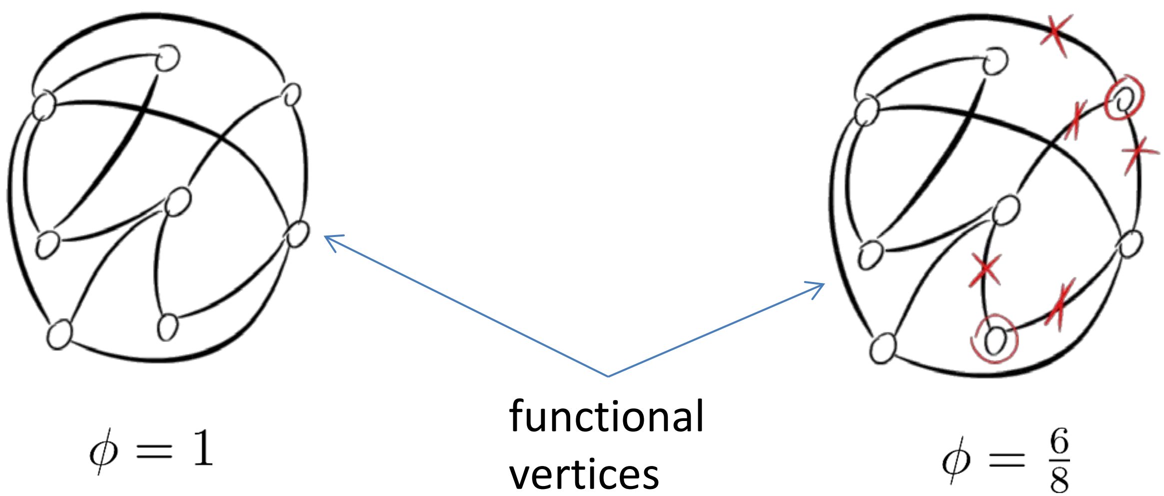
Percolation and Network Resilience

Percolation is a process of removing some fraction of network's nodes with adjacent edges.
(more precisely site/link/cluster percolation)

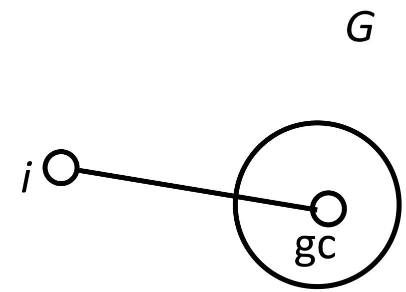
- models real-life phenomena such as router failure, immunization of people, and disasters
- the process is parameterized by **occupation probability** ϕ
- **Percolation transition**: when ϕ is large there is a giant component but as ϕ is decreased then gc breaks into many small components or clusters (similar to phase transition in Poisson random graphs with gc→sc)



Percolation and Configuration Model

Consider a configuration model with
- degree distribution p_k
- occupation probability ϕ

consider node i which



- can belong to gc, i.e., connected to it through some $j \in N(i)$
- is not in gc (i.e., not connected to it via any of $N(i)$) - define the avg probability of it u^k, where $k = deg(i)$ and u is the same prob for one particular neighbor
- avg probability of not being in gc

generating function for the degree distribution

$$\sum_k p_k u^k = g_0(u), \text{ where } g_0(z) = \sum_k^{\infty} p_k z^k \text{ or } \Pr[i \in \text{gc}] = 1 - g_0(u)$$

- total fraction of nodes in gc when percolation is running

$$S = \phi(1 - g_0(u))$$

Let us calculate u, the probability that i is not connected to gc via a particular neighbor. Two cases:

- i is connected to j which is removed with prob $1-\phi$
- or j is not removed with prob ϕ but it is not in gc

j is on and its *k* neighbors are not in gc

$$\Pr[i \notin gc \text{ via } j] = 1-\phi+\phi u^k$$

Node j is reached by following an edge, so average probability

$$u = \sum_{k=0}^{\infty} q_k(1-\phi+\phi u^k) = 1-\phi+\phi+\sum q_k u^k = 1-\phi+\phi g_1(u)$$

(see handout for graphical solution of the equation)

[Cohen, Erez, Ben-Avraham, Halvin]: $\phi_c = \frac{1}{g_1'(1)} = \frac{\langle k\rangle}{\langle k^2\rangle - \langle k\rangle}$

minimum fraction of nodes that must be occupied in configuration model for a gc to exist

