Network Visualization

Hu "*Efficient, High-Quality Force-Directed Graph Drawing*"
Batagelj "Visualization of Large Networks"

Question: How to find a layout for network if nothing is known about its structural properties?

Requirements: flexibility, robustness, clarity

Approach: analogy to physics, i.e., nodes are objects, edges are interactions and forces

Goal: interconnected system at stable configuration = intuitively good layout

One of the solutions: **force-directed methods**

A force-directed method

1. models the graph drawing problem through a physical system of bodies with forces acting between them.
2. The algorithm finds a good placement of the bodies by minimizing the energy of the system.

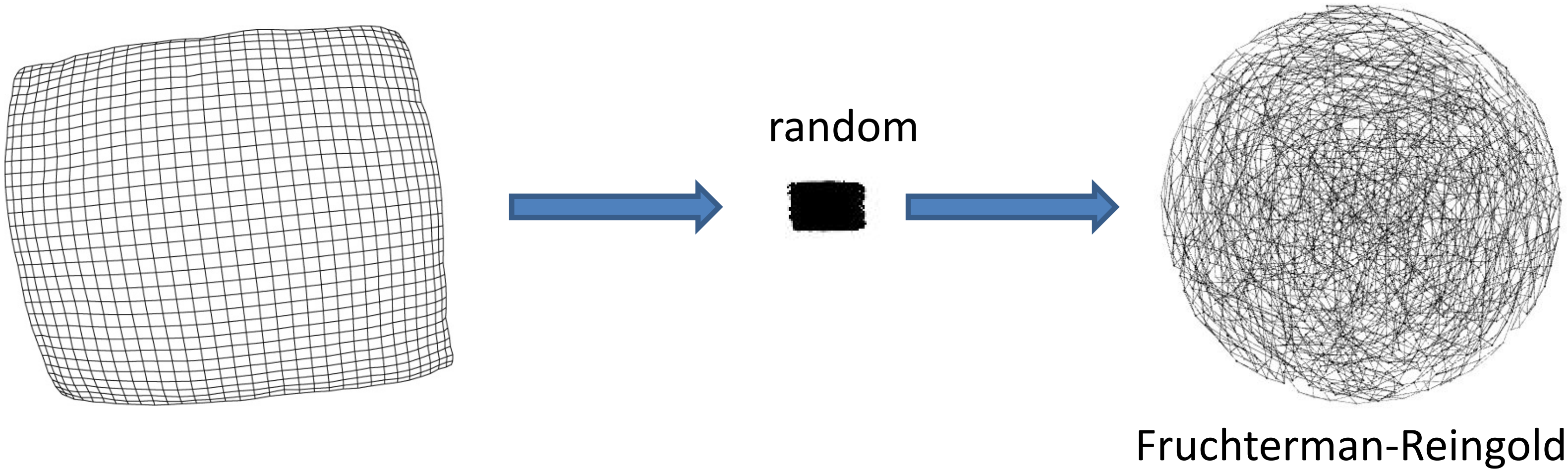
Examples of forces to model

- Fruchterman, Reingold : system of springs between neighbors + repulsive electric forces
- Kamada, Kawai: springs between all vertices with spring length proportional to graph distance

Force-directed methods

Frequent problems that need to be addressed

1. **Many local minimums.** If we start with random configuration we can settle in one of the local minimums already after several iterations



2. **Computational complexity.** Ideally, we should model forces for all pairs of nodes. This gives us complexity O(n^2) per iteration.

Demo: mesh 33 in Gephi with F-R, Force Atlas, Force-Atlas 2

How to overcome these problems? Basic ideas: use multiscale algorithms and limit long-range forces.

