# ELEG403 COMMUNICATIONS SYSTEMS ENGINEERING

## ASSIGNMENT #4 due Monday October 6

1. Haykin Problem 3.9
2. Haykin Problem 3.10 (First read pp. 100-102. This is a problem dealing with equalization, which we did not cover in class.)
3. Haykin Problem 4.1
4. Haykin Problem 4.7
5. Haykin Problems 4.13 and 4.15 (They belong together in one problem.)

SOLUTIONS TO ASSIGNMENT #4

1. Haykin Problem # 3.9

a) $H(f) = |H(f)|\, e^{j\beta(f)}$

$$= \left(a_0 + a_1 \cos\left(\pi \tfrac{f}{B}\right)\right) e^{-j2\pi f t_0}$$

$$= \left[a_0 + \frac{a_1}{2} e^{j\pi \frac{f}{B}} + \frac{a_1}{2} e^{-j\pi\frac{f}{B}}\right] e^{-j2\pi f t_0}$$

$$= a_0 e^{-j2\pi f t_0} + \frac{a_1}{2} e^{-j2\pi f\left(t_0 - \frac{1}{2B}\right)} + \frac{a_1}{2} e^{-j2\pi f\left(t_0 + \frac{1}{2B}\right)}$$

$$\therefore\ y(t) = a_0\, x(t - t_0) + \frac{a_1}{2} x\left(t - t_0 + \tfrac{1}{2B}\right) + \frac{a_1}{2} x\left(t - t_0 - \tfrac{1}{2B}\right)$$

∴ Output composed of
- delayed replica of input modified by $a_0$
- two "echoes", one preceding and the other lagging the main response by $\frac{1}{2B}$ ⟹ represents result of distortion

b.) $H(f) = a_0 \exp\left[-j2\pi f t_0 + j b_1 \sin\left(\pi \tfrac{f}{B}\right)\right]$

Using approximation

$$\exp\left[j b_1 \sin\left(\pi \tfrac{f}{B}\right)\right] \approx 1 + j b_1 \sin\left(\pi \tfrac{f}{B}\right),$$

$$H(f) = a_0 e^{-j2\pi f t_0}\left[1 + j b_1 \sin\left(\pi \tfrac{f}{B}\right)\right]$$

$$= a_0 e^{-j2\pi f t_0} + \frac{a_0 b_1}{2} e^{-j2\pi f\left(t_0 - \frac{1}{2B}\right)} - \frac{a_0 b_1}{2} e^{-j2\pi f\left(t_0 + \frac{1}{2B}\right)}$$

(1. cont'd) $\therefore y(t) = a\,x(t-t_0) + \frac{a_0 b_1}{2} x\left(t-t_0+\frac{1}{2B}\right) - \frac{a_0 b_1}{2} x\left(t-t_0-\frac{1}{2B}\right)$

same interpretation as a)

2. Haykin Problem # 3.10 (Multipath Channel)

From the Figure, the channel output is

$$y(t) = x(t-t_0) + \alpha\, x(t-t_0-\tau)$$

$$Y(f) = X(f)\left[1+\alpha e^{-j2\pi f\tau}\right] e^{-j2\pi f t_0}$$

$$H_c(f) = \frac{Y(f)}{X(f)} = \left[1+\alpha e^{-j2\pi f\tau}\right] e^{-j2\pi f t_0}$$

$$H_{eq}(f)\cdot H_c(f) = K e^{-j2\pi f t_0} \quad \text{distortionless transmission}$$

$$H_{eq}(f) = \frac{K}{1+\alpha e^{-j2\pi f\tau}}$$

$$\underset{\text{approximation}}{\xrightarrow{\text{using}}} K\left[1-\alpha e^{-j2\pi f\tau} + \alpha^2 e^{-j4\pi f\tau}\right]$$

$$H_{eq}(f) = K - K\alpha e^{-j2\pi f\tau} + K\alpha^2 e^{-j4\pi f\tau}$$

For simplicity and without loss of generality, let $K=1$. Then, the three tap weights are

$$w_0 = 1$$
$$w_1 = -\alpha$$
$$w_2 = \alpha^2$$

3. Haykin Problem #4.1

$$g(t) = \begin{cases} e^{-at}, & t>0 \\ 1/2, & t=0 \\ 0, & t<0 \end{cases}$$

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft} dt = \frac{1}{a+j2\pi f}$$

