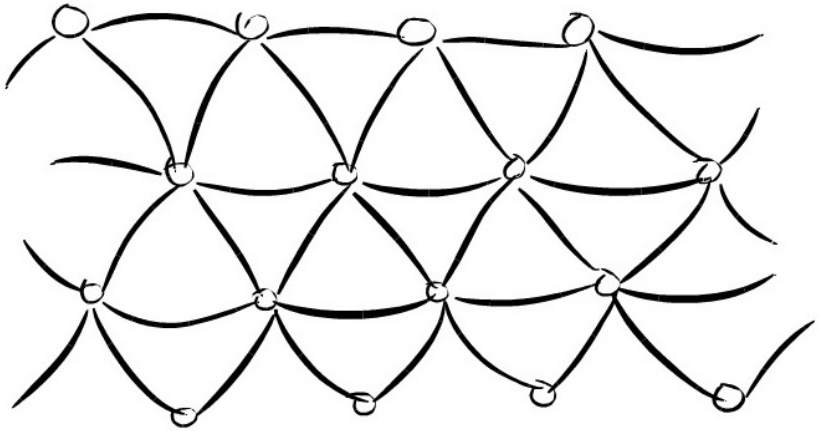
Small Worlds and High Clustering Coefficient

Network	n	z	C measured	C for random graph
Internet [153]	6,374	3.8	0.24	0.00060
World Wide Web (sites) [2]	153,127	35.2	0.11	0.00023
power grid [192]	4,941	2.7	0.080	0.00054
biology collaborations [140]	1,520,251	15.5	0.081	0.000010
mathematics collaborations [141]	253,339	3.9	0.15	0.000015
film actor collaborations [149]	449,913	113.4	0.20	0.00025
company directors [149]	7,673	14.4	0.59	0.0019
word co-occurrence [90]	460,902	70.1	0.44	0.00015
neural network [192]	282	14.0	0.28	0.049
metabolic network [69]	315	28.3	0.59	0.090
food web [138]	134	8.7	0.22	0.065

N. Przulj. Graph theory analysis of protein-protein interactions

Simple Models with High Clustering Coefficient



Triangular lattice

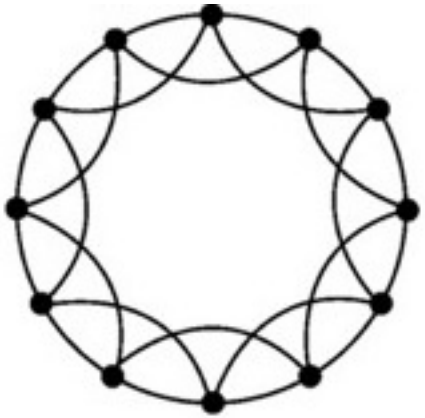
$$c_i = triangles/triples = 6/15 = 0.4$$
$$C \to 0.4$$

One-dimensional line

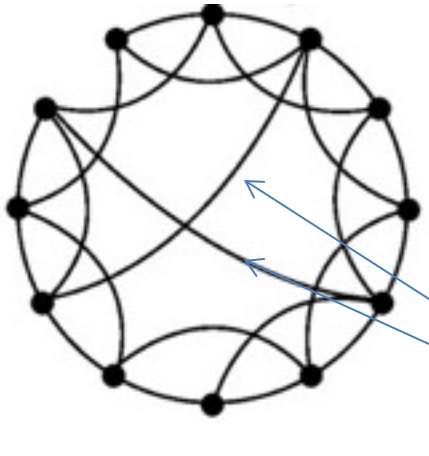
Clustering coefficient depends on the number of connected 1d line neighbors

Small-World Model

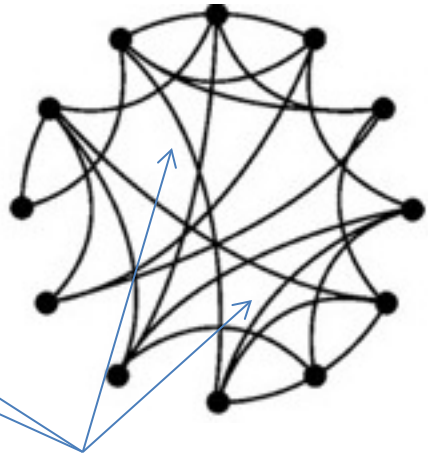
High C, High Diam



High C, Low Diam



Low C, Low Diam



Randomly rerouted (or added) edges
with probability *p* (for each of the edges in circle)