Force-directed methods

$x_i \in \mathbb{R}^2$ or $\mathbb{R}^3$ - coordinates of node i
$||x_i - x_j||$ - 2-norm distance between i and j

We define *spring-electrical* modes with two forces

- the repulsive force between any two nodes i and j

$$f_r = -CK^2/||x_i - x_j||, \; i \neq j$$

- the attractive force between any two neighbors i and j

$$f_a = ||x_i - x_j||^2/K$$

The combined force on vertex i is

$$f(i, x, K, C) = \sum_{i \neq j} \frac{-CK^2}{||x_i - x_j||^2}(x_j - x_i) + \sum_{ij \in E} \frac{||x_i - x_j||}{K}(x_j - x_i)$$

Parameters (mostly for scaling): K is spring length, C strength of f_a and f_r.
Example: two connected nodes, f is minimized when $||x_i - x_j|| = KC^{1/3}$.

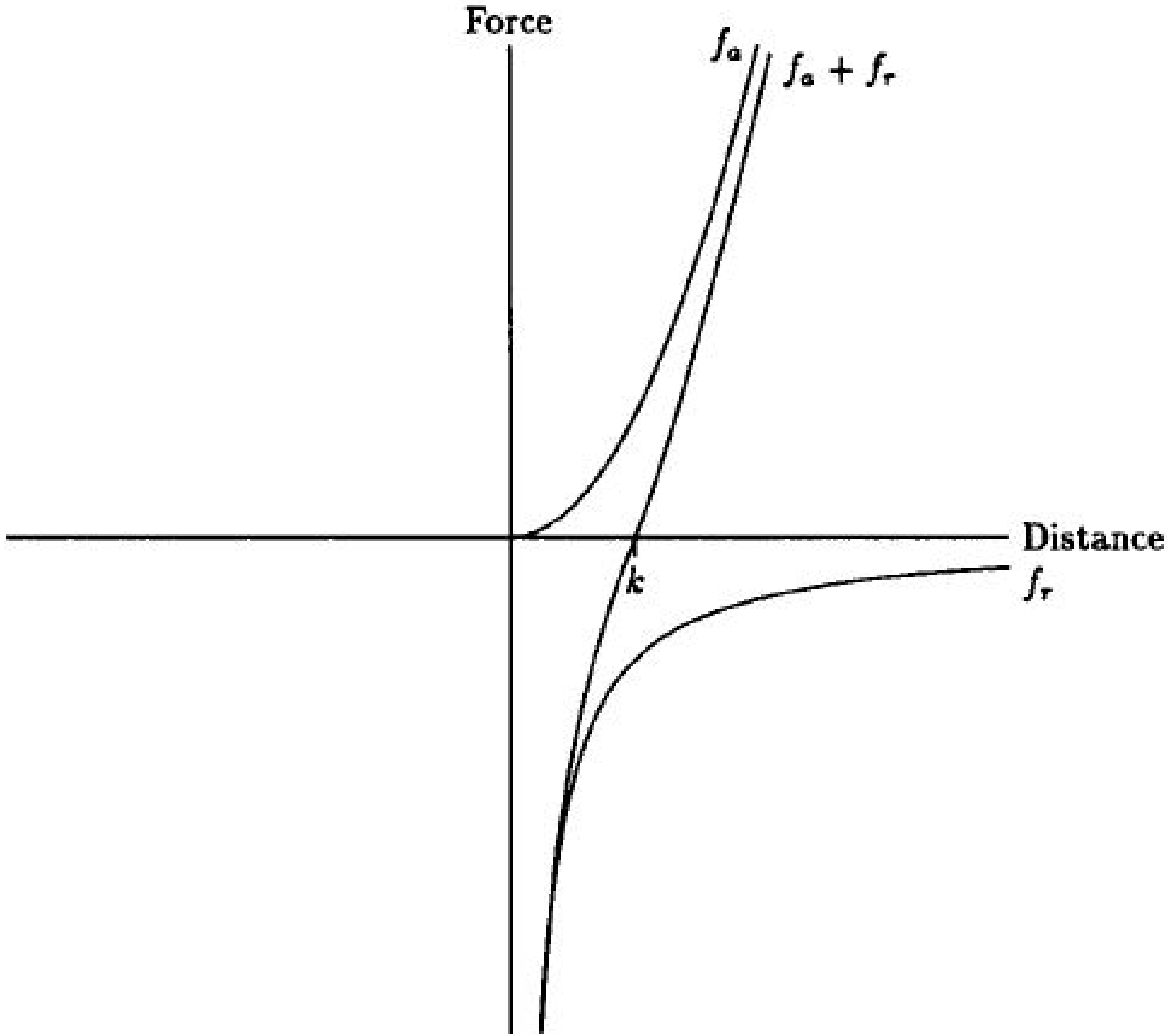


Figure 2. Forces versus distance

Force-directed methods

The total energy of the system is

$$\text{Energy}_{\text{se}}(x, K, C) = \sum_{i \in V} f^2(i, x, K, C)$$

