Supplementary Material

Reformulation of Traditional Poisson Surface/Image Completion Analogous to Our Angular Surface Completion. Our spherical coordinate based surface reconstruction method is based on formulating an over-determined linear system. To show that our formulation is valid, we first prove that traditional spatial-domain Poisson surface integration can be formulated in the same way.

Consider a surface represented as a height field $u = f(x, y)$. Given its gradient field $(p, q) = (u_x, u_y)$, traditional Poisson surface completion aims to find the optimal surface $v = f^*(x, y)$ that satisfies the Poisson Equation $\Delta f^* = p_x + q_y$, where $(p_x, q_y) = (u_{xx}, u_{yy})$. To solve this equation, they then discretize the spatial domain into a $m \times n$ as $(x_i, y_j), i = 1, ..., m; j = 1, ..., n$ and then find height at each grid $v_{i,j}$. The Δ operator can be replaced with the Laplacian and p_x and p_y can be computed using first order differential. Therefore, traditional approach formulates a large linear system $\mathbf{A_P}\mathbf{\Omega} = \mathbf{b_P}$, where:

$$\begin{array}{c} \quad \begin{array}{ccccccccc} ... & col_{i-1,j} & ... & col_{i,j-1} & col_{i,j} & col_{i,j+1} & ... & col_{i+1,j} & ... \end{array} \\ \mathbf{A_P} = \begin{pmatrix} \ddots & ... & \ddots & \ddots & \ddots & ... & \ddots & ... & ... \\ \mathbf{0} & 1 & \mathbf{0} & 1 & -4 & 1 & \mathbf{0} & 1 & \mathbf{0} \\ ... & ... & \ddots & ... & \ddots & \ddots & \ddots & ... & \ddots \end{pmatrix} \end{array}$$

$$\mathbf{\Omega} = \begin{pmatrix} \vdots \\ v_{i-1,j} \\ \vdots \\ v_{i,j-1} \\ v_{i,j} \\ v_{i,j+1} \\ \vdots \\ v_{i+1,j} \\ \vdots \end{pmatrix} \mathbf{b_P} = \begin{pmatrix} \vdots \\ (p_{i,j} - p_{i-1,j}) + (q_{i,j} - q_{i,j-1}) \\ \vdots \end{pmatrix}$$

We show that the problem can be (much easily) reformulated as to solve an over-determined linear system. For every grid point, we have

$$\begin{cases} v_{i+1,j} - v_{i,j} = p_{i,j} \\ v_{i,j+1} - v_{i,j} = q_{i,j} \end{cases} \tag{1}$$

We can then stack all these equations to obtain a linear system $\tilde{\mathbf{A}}_{\mathbf{P}}\mathbf{\Omega} = \tilde{\mathbf{b}}_{\mathbf{P}}$, where

$$\tilde{\mathbf{A}}_{\mathbf{P}} = \begin{array}{c} \\ \vdots \\ row_l \\ row_k \\ row_s \\ row_t \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccccccc} ... & col_{i-1,j} & ... & col_{i,j-1} & col_{i,j} & col_{i,j+1} & ... & col_{i+1,j} & ... \end{array} \\ \left(\begin{array}{ccccccccc} ... & ... & ... & ... & ... & ... & ... & ... & ... \\ ... & -1 & & 0 & 1 & 0 & ... & 0 & ... \\ ... & 0 & ... & -1 & 1 & 0 & ... & 0 & ... \\ ... & 0 & ... & 0 & -1 & 1 & ... & 0 & ... \\ ... & 0 & ... & 0 & -1 & 0 & ... & 1 & ... \\ ... & ... & ... & ... & ... & ... & ... & ... & ... \end{array}\right) \end{array}$$

$$\mathbf{\Omega} = \begin{pmatrix} \vdots \\ v_{i-1,j} \\ \vdots \\ v_{i,j-1} \\ v_{i,j} \\ v_{i,j+1} \\ \vdots \\ v_{i+1,j} \\ \vdots \end{pmatrix} \quad \tilde{\mathbf{b}}_{\mathbf{P}} = \begin{array}{c} \vdots \\ row_l \\ row_k \\ row_s \\ row_t \\ \vdots \end{array} \begin{pmatrix} \vdots \\ p_{i-1,j} \\ q_{i,j-1} \\ p_{i,j} \\ q_{i,j} \\ \vdots \end{pmatrix}$$

In $\tilde{\mathbf{A}}_{\mathbf{P}}$, every two rows correspond to Eq. (1). Since a point only has relation with its four neighbors, every column only has four non-zero elements (1,1,-1,-1). However, $\tilde{\mathbf{A}}_{\mathbf{P}}$ is not a square matrix as every grid maps to two equations, thus forming an over-determined linear system. To solve this system, we can simply apply SVD as $\tilde{\mathbf{A}}_{\mathbf{P}}^{\top}\tilde{\mathbf{A}}_{\mathbf{P}}\mathbf{\Omega} = \tilde{\mathbf{A}}_{\mathbf{P}}^{\top}\tilde{\mathbf{b}}_{\mathbf{P}}$.

