Topological sort

**Theorem:** A directed graph $G$ is acyclic if and only if a DFS of $G$ yields no back edges.

**Proof:**
- If DFS($G$) yields a back edge $(u,v)$, then there is a cycle in $G$.
- Suppose $G$ has a cycle $c$. Edge $(u,v)$ is part of $c$, and $v$ is the first vertex on $c$ discovered by DFS($G$). All other vertices on $c$ form a white path from $v$ to $u$. By White path theorem, $u$ is a descendant of $v$ and edge $(u,v)$ becomes a back edge. (Because DFS-visit($v$) won’t return to $v$ until all reachable vertices are reached. When it reaches $u$, (u,v) is a back edge.)

### Running Time
- DFS part: $O(V+E)$
- Insert each of the $|V|$ vertices onto the front of the linked list: $O(V)$
- Total time: $O(V+E)$

### Correctness:
If $G$ is a dag, then for any edge $(u,v) \in E \Rightarrow f[u] > f[v]$ when $(u,v)$ is explored, $u$ is GRAY.
1. $v$ is also GRAY. Then $(u,v)$ is a back edge. This means $G$ is not a dag.
2. $v$ is WHITE. Then $v$ is a descendant of $u$. Therefore $v$ is finished before $u$, namely $f[v] < f[u]$.
3. $v$ is BLACK. Then $v$ is already finished, i.e., $f[v] < f[u]$.

**Strongly connected component**

**Definition:** A SCC of a digraph $G = (V,E)$ is a maximal set of vertices $U \subseteq V$ such that every pair of vertices are reachable from each other in $G$ (why not just in $G$?).

**Q:** Is it possible that $u$ and $v$ are in a SCC and there are edges $(u,x)$ and $(x,v)$, but $x$ is not in the same SCC?

**SCC($G$)**
1. Call DFS($G$) to compute finishing times $f[u]$ for each vertex $u$.
2. Compute $G^T$.
3. Call DFS($G^T$), but in the main loop of DFS, consider the vertices in order of decreasing $f[u]$ (as computed in step 1), that is, take vertices from the finishStack.
4. Output vertices of each tree (in the depth-first forest of step 3) as a separate SCC.
Exercise: Condensation graph (or called component graph) is a dag.

Exercise: Design an algorithm to determine whether or not a given undirected graph contains a cycle. Your algorithm should run in $O(V)$ time, independent of $|E|$. 