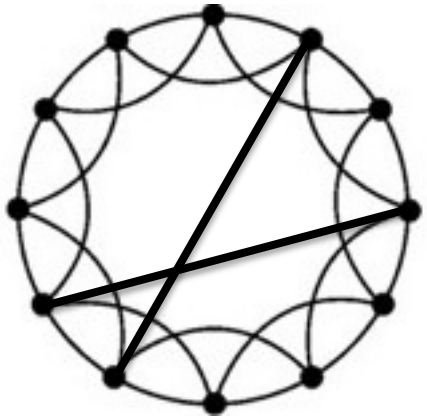
Circle models
are not "small-world"

Random models
have small clustering
coefficients

Watts-Strogatz Model (1998)

- Given circle models with *n* nodes
- Go through all edges, remove each with probability *p*
  and then add new edge uniformly at random

Small-World Model (without edge removal)



If c is a degree in circle model and p is a prob of an edge then

- $\frac{1}{2}ncp$ short-cuts in new graph
- cp ends of new short-cuts at each node on the average
- s number of short-cuts is Poisson distributed over all nodes

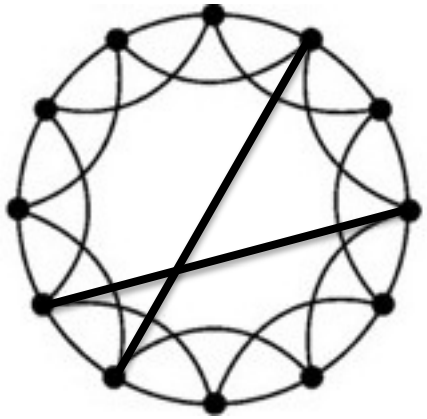
$$p_s = e^{-cp}\frac{(cp)^s}{s!}$$

- node degree $k = s + c$, then degree distribution of small-world models is

$$p_k = e^{-cp}\frac{(cp)^{k-c}}{(k-c)!},$$

where $p_k = 0$ if $k < c$.

Small-World Model (without edge removal)



