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ON THE DIFFERENTIAL REPRESENTATIONS OF QUANTUM LIE SUPERALGEBRAS

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Abstract

In this letter, we study the differential representations of quantum Lie superalgebras. A concrete example of $\mathcal{U}_q gl(1|1)$ is given.

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The notion of quantum groups and quantum universal enveloping algebras originates in the study of quantum inverse scattering methods[1,2]. Now there have been many extensive studies on the representation theory [3]. In this letter, we discuss the differential representations of quantum Lie superalgebras. New feature arises from the gradation of the algebras. It seems that the differential representations are not irreducible anymore.

Discussion of the differential representations in the non-super case can be found in Ref [4]. In the super case, we just have to pay special attention to gradation when considering the tensor product. The duality between the quantum group (QG) and quantum universal enveloping algebra (QUEA) is just the same in the super case [6], which indicates that one can endow a structure of Hopf algebra to the QG $\mathcal{A} = Hom(\mathcal{U}, C)$, which is the full dual space of QUEA $\mathcal{U}$:

1) Multiplication $m_{\mathcal{A}} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$

$$m_{\mathcal{A}}(\Phi)(a) = \Phi(\Delta(a)), \quad for\ \Phi \in \mathcal{A} \otimes \mathcal{A}, \quad a \in \mathcal{U}. \tag{1}$$

2) Coproduct $\Delta_{\mathcal{A}} : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$

$$\Delta_{\mathcal{A}}(\phi)(a \otimes b) = \phi(ab), \quad for\ \phi \in \mathcal{A},\ and\ a, b \in \mathcal{U} \tag{2}$$

3) Counit $\epsilon_{\mathcal{A}} :\ \mathcal{A} \to C$

$$\epsilon_{\mathcal{A}}(\phi) = \phi(1), \quad \phi \in \mathcal{A},\ 1\ is\ the\ unit\ of\ \mathcal{U} \tag{3}$$

4) Antipode $\mathcal{S}_{\mathcal{A}} :\ \mathcal{A} \to \mathcal{A}$ [A

$$\mathcal{S}_{\mathcal{A}}(\phi) = \phi(\mathcal{S}(a)), \quad for\ \phi \in \mathcal{A},\ a \in \mathcal{U} \tag{4}$$

The differential representation of $\mathcal{U}$ on $\mathcal{A}$ is defined by the following linear mappings π_l and π_r

$$[\pi_l(a)\phi](b) = \phi(ba), \quad [\pi_r(a)\phi](b) = \phi(\mathcal{S}(a)b) \quad for \quad a, b \in \mathcal{U};\ \phi \in \mathcal{A}. \tag{5}$$

It is easy to verify that the above linear mappings satisfy the following properties:
1)

$$\pi_l(ab) = \pi_l(a)\pi_l(b), \quad \pi_r(ab) = \pi_r(a)\pi_r(b) \quad for\ a, b \in \mathcal{U} \tag{6}$$

2) Let $\Delta_{\mathcal{A}}(\phi) = \sum_\mu \phi_\mu \otimes \phi^\mu$ ($\phi \in \mathcal{A}$). Then

$$\pi_l(a)\phi = \sum_\mu \phi^\mu(a)\phi_\mu, \quad \pi_r(a)\phi = \sum_\mu \phi_\mu(\mathcal{S}(a))\phi^\mu. \tag{7}$$

3) Let $\Delta(a) = \sum_i a_i \otimes a^i$ $(a \in \mathcal{U})$. Then

$$\pi_l(a)\phi\psi = \sum_i \sum_{\mu,\nu} \phi^\mu(a_i)\psi^\nu(a^i)(-)^{deg(a_i)deg(\psi_\nu)}\phi_\mu\psi_\nu \tag{8}$$

Proof:

$$\begin{aligned}
[\pi_l(a)\phi\psi](b) &= (\phi\psi)(ba) = (\phi\otimes\psi)(\Delta(ba)) \\
&= \textstyle\sum_{i,j}(\phi\otimes\psi)(b_j a_i \otimes b^j a^i)(-)^{deg(a_i)deg(b^j)} \\
&= \textstyle\sum_{ij}\phi(b_j a_i)\psi(b^j a^i)(-)^{deg(a_i)deg(b^j)} \\
&= \textstyle\sum_{i,j}(\sum_\mu \phi^\mu(a_i)\phi_\mu(b_j))(\sum_\nu \psi^\nu(a^i)\psi_\nu(b^j))(-)^{deg(a_i)deg(b^j)} \\
&= \textstyle\sum_{i,j}\sum_{\mu,\nu}\phi^\mu(a_i)\psi^\nu(a^i)(-)^{deg(a_i)deg(b^j)}\phi_\mu(b_j)\psi_\nu(b^j) \\
&= (\textstyle\sum_i\sum_{\mu,\nu}\phi^\mu(a_i)\psi^\nu(a^i)(-)^{deg(a_i)deg(\psi_\nu)})\phi_\mu\psi_\nu(b)
\end{aligned}$$

where use has been made of the "grade conservation" property of the $\check{R}$ matrix which ensures that

$$deg(b^j) = deg(\psi_\nu), \quad for\ \psi_\nu(b^j) \neq 0 \tag{9}$$

Q.E.D.