Reminder: excess degree is the number of edges attached to a vertex other than the edge we arrived along.

$$q_k = \frac{(k+1)\,p_{k+1}}{\langle k\rangle}, \quad \sum_k q_k = 1, \quad g_1(z) = \sum_{k=0}^{\infty} q_k z^k$$

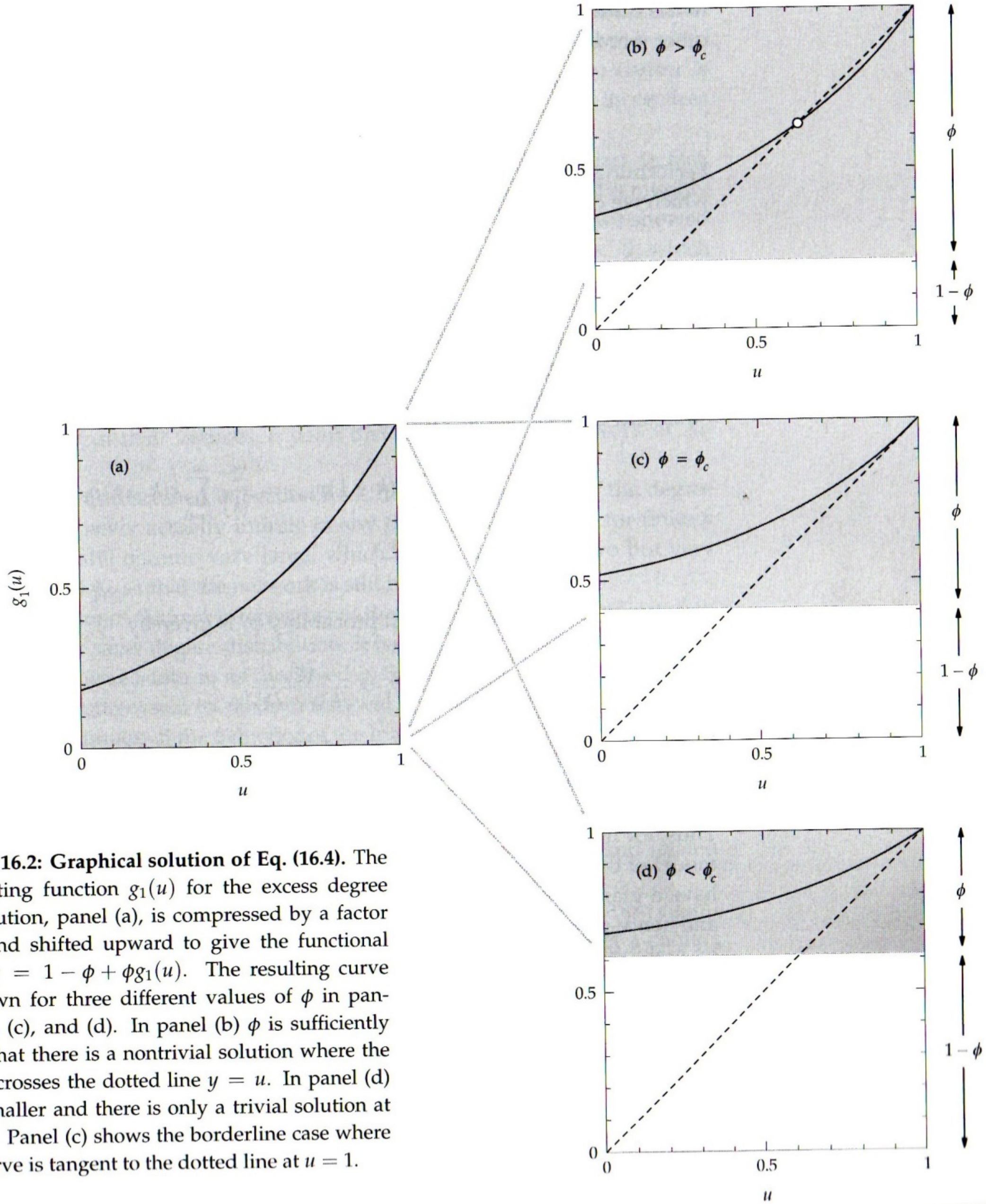


Figure 16.2: Graphical solution of Eq. (16.4). The generating function $g_1(u)$ for the excess degree distribution, panel (a), is compressed by a factor of ϕ and shifted upward to give the functional form $y = 1 - \phi + \phi g_1(u)$. The resulting curve is shown for three different values of ϕ in panels (b), (c), and (d). In panel (b) ϕ is sufficiently large that there is a nontrivial solution where the curve crosses the dotted line $y = u$. In panel (d) ϕ is smaller and there is only a trivial solution at $u = 1$. Panel (c) shows the borderline case where the curve is tangent to the dotted line at $u = 1$.

