CISC 320 Introduction to Algorithms
Fall 2005

Lecture 8
Hash Tables

Problem: to construct a dynamic set that supports the dictionary operations: search, insert and delete.

Examples:

dictionary: word key to definition

compiler: symbol key to semantic data

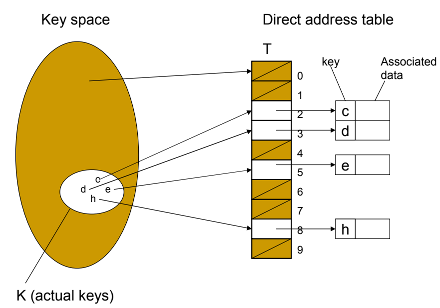
- Key types:
  - Numerical
  - Alphabet
- Key space: the set of all possible keys.

Recall that we can search a sorted array quickly, so the question is

Can we use array?

Case 1: if keys are integer, directly index into array.

Case 2: if keys are string of alphabets, convert to case 1 by first transforming characters to integers (e.g., ASCII).



- Dictionary operations are easily supported in such direct-address model.
  - Each operation takes O(1) time.
  - Problem: key space may be too huge.

  e.g., names of at most 20 letters => size of key space $= 26^{20} \approx 2^{100} \approx 10^{28}$

In practice, while key space is huge, only a small portion is really used, say a few millions of names in our example.

Hashing

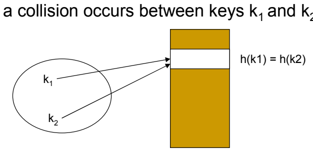
Hash function h

$h: U \rightarrow \{0, 1, \ldots, m-1\}$

where U is the key space and typically $m \ll |U|$.

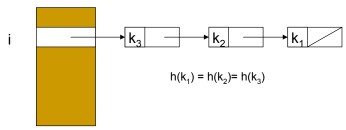
Since m is smaller than |U|, h can not be a one-to-one mapping.

Collisions: a collision occurs between keys k_1 and k_2 if $h(k_1) = h(k_2)$.



Collision resolution by chaining (closed-address)

- Each position in hash table is pointer to head of a linked list.
- To insert elements into the table, add to head of list.



- Chained Hash Insert(T,x)
  insert x at the head of list T[h(key[x])].
  worst-case running time is O(1).
- Chained Hash Search(T,k)
  search for an element with key k in list T[h(k)].
  worst-case running time is proportional to length of list T[h(k)].
- Chained Hash Delete(T,x)
  delete x from the list T[h(key[x])].
  worst-case running time is the time for searching x plus O(1) time for removing it from the list.

- Uniform hashing: each key is equally likely to be hashed into any integer [0, …, m-1].

load factor α: n/m,
where n is the number of keys that will be actually stored in the table. That is, α is the average length of lists. Therefore,
average time for search = O(1+ α).
If n = O(m), then α = O(1).

All dictionary operations can be supported in O(1) time on average.

- Open-address hashing
  - all elements stored in the array of the hash table (no linked lists).
    - More space efficient
    - Less flexible: **load factor α can not be larger than 1.**
  - Rehashing to resolve collisions.
    If a key K is hashed to position i, which is already occupied, K is rehashed to an alternative location:
    rehash(i+d) = (i+d) mod m
    where d is an increment computed from K.
    Linear probing: d = 1
    In linear probing, the alternative to i is the next position i+1.
    When i+1 = m will be mod by m to 0. So rehasing m times will guarantee to probe every slot in the Table.

- Example: h (x) = 5x mod 8
  keys: 1055, 1492, 1776, 1812, 1918, 1945.
  h(1055) = 3
  h(1492) = 4
  h(1776) = 0
  h(1812) = 4
  h(1918) = 6
  h(1945) = 5

0	1	2	3	4	5	6	7
1776			1055	1492 1812	1945	1918	

- Example: h (x) = 5x mod 8, rehash(i) = (i+1) mod 8.
  keys: 1055, 1492, 1776, 1812, 1918, 1945.
  h(1055) = 3
  h(1492) = 4
  h(1776) = 0

0	1	2	3	4	5	6	7
1776			1055	1492	1812	1918	1945

h(1812) = 4, but T[4] is occupied. Rehash(4) = (4+1) mod 8 = 5, which is empty, so 1812 is stored in T[5].
h(1918) = 6
h(1945) = 5, but T[5] is occupied. Rehash(5) = 6, T[6] is also occupied, so 6 is rehashed to 7, which is empty.

```
Search(T,key)
i = h(key);
j = 0; // counter of rehash
inc = hashInc(key); // for a general increment scheme
while (T[i] ≠ nil and j < m)
   if (T[i] = key)
         then return i;   // successful search
   i = rehash(i, inc); // i = i+1 for linear probing
   j = j+1;
return nil;   // unsuccessful search
```


- Theorem 11.6 Given an open-address hash table with load factor $\alpha = n/m < 1$, the expected number of probes in an unsuccessful search is at most $1/(1-\alpha)$, assuming uniform hashing.

e.g., In a half full table, 1/(1-.5) = 2; In a 90% full table, 1/(1-.9) = 10.

- Theorem 11.8 Given an open-address hash table with load factor $\alpha < 1$, the expected number of probes in a successful search is at most $1/\alpha \ln [1/(1-\alpha)]$, assuming uniform hashing and assuming that each key in the table is equally likely to be searched for.

e.g., in a half full table, it's <1.387; in a 90% full table, it's < 2.559

Choice of Hash Functions

- Distribute keys uniformly into integer range [0, 1, …, m].
- Low collision rate.
- Hashing method I: division

  h(k) = k mod m

  - Must avoid certain values of m.
    - Powers of 2. If $m = 2^p$, h(k) is p lowest order bits of k.

      e.g., $m = 8 = 2^3$, $0 \le k \le 128$

      k = (107) = 1101<u>011</u>, h(k) = 011 = 3

      k = (43) = 0101<u>011</u>, h(k) = 011 = 3

      … xxxx<u>011</u>,

      there are 16 collisions on h(k) = 3.
    - Powers of 10. similar argument.
  - Good values for m are primes not too close to exact power of 2.

- Hashing method II: multiplication

  $h(k) = \lfloor m(k A \bmod 1) \rfloor$

where A is a constant, 0< A < 1, and (k A mod 1) is the fractional part of kA, namely, $kA - \lfloor kA \rfloor$.

e.g., $A = (\sqrt{5} - 1)/2 \approx 0.6180339887…$

m = 10000

$h(123456) = \lfloor 10000 \times (123456 \times 0.61803… \bmod 1) \rfloor$

$= \lfloor 10000 \times (76300.0041151.. \bmod 1) \rfloor$

$= \lfloor 10000 \times 0.0041151.. \rfloor$

$= \lfloor 41.151… \rfloor$

= 41.

- Optimal choice of A depends on characteristics of data (Knuth suggests the golden ratio)
- Choose m as power of 2.

Summary

- Hash tables are an effective data structure for implementing dictionaries.
- Worst-case: search may take as long as $\Theta(n)$ time.
- Average-case: O(1).