Theorem 1. Let $x^* = \{x_i^* \mid i \in V\}$ minimizes the energy of the spring-electrical model $\text{Energy}_{\text{se}}(x, K, C)$, then sx^* minimizes $\text{Energy}_{\text{se}}(x, K', C')$, where $s = (K' / K)(C' / C)^{1/3}$. Here K, C, K' and C' are all positive real numbers.

Proof: This follows simply by the relationship

$$f(i, x, K, C) = \sum_{i \neq j} \frac{-C K^2}{\| x_i - x_j \|^2} (x_j - x_i) + \sum_{i \leftrightarrow j} \frac{\| x_i - x_j \|}{K} (x_j - x_i)$$

$$= \left(\frac{C}{C'}\right)^{2/3} \frac{K}{K'} \left(\sum_{i \neq j} \frac{-C' (K')^2}{\| s x_i - s x_j \|^2} (s x_j - s x_i) \right.$$

$$\left. + \sum_{i \leftrightarrow j} \frac{\| s x_i - s x_j \|}{K'} (s x_j - s x_i) \right)$$

$$= \left(\frac{C}{C'}\right)^{2/3} \frac{K}{K'} f(i, s x, K', C'),$$

where $s = (K' / K)(C' / C)^{1/3}$. Thus,

$$\text{Energy}_{\text{se}}(x, K, C) = \left(\frac{C}{C'}\right)^{4/3} \left(\frac{K}{K'}\right)^2 \text{Energy}_{\text{se}}(s x, K', C').$$