[Cohen, Erez, Ben-Avraham, Halvin]: $\phi_c = \frac{1}{g_1'(1)} = \frac{\langle k \rangle}{\langle k^2 \rangle - \langle k \rangle}$

- In configuration model fixing low ϕ_c leads to gc, for example $\langle k^2 \rangle > \langle k \rangle$
- Example: given Poisson degree distribution with c - mean degree

$$p_k = e^{-c}\frac{c^k}{k!} \;\Rightarrow\; \langle k \rangle = c,\; \langle k^2 \rangle = c(c+1) \;\Rightarrow\; \phi_c = \frac{1}{c}$$

i.e., $c = 4$ means that $\frac{3}{4}$ vertices will fail before gc disappears.

[Cohen, Erez, Ben-Avraham, Halvin]: $\phi_c = \frac{1}{g_1'(1)} = \frac{\langle k \rangle}{\langle k^2 \rangle - \langle k \rangle}$

- Example: Exponential degree distribution $p_k = (1 - e^{-\lambda}) e^{-\lambda k}$, $\lambda > 0$

$$g_0(z) = \frac{e^\lambda - 1}{e^\lambda - z},\ g_1(z) = \left(\frac{e^\lambda - 1}{e^\lambda - z}\right)^2 \Rightarrow$$

$$u\left(e^\lambda - u\right)^2 - (1-\phi)\left(e^\lambda - u\right)^2 - \phi\left(e^\lambda - 1\right)^2 = 0 \Rightarrow$$

u=1 is always a solution and (u-1) is always factor

$$u = e^\lambda - \frac{1}{2}\phi - \sqrt{\frac{1}{4}\phi^2 + \phi(e^\lambda - 1)} \Rightarrow$$

size of gc

$$S = \frac{3}{2}\phi - \sqrt{\frac{1}{4}\phi^2 + \phi(e^\lambda - 1)} \Rightarrow$$