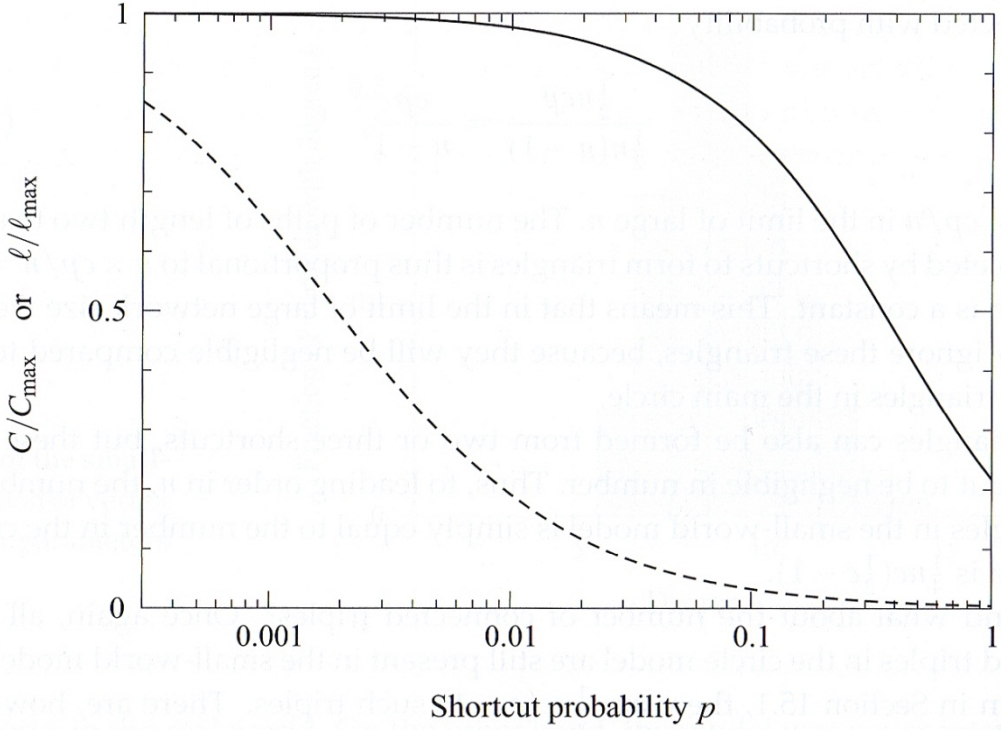


Figure 15.5: Clustering coefficient and average path length in the small-world model. The solid line shows the clustering coefficient, Eq. (15.7), for a small-world model with $c = 6$ and $n = 600$, as a fraction of its maximum value $C_{\max} = \frac{3}{4}(c-2)/(c-1) = 0.6$, plotted as a function of the parameter p. The dashed line shows the average geodesic distance between vertices for the same model as a fraction of its maximum value $\ell_{\max} = n/2c = 50$, calculated from the mean-field solution, Eq. (15.14). Note that the horizontal axis is logarithmic.

Homework (review by 4/29/2014): Watts and Strogatz “Collective dynamics of ‘small-world’ networks”, 1998

Additional material: Kleinberg “Small-world phenomenon: an algorithmic perspective”, 2000

Exponential Random Graphs Model

Instead of analyzing one network with fixed parameters, it is useful to consider ensembles of networks that are similar to the original.

Let us fix *average* values of some network properties (such as clustering and modularity). Possible property of an ensemble: values closer to the averages have higher probability. Define

$$\sum_{G \in \mathcal{G}} \Pr(G) = 1$$

graphs with *n* nodes

For network measure x_i, $1 \leq i \leq M (\ll 2^{n(n-1)/2})$

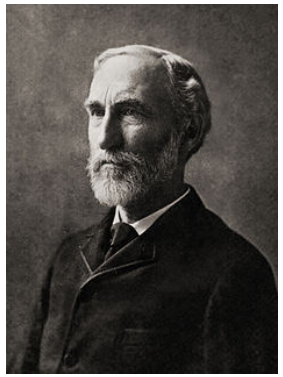
$$\langle x_i \rangle = \sum_{G \in \mathcal{G}} \Pr(G) x_i(G)$$

i.e., if $\Pr(G)$ are variables then such systems do not describle the system completely.

How to choose $\Pr(G)$?

Best choice of probability distribution given a small number of constraints maximizes Gibbs entropy

$$S = -\sum_{G \in \mathcal{G}} \Pr(G) \ln \Pr(G)$$



J. Willard Gibbs
1839-1903

Maximization of entropy with Lagrange multipliers

$$\max \quad -\sum_{G \in \mathcal{G}} \Pr(G) \ln \Pr(G) - \alpha(1 - \sum_{G \in \mathcal{G}} \Pr(G)) - \sum_i \beta_i (\langle x_i \rangle - \sum_{G \in \mathcal{G}} \Pr(G) x_i(G))$$

Differentiate wrt $P(G)$ of a particular G

$$-\ln \Pr(G) - 1 + \alpha + \sum_i \beta_i x_i(G) = 0$$

or

$$\Pr(G) = \exp(\alpha - 1 + \sum_i \beta_i x_i(G)) \;\Rightarrow\; \Pr(G) = \frac{e^{H(G)}}{Z},$$

where $Z = e^{1-\alpha}$ and $H(G) = \sum_i \beta_i x_i(G)$ is the graph Hamiltonian.

Z is solved by normalization

$$\sum_{G\in\mathcal{G}} \Pr(G) = \frac{1}{Z}\sum_{G\in\mathcal{G}} e^{H(G)} = 1$$

β_i are solved by substituting $\Pr(G) = \frac{e^{H(G)}}{Z}$ into $\sum_{G\in\mathcal{G}} \Pr(G)x_i(G) = \langle x_i \rangle$

In general β_i can play a role of importance coefficients.