We prove that the linear system obtained by SVD based approach is identical to the Poisson solution, i.e., $\tilde{\mathbf{A}}_{\mathbf{P}}^{\top}\tilde{\mathbf{A}}_{\mathbf{P}} = \mathbf{A}_{\mathbf{P}}$ and $\tilde{\mathbf{A}}_{\mathbf{P}}^{\top}\tilde{\mathbf{b}}_{\mathbf{P}} = \mathbf{b}_{\mathbf{P}}$.

$$
\tilde{\mathbf{A}}_{\mathbf{P}}^{\top} = \begin{matrix} \\ \vdots \\ row_{i-1,j} \\ \vdots \\ row_{i,j-1} \\ row_{i,j} \\ row_{i,j+1} \\ \vdots \\ row_{i+1,j} \\ \vdots \end{matrix} \overset{\begin{matrix} \cdots & col_l & col_k & col_s & col_t & \cdots \end{matrix}}{\begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & -1 & 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & 0 & -1 & 0 & 0 & \cdots \\ \cdots & 1 & 1 & -1 & -1 & \cdots \\ \cdots & 0 & 0 & 1 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & 0 & 0 & 0 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}}
$$

Without loss of generality, let us randomly pick a row, e.g., $row_{i,j}$ in $\tilde{\mathbf{A}}_{\mathbf{P}}^{\top}$ for computing $\tilde{\mathbf{A}}_{\mathbf{P}}^{\top}\tilde{\mathbf{A}}_{\mathbf{P}}$. Notice that there are only four non-zero elements (1,1,-1,-1) in $row_{i,j}$ at $col_l, col_k, col_s, col_t$. As for $\tilde{\mathbf{A}}_{\mathbf{P}}$, only $col_{i-1,j}, col_{i,j-1}, col_{i,j}, col_{i,j+1}, col_{i+1,j}$ have non-zero elements on $row_l, row_k, row_s, row_t$. We therefore only need to consider these five column in $\tilde{\mathbf{A}}_{\mathbf{P}}$ when multiplying it with $row_{i,j}$ in $\tilde{\mathbf{A}}_{\mathbf{P}}$ to produce non-zero elements in $\tilde{\mathbf{A}}_{\mathbf{P}}^{\top}\tilde{\mathbf{A}}_{\mathbf{P}} = \mathbf{A}_{\mathbf{P}}$. We therefore have:

$$
\tilde{\mathbf{A}}_{\mathbf{P}}^{\top}\tilde{\mathbf{A}}_{\mathbf{P}} = \begin{matrix} \\ \vdots \\ row_{i,j} \\ \vdots \end{matrix} \overset{\begin{matrix} \ldots & col_{i-1,j} & \ldots & col_{i,j-1} & col_{i,j} & col_{i,j+1} & \ldots & col_{i+1,j} & \ldots \end{matrix}}{\begin{pmatrix} \ddots & \ldots & \ddots & \ddots & \ddots & \ldots & \ddots & \ldots & \ldots \\ \mathbf{0} & -1 & \mathbf{0} & -1 & 4 & -1 & \mathbf{0} & -1 & \mathbf{0} \\ \ldots & \ldots & \ddots & \ldots & \ddots & \ddots & \ddots & \ldots & \ddots \end{pmatrix}}
$$

For $\tilde{\mathbf{A}}_{\mathbf{P}}^{\top}\tilde{\mathbf{b}}_{\mathbf{P}}$, we also take $row_{i,j}$ in $\tilde{\mathbf{A}}_{\mathbf{P}}$ to multiply with $\tilde{\mathbf{b}}_{\mathbf{P}}$ and we have:

$$
\tilde{\mathbf{A}}_{\mathbf{P}}^{\top}\tilde{\mathbf{b}}_{\mathbf{P}} = \begin{matrix} \vdots \\ row_{i,j} \\ \vdots \end{matrix} \begin{pmatrix} \vdots \\ (p_{i-1,j} - p_{i,j}) + (q_{i,j-1} - q_{i,j}) \\ \vdots \end{pmatrix}
$$

This reveals that the linear system obtained by the SVD solution is identical to the one from the Poisson equation.

The proof illustrates that a simple solution for solving the shape from gradient/normal field is to formulate an over-determined linear system. In the same vein, we formulate our spherical coordinate based shape from normal as a linear system and solve for the optimal radius at each angular using SVD.