percolation threshold

$$\phi_c = \frac{1}{2}(e^\lambda - 1)$$

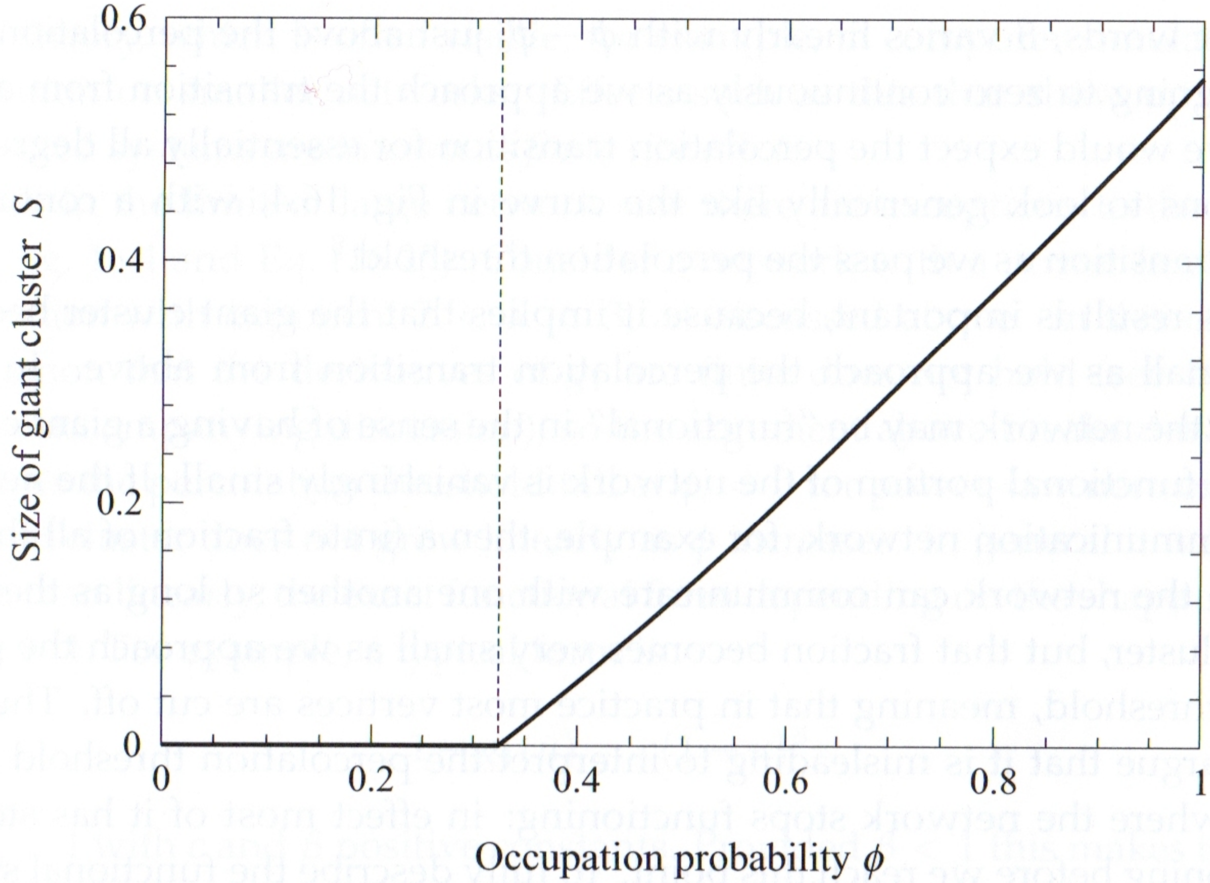


Figure 16.4: Size of the giant cluster for site percolation in the configuration model. The curve indicates the size of the giant cluster for a configuration model with an exponential degree distribution of the form (16.12) with $\lambda = \frac{1}{2}$, as given by Eq. (16.18). The dotted line indicates the position of the percolation transition, Eq. (16.20).