Practice

If we have $\Pr(G)$ over graphs let us estimate useful quantities. For property y

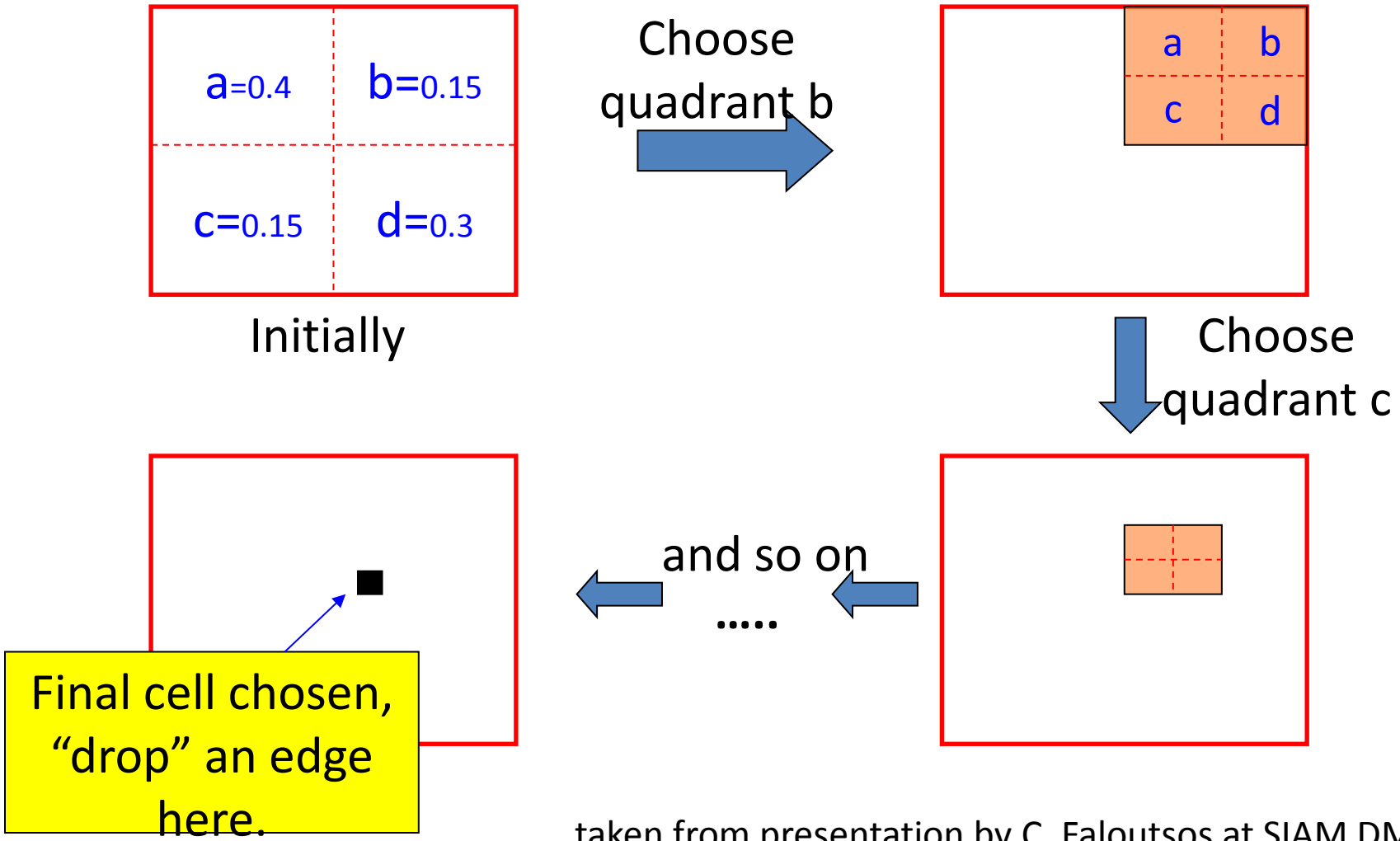
$$\langle y \rangle = \sum_{G\in\mathcal{G}} \Pr(G)y(G) = \frac{1}{Z}\sum_{G\in\mathcal{G}} e^{H(G)}y(G)$$

Example: Fix the expected number of edges only. Then $H = \beta m$ and individual graphs appear with prob

$\Pr(G) = \frac{e^{\beta m}}{Z}$, where $Z = \sum_G e^{\beta m}$ $\Rightarrow$ higher β correspond to denser networks

