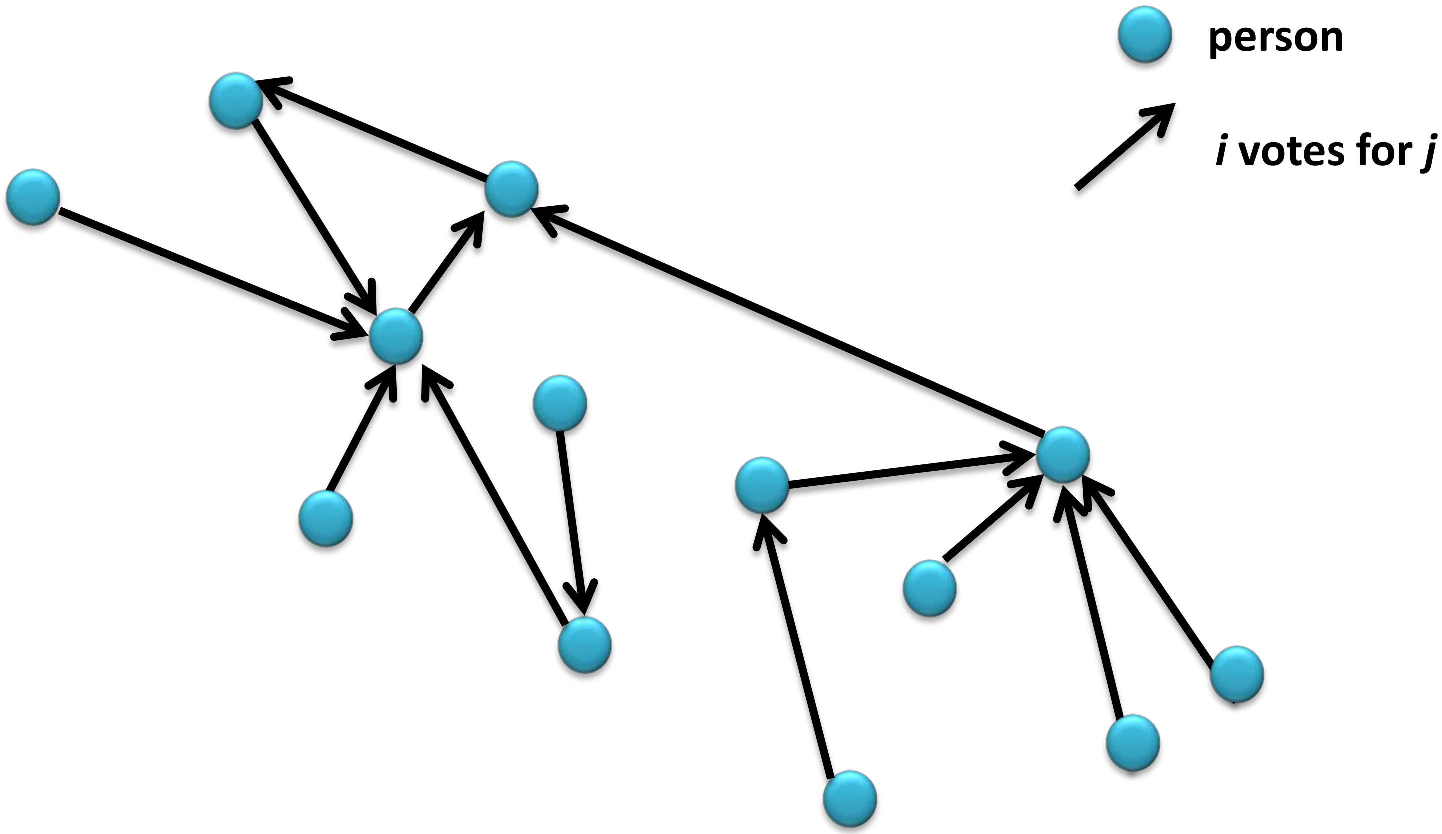
Some ideas behind the feedback centralities



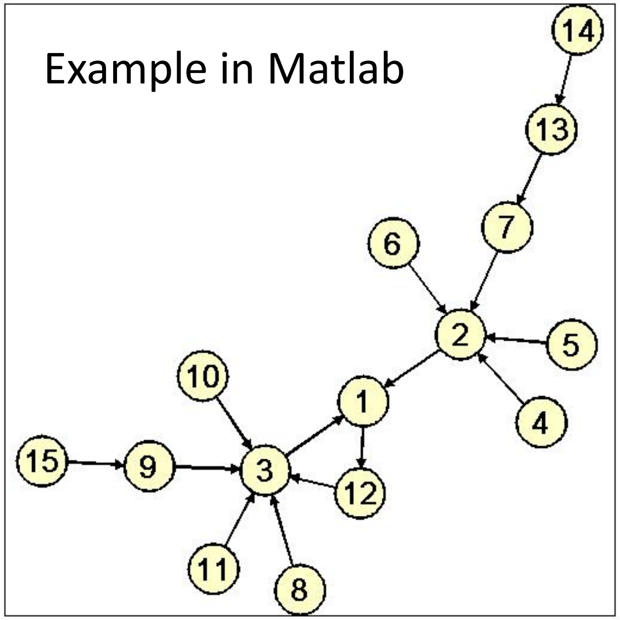
Counting All Paths

Simple, directed $G = (V, E)$, no loops

$$\forall i \in V \;\; c_K(i) = \sum_{k=1}^{\infty} \sum_{j=1}^{n} \alpha_k (A^k)_{ji}$$

The sum converges with restricted α_k.