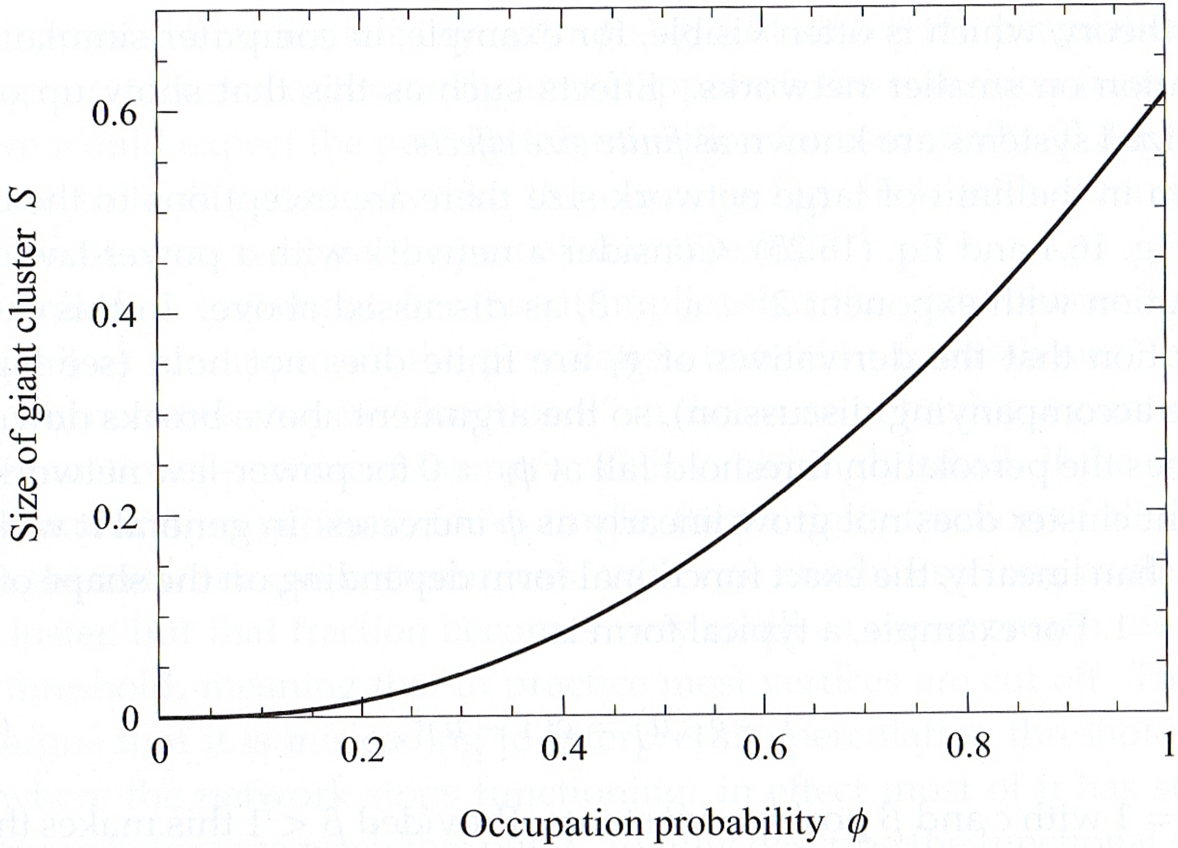


Figure 16.5: Size of the giant cluster for a network with power-law degree distribution. The size of the giant cluster for a scale-free configuration model network with exponent $\alpha = 2.5$, a typical value for real-world networks. Note the non-linear form of the curve near $\phi = 0$, which means that S, while technically non-zero, becomes very small in this regime. Contrast this figure with Fig. 16.4 for the giant cluster size in a network with an exponential degree distribution.