R-Mat Generator

by Chakrabarti, Zhang, Faloutsos



taken from presentation by C. Faloutsos at SIAM DM04

R-Mat Generator

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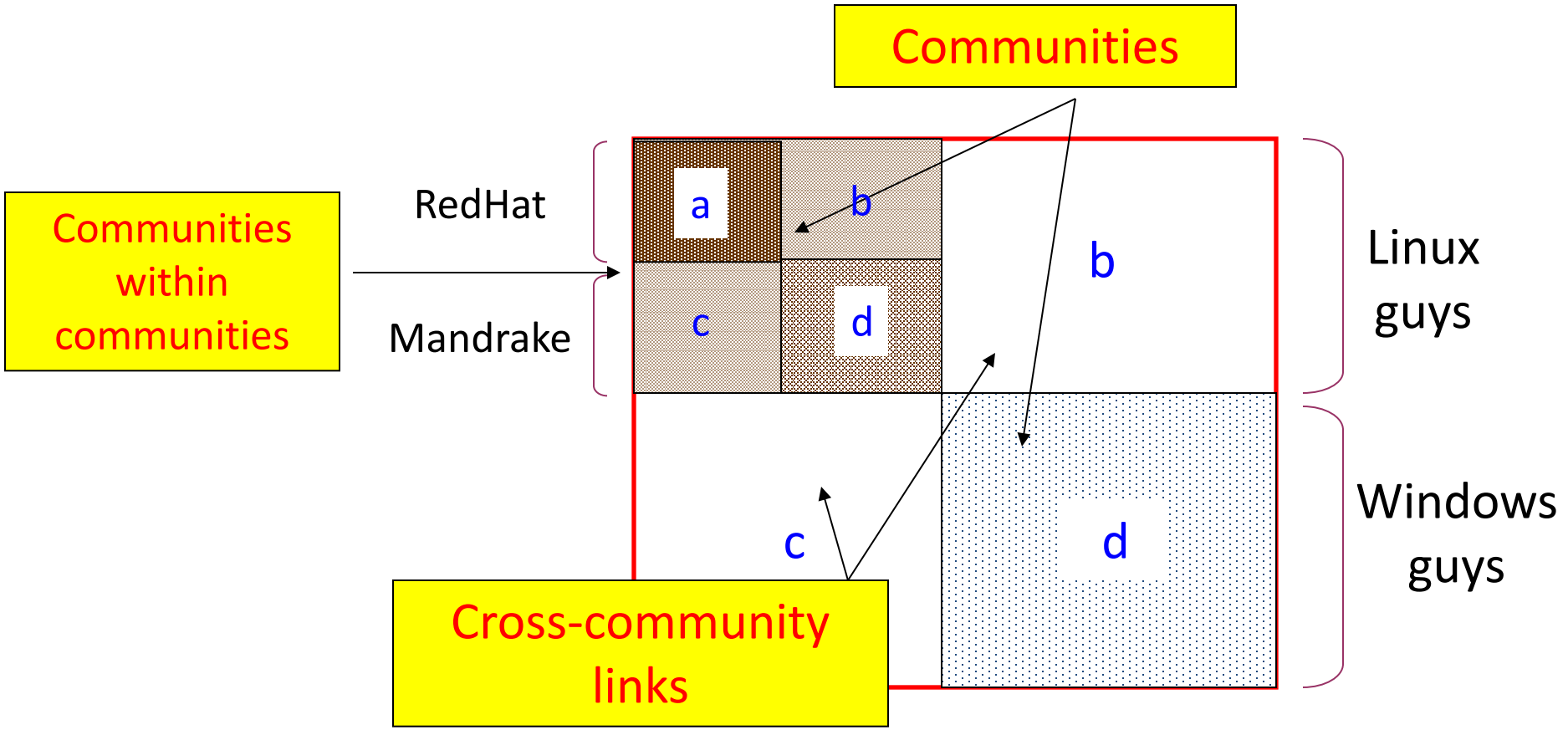


taken from presentation by C. Faloutsos at SIAM DM04

Kronecker Graphs

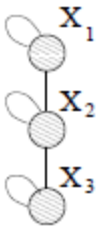
Definition 1 (Kronecker product of matrices) *Given two matrices* $\mathbf{A} = [a_{i,j}]$ *and* $\mathbf{B}$ *of sizes* $n \times m$ *and* $n' \times m'$ *respectively, the Kronecker product matrix* $\mathbf{C}$ *of dimensions* $(n \cdot n') \times (m \cdot m')$ *is given by*