Example in Matlab



Theorem. *If A is the adjacency matrix of G, $\alpha > 0$, and λ_1 the largest eigenvalue of A, then*

$$\lambda_1 < 1/\alpha \iff \sum_{k=1}^{\infty} \alpha^k A^k \textit{ converges}$$

and $c_K = (I - \alpha A)^{-1} \cdot \mathbf{1}_n$.

Leo Katz "A new status index derived from sociometric analysis"

Network Flow

A flow network is given by a directed graph $G = (V, E)$, edge capacity fucntion $u : E \to \mathbb{R}_{\geq 0}$, and two distinct nodes $s, t \in V$. A flow from s to t is a function $f : E \to \mathbb{R}$ satifying the following constraints

- Capacity: $\forall e \in E : \; 0 \leq f(e) \leq u(e)$
- Balance: $\forall v \in V \setminus \{s, t\}$:

$$\sum_{e \in \Gamma^-(v)} f(e) = \sum_{e \in \Gamma^+(v)} f(e)$$

The value of the flow f is defined as $\sum_{e \in \Gamma^+(s)} f(e) - \sum_{e \in \Gamma^-(s)} f(e)$. Computing a flow of a maximum value is important problem. Goldberg and Tarjan solved it in $O(nm \log(n^2/m))$. Ford-Fulkerson theorem says that the value of a maximum s-t-flow = the capacity of minimum s-t-cut.

Vitality (robustness)

Let $\mathcal{G}$ be the set of all simple, undirected and unweighted graphs $G = (V, E)$ and $f : \mathcal{G} \to \mathbb{R}$ be any real-valued function on $G \in \mathcal{G}$. A vitality index $\mathcal{V}(G, x)$ is the difference of the values of f on G and on G without element x, i.e., $\mathcal{V}(G, x) = f(G) - f(G \setminus x)$.

Max-flow Betweenness Vitality

Q: How much flow must go over a vertex *i* in order to obtain the maximum flow value? How does the objective function value change if we remove *i* from the network?

$$c_{mf}(i) = \sum_{\substack{s,t \in V \\ i \neq s, i \neq t \\ f_{st} > 0}} \frac{f_{st}(i)}{f_{st}}, \text{ where } f_{st}(i) = f_{st} - \text{ max s-t-flow in } G \setminus i$$

.

?

Examples of vitality: power grids, social networks with no leader, collaboration networks

Closeness Vitality

Wiener index of a network

i-j shortest path

$$I_W(G) = \sum_{i,j \in V} \delta_{ij}$$

or in terms of closeness centrality

$$I_W(G) = n \cdot \sum_{i \in V} \frac{1}{C_i}$$

Closeness vitality is defined on both vertices and edges

$$c_{CV}(x) = I_W(G) - I_W(G \setminus \{x\})$$

Computational problem with this vitality index?

Stress Centrality as a Vitality Index

$\sigma_{st}(i)$ is a number of *s*-*t* shortest paths containing *i*

σ_{st} is a number of all *s*-*t* shortest-paths

- **Stress Centrality**

$$c_S(i) = \sum_{s \neq i} \sum_{t \neq i} \sigma_{st}(i)$$

for nodes

$$c_S(ij) = \sum_{s \in V} \sum_{t \in V} \sigma_{st}(ij)$$

for edges

Can be interpreted as the number of shortest paths that are lost if the vertex or edge is removed from the graph. However, … (what can be the problem?)

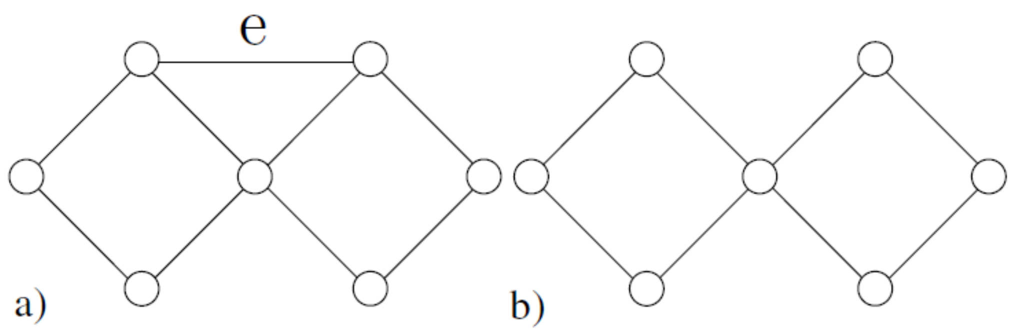


Fig. 3.9. The figure shows that the removal of an edge can actually increase the number of shortest paths in a graph

