mathbf{N}(\theta,\phi) \\ &= -jkf(r)\mathbf{u}_r\cdot\mathbf{N}(\theta,\phi)\end{aligned}$$

because

- $\nabla\cdot\mathbf{N} \propto r^{-1}$, and
- We showed it two slides ago.

The Electric Field

$$\begin{aligned}
\nabla\nabla \cdot [f(r)\mathbf{N}(\theta,\phi)] &\approx -jk\nabla\left[f(r)N_r(\theta,\phi)\right] \\
&= -jk\left[f(r)\nabla N_r(\theta,\phi) + N_r(\theta,\phi)\nabla f(r)\right] \\
&\approx -jkN_r(\theta,\phi)\nabla f(r) \\
&\approx -k^2\mathbf{u}_r f(r)N_r(\theta,\phi) \\
&= -k^2\mathbf{u}_r\mathbf{u}_r \cdot f(r)\mathbf{N}(\theta,\phi)
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathbf{E}(\mathbf{r}) &= -j\omega\mu\left\{f(r)\mathbf{N}(\theta,\phi) + \frac{1}{k^2}\nabla\nabla \cdot [f(r)\mathbf{N}(\theta,\phi)]\right\} \\
&= -j\omega\left[\mathbf{A}(\mathbf{r}) - \mathbf{u}_r\mathbf{u}_r \cdot \mathbf{A}(\mathbf{r})\right]
\end{aligned}$$

Relationship Between Far Electric and Magnetic Field

Notice that

$$
\begin{aligned}
\frac{1}{\eta}\mathbf{u}_r \times \mathbf{E}(\mathbf{r}) &= -j\omega\sqrt{\frac{\epsilon}{\mu}}\mathbf{u}_r \times [\mathbf{A}(\mathbf{r}) - \mathbf{u}_r\mathbf{u}_r \cdot \mathbf{A}(\mathbf{r})] \\
&= \frac{-j\omega\sqrt{\epsilon\mu}}{\mu}\mathbf{u}_r \times \mathbf{A}(\mathbf{r}) \\
&= \frac{-jk}{\mu}\mathbf{u}_r \times \mathbf{A}(\mathbf{r}) \\
&= \mathbf{H}(\mathbf{r})
\end{aligned}
$$

Far Field Summary

Far Field Vector Potential

$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi}\frac{e^{-jkr}}{r}\iiint \mathbf{J}(\mathbf{r}')e^{jk\mathbf{u}_r\cdot\mathbf{r}'}\mathrm{d}V'$$

Far Fields

$$\begin{aligned}\mathbf{E}(\mathbf{r}) &= -j\omega\left[\mathbf{A}(\mathbf{r}) - \mathbf{u}_r\mathbf{u}_r\cdot\mathbf{A}(\mathbf{r})\right]\\ \mathbf{H}(\mathbf{r}) &= \frac{1}{\eta}\mathbf{u}_r\times\mathbf{E}(\mathbf{r})\end{aligned}$$

$$\Rightarrow \begin{cases} E_r \approx 0 \\ E_\theta \approx -j\omega A_\theta \\ E_\varphi \approx -j\omega A_\varphi \end{cases} \qquad \vec{H}_A = \frac{1}{\mu}\nabla\times\vec{A} \quad \begin{cases} H_r \approx 0 \\ H_\theta \approx -\dfrac{E_\varphi}{\eta} = \dfrac{j\omega}{\eta}A_\varphi \\ H_\varphi \approx \dfrac{E_\theta}{\eta} = -\dfrac{j\omega}{\eta}A_\theta \end{cases}$$

Far-Field Approximation (3)

For $\vec{E}_F$ and $\vec{H}_F$:

$$\begin{cases} H_r \approx 0 \\ H_\theta \approx -j\omega F_\theta \\ H_\varphi \approx -j\omega F_\varphi \end{cases} \qquad \begin{cases} E_r \approx 0 \\ E_\theta \approx \eta H_\varphi = -j\omega\eta F_\varphi \\ E_\varphi \approx -\eta H_\theta = j\omega\eta F_\theta \end{cases}$$

Total fields:

$$E_r \approx 0$$

$$E_\theta \approx -j\omega A_\theta - j\omega\eta F_\varphi = -j\omega(A_\theta + \eta F_\varphi)$$

$$E_\varphi \approx -j\omega A_\varphi + j\omega\eta F_\theta = -j\omega(A_\varphi - \eta F_\theta)$$

Far-Field Approximation (4)

$$H_r \approx 0$$

$$H_\theta \approx \frac{j\omega}{\eta} A_\varphi - j\omega F_\theta = \frac{j\omega}{\eta}(A_\varphi - \eta F_\theta)$$

or

$$H_\varphi \approx -\frac{j\omega}{\eta} A_\theta - j\omega F_\varphi = -\frac{j\omega}{\eta}(A_\theta + \eta F_\varphi)$$

$$E_r \approx 0$$

$$E_\theta \approx -\frac{jke^{-jkr}}{4\pi r}(L_\varphi + \eta N_\theta)$$

$$E_\varphi \approx \frac{jke^{-jkr}}{4\pi r}(L_\theta - \eta N_\varphi)$$

$$H_r \approx 0$$

$$H_\theta \approx \frac{jke^{-jkr}}{4\pi r}(N_\varphi - \frac{L_\theta}{\eta})$$

$$H_\varphi \approx -\frac{jke^{-jkr}}{4\pi r}(N_\theta + \frac{L_\varphi}{\eta})$$