energy spectral density $\quad \psi_g(f) = |G(f)|^2 = \frac{1}{a^2+(2\pi f)^2}$

$$E_W = \int_{-W}^{W} \psi_g(f) df = \text{energy within } [-W,W] \text{ where } W = \frac{a}{2\pi}$$

$$E_{total} = \int_{-\infty}^{\infty} \psi_g(f) df = \int_{-\infty}^{\infty} \frac{1}{a^2+(2\pi)^2 f^2} df$$

$$E_W = \int_{-W}^{W} \frac{1}{a^2+(2\pi f)^2} df = \frac{1}{2\pi a} \tan^{-1}\left(\frac{2\pi f}{a}\right)\Bigg|_{-a/2\pi}^{a/2\pi} = \frac{1}{2\pi a}\cdot\frac{\pi}{2}$$

$$= \frac{1}{4a}$$

$$E_{total} = \frac{1}{2\pi a} \tan^{-1} u \Big|_{-\infty}^{\infty} = \frac{1}{2\pi a}\cdot \pi = \frac{1}{2a}$$

$$\frac{E_W}{E_{total}} = \frac{1}{2} \Rightarrow 50\%$$

4. Haykin Problem #4.7

$$g(t) = A_0 + A_1 \cos(2\pi f_1 t + \theta_1) + A_2 \cos(2\pi f_2 t + \theta_2)$$

4. cont'd) a.) $R_g(\tau)$ and $\psi_g(f)$ are a Fourier transform pair

$$G(f) = A_0\delta(f) + \frac{A_1}{2}\left[\delta(f-f_1)e^{j\theta_1} + \delta(f+f_1)e^{-j\theta_1}\right] + \frac{A_2}{2}\left[\delta(f-f_2)e^{j\theta_2} + \delta(f+f_2)e^{-j\theta_2}\right]$$

$$|G(f)|^2 = A_0^2\delta(f) + \frac{A_1^2}{4}\left[\delta(f-f_1) + \delta(f+f_1)\right] + \frac{A_2^2}{4}\left[\delta(f-f_2) + \delta(f+f_2)\right]$$

- in deriving this, note that $\delta(f-\alpha)\cdot\delta(f+\alpha) = 0$

$$\therefore R_g(\tau) = \mathcal{F}^{-1}\left[|G(f)|^2\right] = A_0^2 + \frac{A_1^2}{2}\cos(2\pi f_1\tau) + \frac{A_2^2}{2}\cos(2\pi f_2\tau)$$

b.) $R_g(0) = A_0^2 + \frac{A_1^2}{2} + \frac{A_2^2}{2}$

c.) $R_g(\tau)$ depends only on the amplitudes $A_0$, $A_1$, and $A_2$ of the dc and sinusoidal components and the frequencies $f_1$ and $f_2$. The phase information contained in the phase angles of the two sinusoidal components is completely lost.

5. Haykin Problems #4.13 and 4.15

#4.13

$$g_T(t) = A \exp(j2\pi f_c t)\,\mathrm{rect}\left(\frac{t}{2T}\right)$$

$$G_T(f) = \int_{-\infty}^{\infty} g_T(t) e^{-j2\pi f t}\,dt = \int_{-T}^{T} A e^{j2\pi f_c t} e^{-j2\pi f t}\,dt$$

$$= \frac{A}{\pi(f_c - f)} \sin[2\pi(f_c - f)T]$$

$$= 2AT\,\mathrm{sinc}[2(f_c - f)T]$$

$$S_g(f) \text{ for finite } T = \frac{1}{2T}|G_T(f)|^2$$

$$= 2A^2T\,\mathrm{sinc}^2[2(f - f_c)T]$$

As $T \to \infty$,

$$\lim_{T\to\infty} 2T\,\mathrm{sinc}^2(2fT) = \delta(f)$$

$$\therefore\ S_g(f) = A^2\delta(f - f_c)$$

#4.15

$$R_g(\tau) = \mathcal{F}^{-1}[S_g(f)]$$

$$= A^2 \exp[j2\pi f_c \tau]$$