Force-directed methods

Another example Kamada-Kawai *spring* model

- the repulsive force between any two nodes i and j

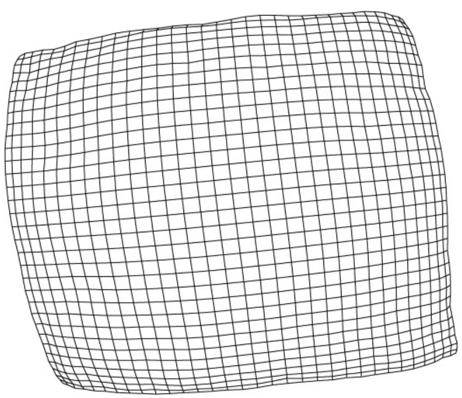
graph distance

$$f_r(i,j) = f_a(i,j) = ||x_i - x_j|| - d(i,j), \; i \neq j$$

2-norm distance

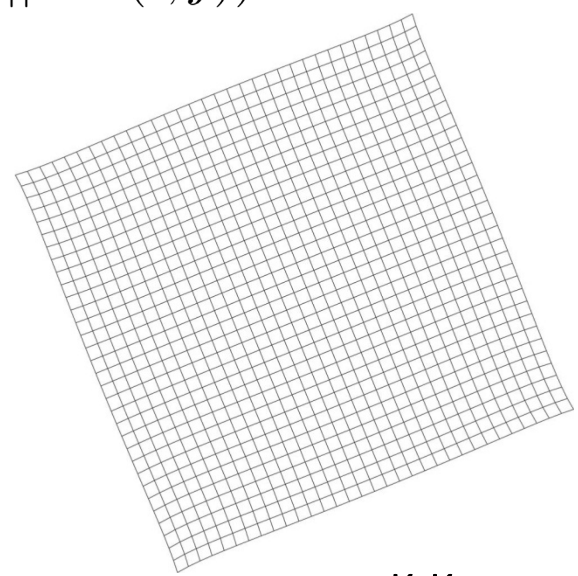
The combined energy of the system is

$$\text{Energy}_s(x) = \sum_{i \neq j} (||x_i - x_j|| - d(i,j))^2$$



Peripheral effect

F-R



K-K

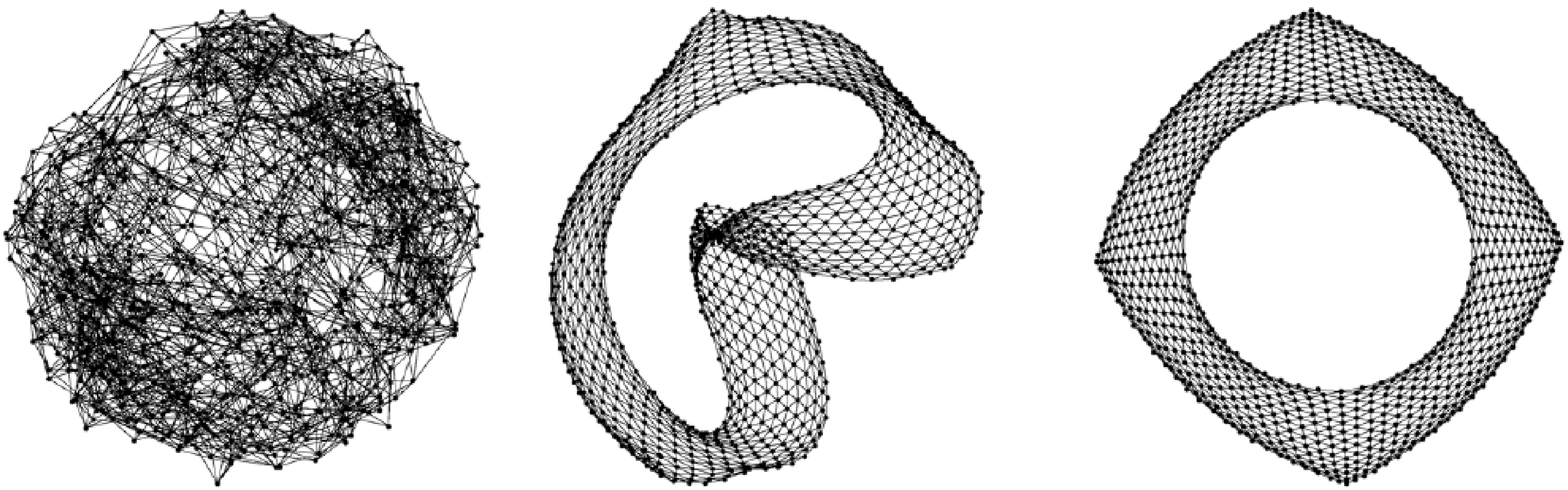
- ForceDirectedAlgorithm(G, x, tol) {
  - converged = FALSE;
  - step = initial step length;
  - Energy = Infinity
  - while (converged equals FALSE) {
    - $x^0 = x$;
    - Energy^0 = Energy; Energy = 0;
    - for $i \in V$ {
      - $f = 0$;
      - for $(j \leftrightarrow i)$ $f := f + \frac{f_a(i,j)}{\| x_j - x_i \|}(x_j - x_i)$;
      - for $(j \neq i,\ j \in V)$ $f := f + \frac{f_r(i,j)}{\| x_j - x_i \|}(x_j - x_i)$;
      - $x_{i:} = x_i + \text{step} * (f / \| f \|)$;
      - Energy := Energy + $\| f \|^2$;
    - }
    - step := update_steplength (step, Energy, Energy^0);
    - if $(\| x - x^0 \| < K \text{ tol})$ converged = TRUE;
  - }
  - return x;
- }

Algorithm 1. An iterative force-directed algorithm.