The same things hold for $\pi_r(a)$.

It is in order of showing that the space spanned by the polynomials of $\phi \in \mathcal{A}$ carries a representation of $\mathcal{U}$. The key for this is to check that property 3) in Eq(8) is an algebraic homomorphism, i.e.

$$\pi_l(ab)\phi\psi = \pi(a)\pi(b)\phi\psi \tag{10}$$

Proof: According to Eq(8),

$$\begin{aligned}
LHS &= \textstyle\sum_{\mu,\nu}\phi^\mu((ab)_k)\psi^\nu((ab)^k)(-)^{deg((ab)_k)deg(\psi_\nu)}[\phi_\mu\psi_\nu] \\
&= \textstyle\sum_{i,j}\sum_{\mu,\nu}\phi^\mu(a_i b_j)\psi^\nu(a^i b^j)(-)^{deg(a_i)+deg(b_j)deg(\psi_\nu)}(-)^{deg(a^i)deg(b_j)} \\
&= \textstyle\sum_{i,j}\sum_{\mu,\nu}\sum_{\lambda\sigma}(\phi^\mu)_\lambda(a_i)(\phi^\mu)^\lambda(b_j)(\psi^\nu)_\sigma(a^i)(\psi^\nu)^\sigma(b^j) \\
&\quad (-)^{(deg(a_i)+deg(b_j))deg(\psi_\nu)+deg(a^i)deg(b_j)}\phi_\mu\psi_\nu \\
&= \textstyle\sum_{i,j}\sum_{\mu,\nu}\sum_{\lambda,\sigma}\phi^\mu(b_j)\psi^\nu(b^j)(-)^{deg(b_i)deg(\psi_\nu)}(\phi_\mu)^\lambda(a_i)(\psi_\nu)^\sigma(a^i) \\
&\quad (-)^{deg(a_i)deg((\psi_\nu)_\sigma))}(\phi_\mu)_\lambda(\psi_\nu)_\sigma \\
&= \textstyle\sum_{\mu,\nu}\sum_j\phi^\mu(b_j)\psi^\nu(b^j)(-)^{deg(b_j)deg(\psi_\nu)}\pi(a)[\phi_\mu\psi_\nu] \\
&= \pi(a)\pi(b)\phi\psi
\end{aligned}$$

where we have used the homomorphism of the coproduct

$$\Delta(ab) = \sum_k (ab)_k \otimes (ab)^k = \sum_{i,j}(a_i \otimes a^i)(b_j \otimes b^j)$$

$$= \sum_{i,j}(a_i b_j) \otimes (a^i b^j)(-)^{deg(a^i)deg(b_j)} = \Delta(a)\Delta(b)$$

and the "grade conservation" in the super case. It should be kept in mind that the compact form of indices is adopted in the proof, i.e., μ etc. is just a shorthand notation of one pair indices of ϕ etc., because ϕ, ψ and a, b are indeed elements of matrix (refer to Eq(15, 19) in the following example). .

Q.E.D.

A concrete example of using the above procedure is given in the case of $\mathcal{U}_q(gl(1|1))$. QUEA $\mathcal{U}_q(gl(1|1))$ has two boson generators k, l and two fermion generators x, y, which satisfy the following relations [6]

$$[\ k\ ,\ l\ ] = 0, \tag{11a}$$

$$k\ x\ k^{-1} = \ q\ x, \quad l\ x\ l^{-1} = \ q^{-1}x, \tag{11b}$$

$$k\ y\ k^{-1} = \ q^{-1}y, \quad l\ y\ l^{-1} = \ q\ y, \tag{11c}$$

$$x^2 = y^2 = 0, \tag{11d}$$

$$\{\ x\ ,\ y\ \} = \frac{1}{q-q^{-1}}(\ k\ l - k^{-1}l^{-1}). \tag{11e}$$

The coproduct is defined as

$$\Delta(k^{\pm 1}) = k^{\pm 1} \otimes k^{\pm 1}, \ \Delta(l^{\pm 1}) = l^{\pm 1} \otimes l^{\pm 1}, \tag{12a}$$

$$\Delta(x) = x \otimes k + l^{-1} \otimes x, \ \Delta(y) = y \otimes l + k^{-1} \otimes y. \tag{12b}$$