Homework (grads only; bonus for undergrads): 1) check at home; 2) when will the removal of an edge lead to an increase in the edge number? 3) any solution to this problem? (submit by 2/3/2014)

Current Flow

- Electrical network is an undirected, simple, connected graph *G = (V, E)*
- Conductance function *c:E* → R
- Supply function *b:V* → R (external electrical current enters and leaves network)
- Positive *b* = entering current
- Negative *b* = leaving current

  $\sum_{i \in V} b(i) = 0$

- Direction of the current: each edge *ij* in *E* is oriented arbitrarily

  Fucntion $x : \overline{E} \to \mathbb{R}$ is called current if

$$\sum_{ij \in \overline{E}} x_{ij} - \sum_{ji \in \overline{E}} x_{ji} = b(i) \text{ and } \sum_{ij \in C} x_{\overline{ij}} = 0$$

for every cycle $C \subset E$. ← undirected

A function $p : V \to \mathbb{R}$ is a potential if $p(i) - p(j) = x_{ij}/c_{ij}$ for all $ij \in \overline{E}$. As an electrical network $N = (G, c)$ has a unique current x for any supply b, it also has a potential p that is unique up to an additive factor.

Given edge weights *c(i)* , *w*e define electrical network Laplacian *L.*

We can find *p* and *b* by solving *Lp=b.*

Current-Flow Betweenness Centrality

Unit $s-t$-supply b_{st} is a supply of one unit that enters the network at s and leaves at t, that is, $b_{st}(s)=1$, $b_{st}(t)=-1$, and $b_{st}(i)=0$ for all $i \in V \setminus \{s,t\}$.

Throughput of $i \in V$ with respect to a unit $s-t$-supply b_{st} is defined as

$$\tau_{st}(i) = \frac{1}{2}\Big(-|b_{st}(i)| + \sum_{ij \ni i} |x(-\overline{ij})| \Big)$$

$$c_{CB}(i) = \frac{1}{(n-1)(n-2)} \sum_{s,t \in V} \tau_{st}(i)$$

Paper review: M. Newman "A measure of betweenness centrality based on random walks"
Submit by 2/4/2014

How to compare different centrality concepts?

Normalization in one network

p-norm of the centrality vector for concept $\mathbf{X}$

$$||c_{\mathbf{X}}||_p = \begin{cases} (\sum_{i=1}^{n} |c_{\mathbf{X}i}|^p)^{1/p} & 1 \leq p \leq \infty \\ \max_i\{|c_{\mathbf{X}i}|\} & p = \infty \end{cases} \implies \frac{c_{\mathbf{X}}}{||c_{\mathbf{X}}||_p} \implies c_{\mathbf{X}i} \leq 1$$

separation of positive and negative values of $c_{\mathbf{X}}$

$$c'_{\mathbf{X}} = \begin{cases} c_{\mathbf{X}i}/(\sum_{j:c_{\mathbf{X}j}>0} |c_{\mathbf{X}j}|^p)^{1/p} & c_{\mathbf{X}i} > 0 \\ 0 & c_{\mathbf{X}i} = 0 \\ c_{\mathbf{X}i}/(\sum_{j:c_{\mathbf{X}j}<0} |c_{\mathbf{X}j}|^p)^{1/p} & c_{\mathbf{X}i} < 0 \end{cases}$$

Exercise (do not submit): Is $c'_{\mathbf{X}}$ a norm? Prove or disprove.

Freeman "Centrality in social networks: Conceptual clarification"

Normalization for different networks

Point-centrality

$$c''_{\mathbf{X}i} = c_{\mathbf{X}i} / \big(\max_{G \in \mathcal{G}_n} \max_{i \in V(G)} c_{\mathbf{X}i} \big)$$

set of all graphs with *n* vertices

Examples

- Degree centrality = normalization by factor *(n-1)*
- Shortest paths betweenness centrality $c_B(i) = \sum_{s \neq i} \sum_{t \neq i} \frac{\sigma_{st}(i)}{\sigma_{st}}$

  What is the upper bound (or normalization factor)?

  Star graph, $c_B(i) = (n-1)(n-2)/2$
- Closeness centrality

  $\forall i \in V \;\; C_i = 1/l_i$, where $l_i = \frac{1}{n} \sum_{j \in V} \delta_{ij}$, δ_{ij} = length of $i - j$ shortest path

  What is the upper bound (or normalization factor)? It is *1/(n-1)*