$$\mathbf{C} = \mathbf{A} \otimes \mathbf{B} \doteq \begin{pmatrix} a_{1,1}\mathbf{B} & a_{1,2}\mathbf{B} & \dots & a_{1,m}\mathbf{B} \\ a_{2,1}\mathbf{B} & a_{2,2}\mathbf{B} & \dots & a_{2,m}\mathbf{B} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1}\mathbf{B} & a_{n,2}\mathbf{B} & \dots & a_{n,m}\mathbf{B} \end{pmatrix}$$

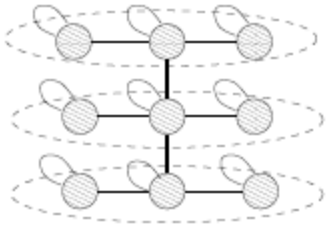
We then define the Kronecker product of two graphs simply as the Kronecker product of their corresponding adjacency matrices.

Definition 2 (Kronecker product of graphs (Weichsel, 1962)) *If* G *and* H *are graphs with adjacency matrices* $A(G)$ *and* $A(H)$ *respectively, then the Kronecker product* $G \otimes H$ *is defined as the graph with adjacency matrix* $A(G) \otimes A(H)$.

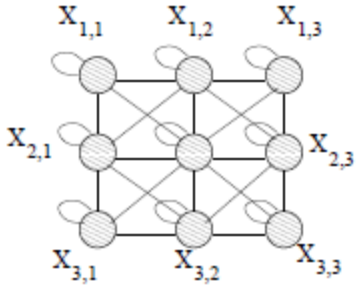
from paper Kronecker Graphs: An approach to modeling networks
by J. Leskovec, D. Chakrabarti, J. Kleinberg, C. Faloutsos, Z. Ghahramani



