CISC 320 Introduction to Algorithms
Fall 2005

Lecture 3

Recurrences and Master theorem

General scheme for time complexity analysis

1. For a sequence of blocks, add up the cost of individual blocks
    1. For a loop, the worst case = the loop range times the cost of a single iteration
2. With alternation, take the cost of the most costly branch
3. If recursive procedure called, add T(n'), where n' is the size at call.

Recursion for computation

A computation model is *Turing complete* when it can compute everything that can be computed by a Turing machine.

Pragmatically, a model (or a language) is Turing complete if it can do

- sequence
- branch
- repetition (either as loop or as recursion)

Recursion

- is as *powerful* as iteration in establishing a Turing complete model.
- is ***proof-friendly*** for proving correctness of algorithms. (Thus promoted in functional programming languages, such as ML).
    - Why? (Free of "Computing by Side Effect" problems using iterations)
- is also efficient.
    - Myth: Loop is much faster than recursion
    - Truth: recursion can be as efficient as iteration.

Note: any algorithm using recursion can be converted to using iterations, and vice versa.

Iterations can be converted as recursions

For example, Sequential Search can be implemented recursively

```
int seqSearchRec(int[] E, int m, int num, int K)
    int ans;
1   if (m >= num)
2     ans = -1;
3   else if (E[m] == K)
4     ans = m;
5   else
6     ans = seqSearchRec(E, m+1, num, K);
7   return ans;
```


For example, the recursive Sequential Search can be analyzed using this scheme

```
int seqSearchRec(int[] E, int m, int num, int K)
    int ans;
1   if (m >= num)
2     ans = -1;
3   else if (E[m] == K)
4     ans = m;
5   else
6     ans = seqSearchRec(E, m+1, num, K);
7   return ans;
```

Let n = num –m as the initial size

T(n) = 1 + max(0, 1+ max(0, T(num-(m+1)))) + 0 = T(n-1) + 2

(Line1, Line2, Line 3, Line4, Line6, Line7)

Divide and Conquer

E.g., Binary search of an ordered array.

Modify the seqSearchRec to do binary search. If the recursive implementation of sequential search is superficial, a recursive implementation of binary search is a real convenience (as compared to a loop based implementation).

$T(n) = T(n/2) + \Theta(1)$.

In general, the cost of solving a problem of size n is shared by the cost of a subproblems of size n/b, plus non-recursive overhead cost f(n):

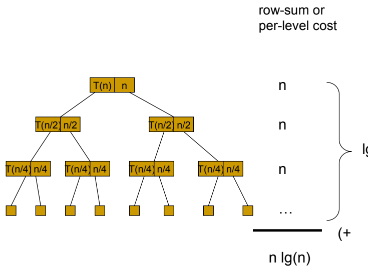
$$T(n) = a\,T(n/b) + f(n)$$

This is a recurrence equation.

How to evaluate the cost T(n)?

Recursion-tree method

Example: $T(n) = T(n/2) + T(n/2) + n$



Observations of recursion-tree method

1. $T(n)$ = the sum of the nonrecursive costs of all nodes in the tree, which is the sum of the per-level costs at all levels;
2. Depth of the tree is $D = \log_b n$;
3. Number of leaves is approximately $L = a^D = n^E$ where $E = \log_b a$;
4. If the per-level costs remain about constant at all depth, then $T(n) \in \Theta(f(n) \log(n))$.
5. If the per-level costs grow *fast*, the cost at the leaves would dominate, therefore $T(n) \in \Theta(n^E)$;
6. If the per-level costs decrease *fast*, the cost at the root would dominate, therefore $T(n) \in \Theta(f(n))$;
7. And more formally,

The Master theorem (Theorem 4.1)

The recurrence equation

$$T(n) = a\,T(n/b) + f(n),$$

where $a \geq 1$, $b > 1$, and n/b interpreted as either $\lceil n/b \rceil$ or $\lfloor n/b \rfloor$. Then $T(n)$ can be bounded asymptotically as follows:

1. If $f(n) = O(n^{E-\varepsilon})$ for constant $\varepsilon > 0$, then $T(n) = \Theta(n^E)$ where $E = \log_b a$, called critical exponent. *(Note: this means n^E is polynomially faster than f(n).)*
2. If $f(n) = \Theta(n^E)$, then $T(n) = \Theta(f(n) \log(n))$.
3. If $f(n) = \Omega(n^{E+\varepsilon})$ for $\varepsilon > 0$, and if $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$. *(Note: this means f(n) is polynomially faster than n^E.)*

Example 1

$T(n) = 7T(n/2) + n^2$

1. Recognize a, b, and f(n):
 $a = 7$, $b = 2$, and $f(n) = n^2$
2. Compute $E = \log_b a = \lg(7)$
3. Compare f(n) and n^E asymptotically
 $f(n) = n^{\lg 7 + (2 - \lg 7)} = n^{\lg 7 - 0.8} = O(n^{E-0.8})$
4. Apply appropriate case of Master Theorem
 case 1 applies: $T(n) = \Theta(n^{\lg 7})$

Example 2

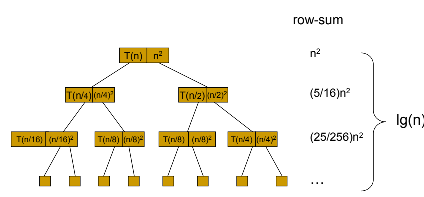
$T(n) = 4T(n/2) + n^2 \lg(n)$

1. Recognize a, b, and f(n):
 $a = 4$, $b = 2$, and $f(n) = n^2\lg(n)$
2. Compute $E = \log_b a = \lg(4) = 2$
3. Compare f(n) and n^E asymptotically
 $f(n)/ n^E = n^2 \lg(n) / n^2 = \lg(n)$
4. Determine appropriate case of Master Theorem and apply
 case 1 : $f(n)/ n^E = \lg(n)$ =? $O(n^{-\varepsilon})$ for some $\varepsilon > 0$ NO
 case 2 : $f(n)/ n^E = \lg(n)$ =? $\Theta(1)$ NO
 case 3 : $f(n)/ n^E = \lg(n)$ =? $\Omega(n^{\varepsilon})$ for some $\varepsilon > 0$ NO
 Note: $\lg(x)$ is faster than $\Theta(1)$ but slower than x^{ε} for any $\varepsilon > 0$ (Exercise).

Lesson: **There are gaps between cases in Master Theorem, therefore Master Theorem does not cover all recurrence equations of that form**.

Example 3

$T(n) = T(n/4) + T(n/2) + n^2$



Exercise: $T(n) = n^2(1 + 5/16 + (5/16)^2 + \ldots) \leq (16/11)\,n^2 = \Theta(n^2)$

Substitution method

- Make a guess
- Substitute it into the recurrence
- Prove the recurrence hold by mathematical induction

Example: $T(n) = T(n/4) + T(n/2) + n^2$.

From decision tree method, we have a good guess that $T(n) = O(n^2)$.

Let $T(n) \le cn^2$, where c is a suitable positive constant.

Plug it into the RHS of the recurrence.

$T(n) \le c (n/4)^2 + c(n/2)^2 + n^2 = (c/16 + c/4 + 1) n^2 \le cn^2$, when $c \ge 16/5$

Induction Proofs

A mechanic procedure with mainly 3 steps

Step 1: prove base case(s), e.g., n=0.

Step 2: assume the goal is true for arbitrary n, say n=k.

Step 3: then prove it is also true for n=k+1.

Example: $\Sigma_{i=1}^{n} i = n(n+1)/2$

Base case n = 1

LHS = 1 and RHS = 1(1+1)/2 = 1

Note: we can do this manually for n =2, 3, …

Let's assume it holds for arbitrary $n \ge 1$, we now prove it also holds for n+1.

$$\begin{aligned} LHS(n+1) &= \Sigma_{i=1}^{n+1} i = (\Sigma_{i=1}^{n} i) + (n+1) \\ &= n(n+1)/2 + (n+1) \\ &= [n(n+1) + 2(n+1)]/2 \\ &= (n+1)[n+2]/2 \\ &= RHS(n+1) \end{aligned}$$

Since we have proved manually it is true when n=1. Now we know if it is true for n=1 it must be true for n =2, and if it is true for n=2 it must be true for n=3, and on and on.

Note: such a procedure is like to unravel a recursive call in a reversed order, i.e., from base case to more general cases.