Geometric graph: percolation example

Non-uniform Removal of Vertices

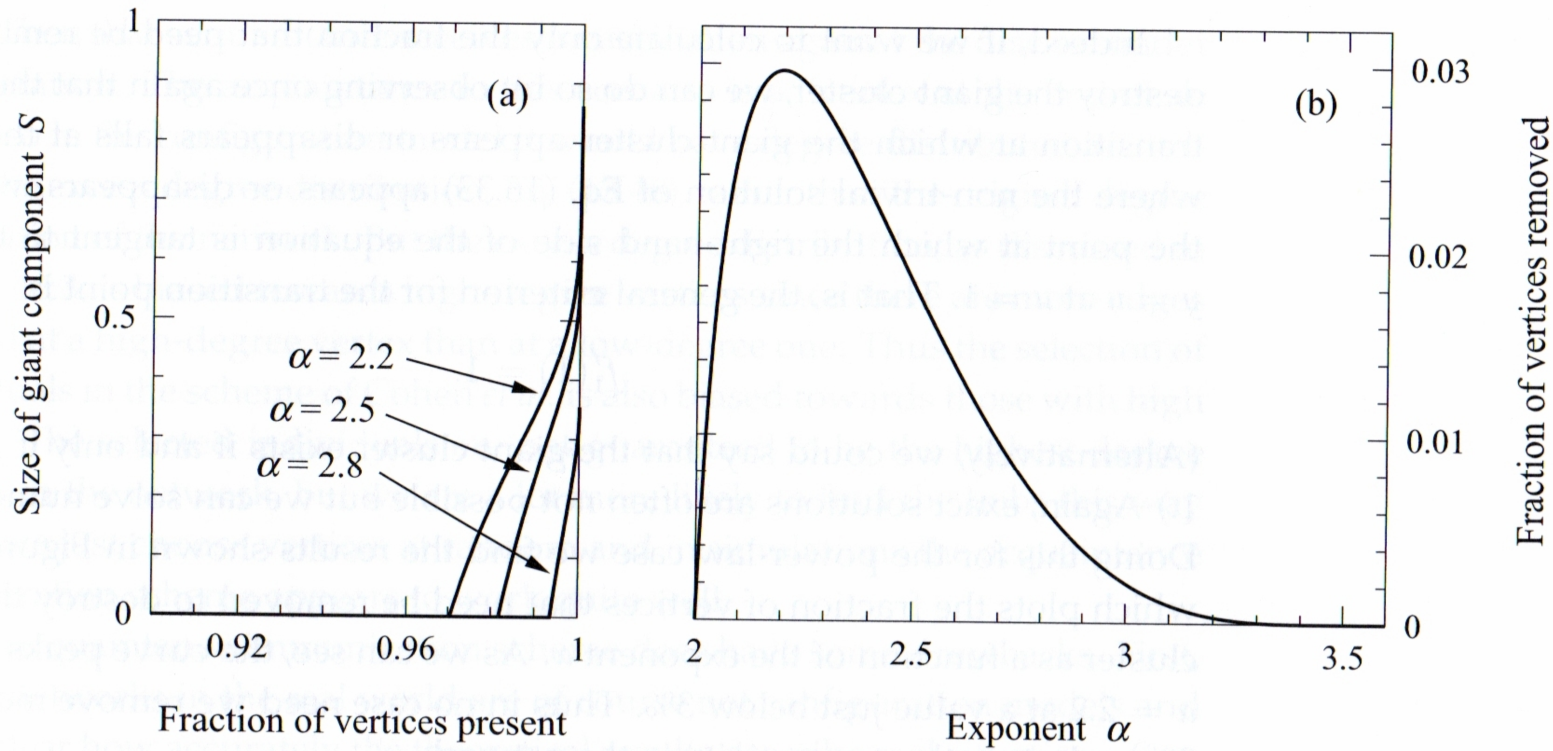


Figure 16.7: Removal of the highest-degree vertices in a scale-free network. (a) The size of the giant cluster in a configuration model network with a power-law degree distribution as vertices are removed in order of their degree, starting with the highest-degree vertices. Only a small fraction of the vertices need be removed to destroy the giant cluster completely. (b) The fraction of vertices that must be removed to destroy the giant cluster as a function of the exponent α of the power-law distribution. For no value of α does the fraction required exceed 3%.

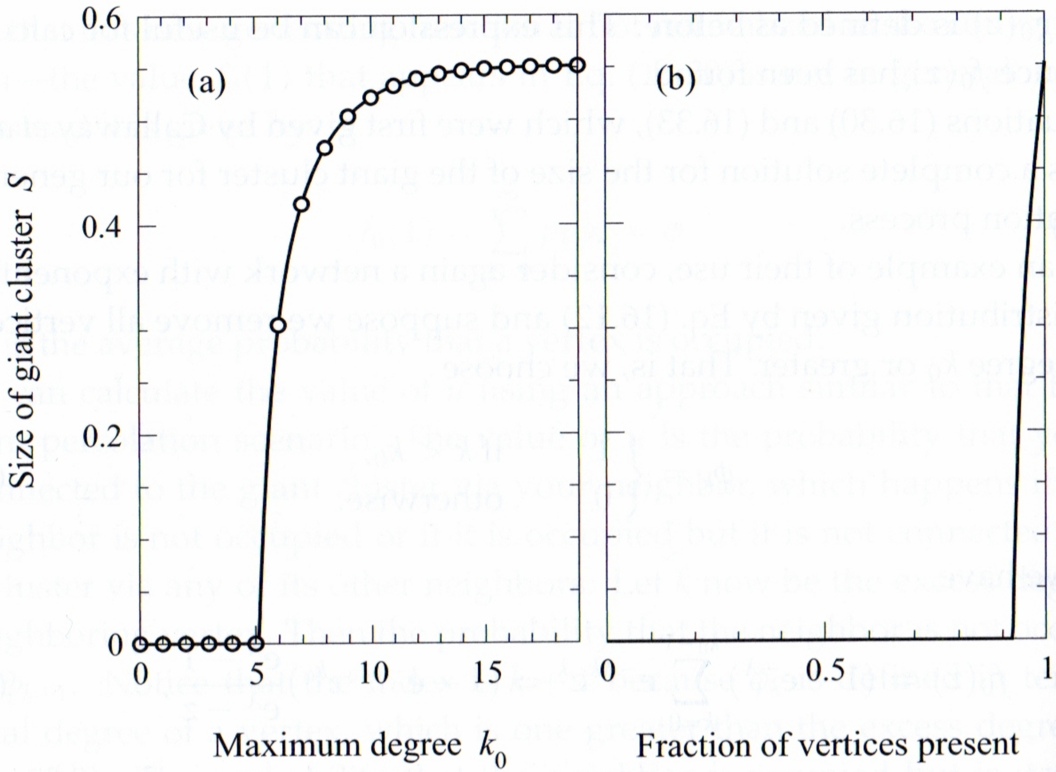


Figure 16.6: Size of the giant percolation cluster as the highest degree vertices in a network are removed. (a) The size of the giant cluster in a network with an exponential degree distribution $p_k \sim e^{-\lambda k}$ with $\lambda = \frac{1}{2}$ as vertices are removed in order of degree, starting from those with the highest degree. The curve is shown as a function of the degree k_0 of the highest-degree vertex remaining in the network. Technically, since k_0 must be an integer, the plot is only valid at the integer points marked by the circles; the curves are just an aid to the eye. (b) The same data plotted now as a function of the fraction $\overline{\phi}$ of vertices remaining in the network.