(a) Graph K_1



(b) Intermediate stage



(c) Graph $K_2 = K_1 \otimes K_1$

1	1	0
1	1	1
0	1	1

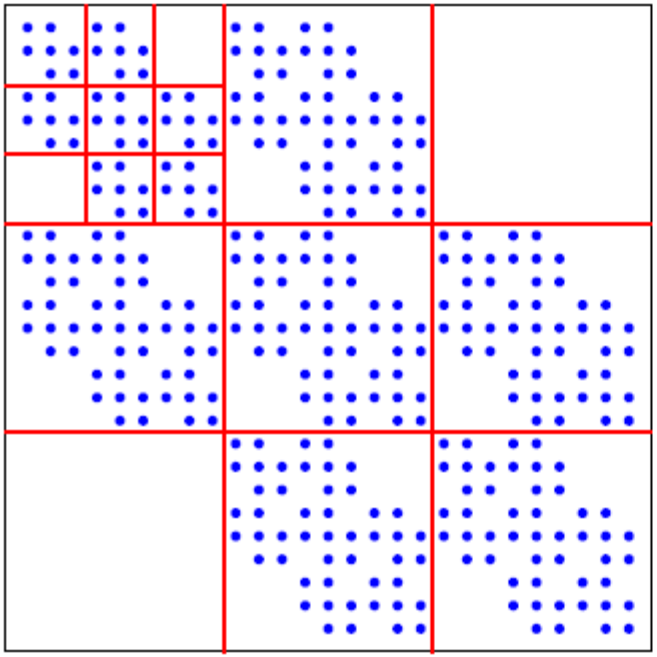
(d) Adjacency matrix of K_1

K_1	K_1	0
K_1	K_1	K_1
0	K_1	K_1

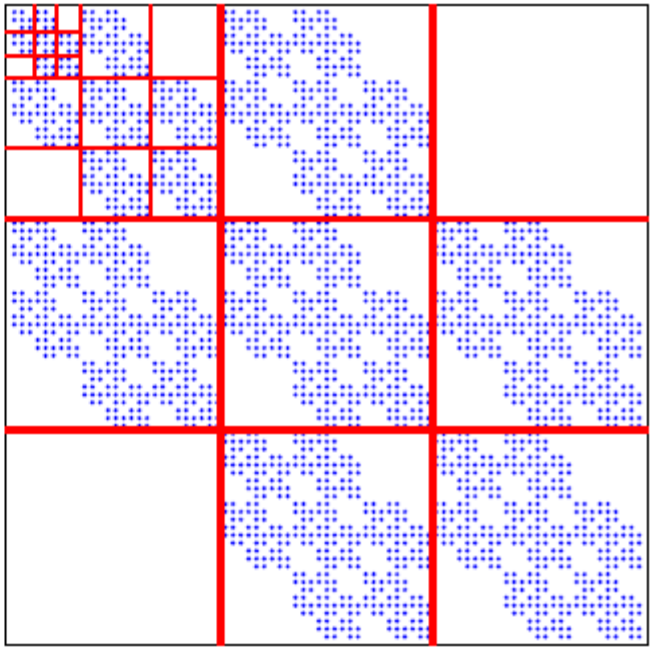
(e) Adjacency matrix of $K_2 = K_1 \otimes K_1$

Example of Kronecker multiplication: Top: a "3-chain" initiator graph and its Kronecker product with itself. Each of the X_i nodes gets expanded into 3 nodes, which are then linked using Observation 1. Bottom row: the corresponding adjacency matrices. See figure 2 for adjacency matrices of K_3 and K_4.

from paper Kronecker Graphs: An approach to modeling networks
by J. Leskovec, D. Chakrabarti, J. Kleinberg, C. Faloutsos, Z. Ghahramani



(a) K_3 adjacency matrix (27×27)



(b) K_4 adjacency matrix (81×81)

Figure 2: Adjacency matrices of K_3 and K_4, the 3^{rd} and 4^{th} Kronecker power of K_1 matrix as defined in Figure 1. Dots represent non-zero matrix entries, and white space represents zeros. Notice the recursive self-similar structure of the adjacency matrix.

from paper Kronecker Graphs: An approach to modeling networks
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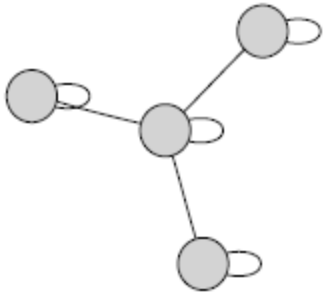
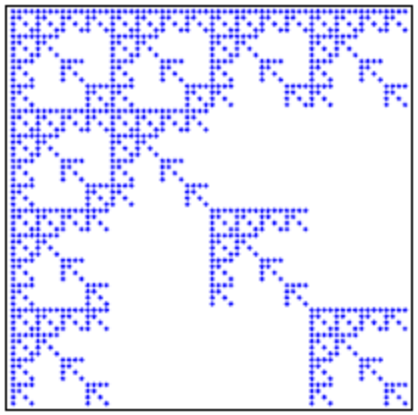
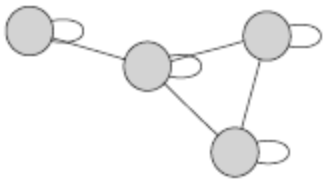
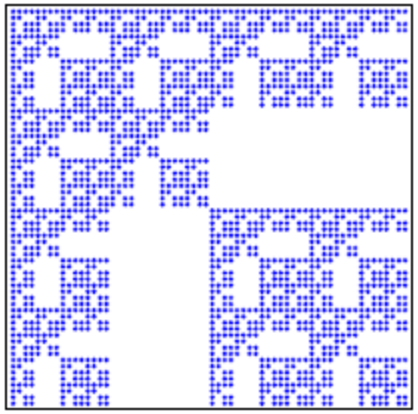


1	1	1	1
1	1	0	0
1	0	1	0
1	0	0	1





1	1	1	1
1	1	0	0
1	0	1	1
1	0	1	1



Initiator K_1 | K_1 adjacency matrix | K_3 adjacency matrix

Figure 3: Two examples of Kronecker initiators on 4 nodes and the self-similar adjacency matrices they produce.

from paper [Kronecker Graphs: An approach to modeling networks]
by J. Leskovec, D. Chakrabarti, J. Kleinberg, C. Faloutsos, Z. Ghahramani