And the counit ϵ and antipode $\mathcal{S}$ are defined as

$$\epsilon(x) = \epsilon(y) = 0, \ \epsilon(k) = \epsilon(l) = 1, \tag{13a}$$

$$\mathcal{S}(k) = k^{-1}, \ \mathcal{S}(l) = l^{-1}, \ \mathcal{S}(x) = -l\ x\ k^{-1}, \ \mathcal{S}(y) = -k\ y\ l^{-1}.B \tag{13b}$$

The quantum group $GL_q(1|1)$ is composed of bosonic elements u, v and fermionic elements ϕ, ψ, satisfying the relations [6]

$$\mathcal{R}_{12}T_1\eta_{12}T_2\eta_{12} = \eta_{12}T_2\eta_{12}T_1\mathcal{R}_{12} \tag{14}$$

where

$$T = \begin{pmatrix} u, & \phi \\ \psi, & v \end{pmatrix}, \tag{15}$$

$\eta_{12} = diag(1,1,1,-1)$ and

$$\mathcal{R} = \begin{pmatrix} q & & & \\ & 1 & & \\ & q-q^{-1} & 1 & \\ & & & q^{-1} \end{pmatrix}. \tag{16}$$

The coproduct Δ' is defined as

$$\Delta'(T) = T\dot{\otimes}T \tag{17}$$

And the counit ϵ' and antipode $\mathcal{S}'$ are defined as

$$\epsilon'(T) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{S}'(T) = \begin{pmatrix} u^{-1}+u^{-1}\phi v^{-1}\psi u^{-1} & -u^{-1}\phi v^{-1} \\ -v^{-1}\psi u^{-1} & v^{-1}-v^{-1}\phi u^{-1}\psi v^{-1} \end{pmatrix}. \tag{18}$$

Writing the generators of $\mathcal{U}_q gl(1|1)$ in the matrix form

$$L^{(+)} = \begin{pmatrix} k^{-1} & (q-q^{-1})x \\ 0 & l \end{pmatrix}, \quad L^{(-)} = \begin{pmatrix} k & 0 \\ (q-q^{-1})y & l^{-1} \end{pmatrix}, \tag{19}$$

the duality between L and T is just the same as in the non-super case:

$$< L_1^{(\pm)},\ T_2 >= \mathcal{R}_{12}^{(\pm)}, \tag{20}$$

where

$$\mathcal{R}_{12}^{(+)} = \eta_{12}\mathcal{R}_{21}\eta_{12}, \quad \mathcal{R}_{12}^{(-)} = \mathcal{R}_{12}^{-1}. \tag{21}$$

or explicitly

$$< \begin{pmatrix} k^{-1} & (q-q^{-1})x \\ 0 & l \end{pmatrix}, \begin{pmatrix} u & \phi \\ \psi & v \end{pmatrix} >= \begin{pmatrix} q & & & \\ & 1 & q-q^{-1} & \\ & & 1 & \\ & & & q^{-1} \end{pmatrix}, \tag{22a}$$

and

$$< \begin{pmatrix} k & 0 \\ (q-q^{-1})y & l^{-1} \end{pmatrix}, \begin{pmatrix} u & \phi \\ \psi & v \end{pmatrix} >= \begin{pmatrix} q^{-1} & & & \\ & 1 & & \\ & -q+q^{-1} & 1 & \\ & & & q \end{pmatrix}. \tag{22b}$$

From the duality, it is easy to read out explicitly the realization of the linear mapping given in Eq(5) as

$$k\begin{pmatrix} u & \phi \\ \psi & v \end{pmatrix} = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}, \quad l\begin{pmatrix} u & \phi \\ \psi & v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & q^{-1} \end{pmatrix},$$
$$x\begin{pmatrix} u & \phi \\ \psi & v \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad y\begin{pmatrix} u & \phi \\ \psi & v \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix},$$

and check the properties in Eqs(6-8). T is thus a natural representation of $\mathcal{U}$, and with the "twisted Leibniz" rule, infinite representation from the polynomials of elements in $GL_q(1|1)$ follows.

In the above, we present a differential representation for the quantum Lie superalgebras, with a concrete example in the case of $\mathcal{U}_q gl(1|1)$. The discussion is general, so it is straightforward to use it to other algebras, even to the multiparameter quantum Lie superalgebras. The third property, called "twisted Leibniz rule" in the Ref [4], is different from that one in [5], where graded duality was adopted. It seems that there exist some freedom in defining the "twisted Leibniz rule". It is an open question for classifying the category. In addition, it is interesting to see the role played by the differential operators in the quantum super plane in realizing the representation, we will discuss it in a forthcoming paper.

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