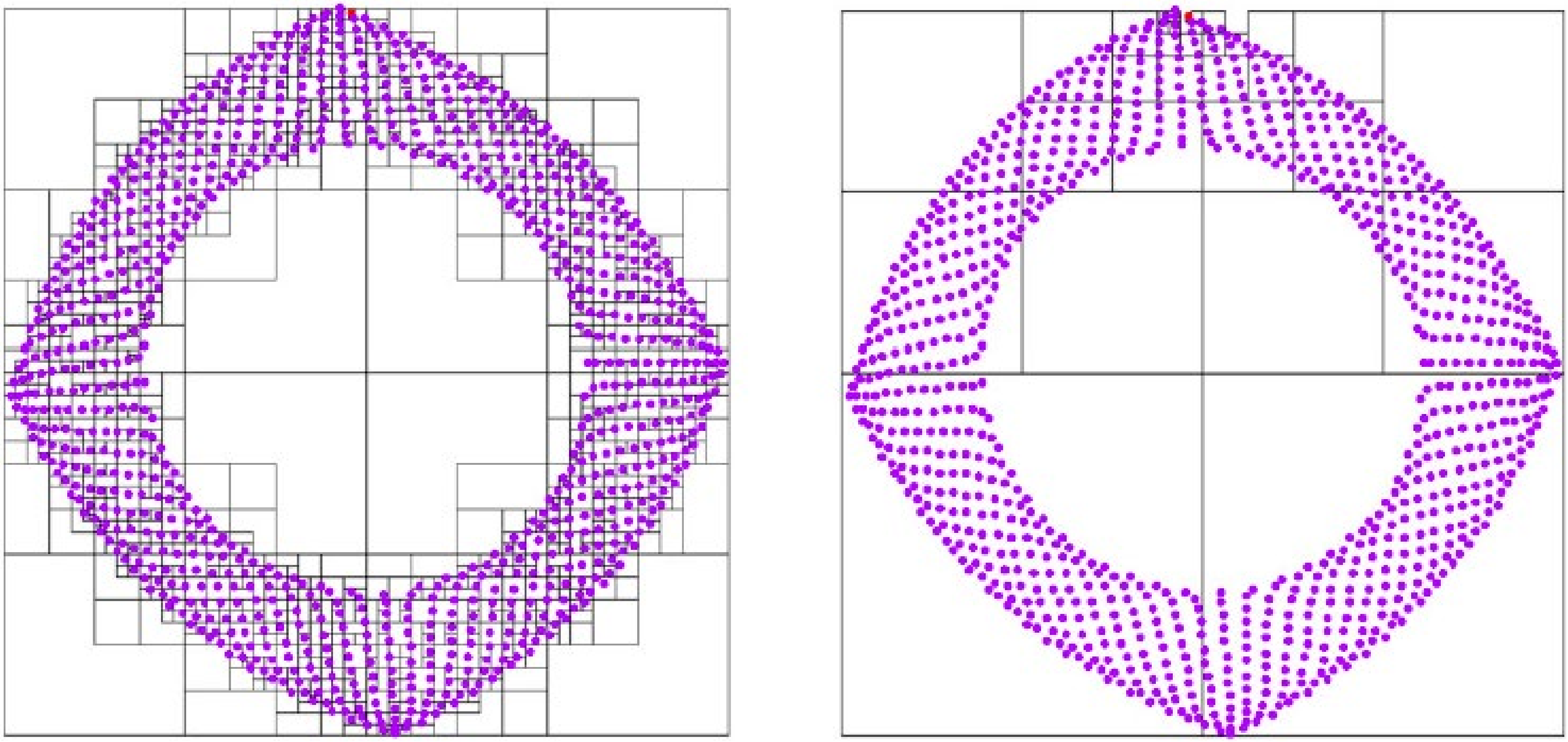
$\text{step} := t\ \text{step}$

```
• function update_steplength (step, Energy, Energy^0)
• if (Energy < Energy^0) {
  – progress = progress + 1;
  – if (progress > = 5) {
     * progress = 0;
     * step := step / t;
  – }
• } else {
  – progress = 0;
  – step := t step;
```

Best minimized layout



70 iterations

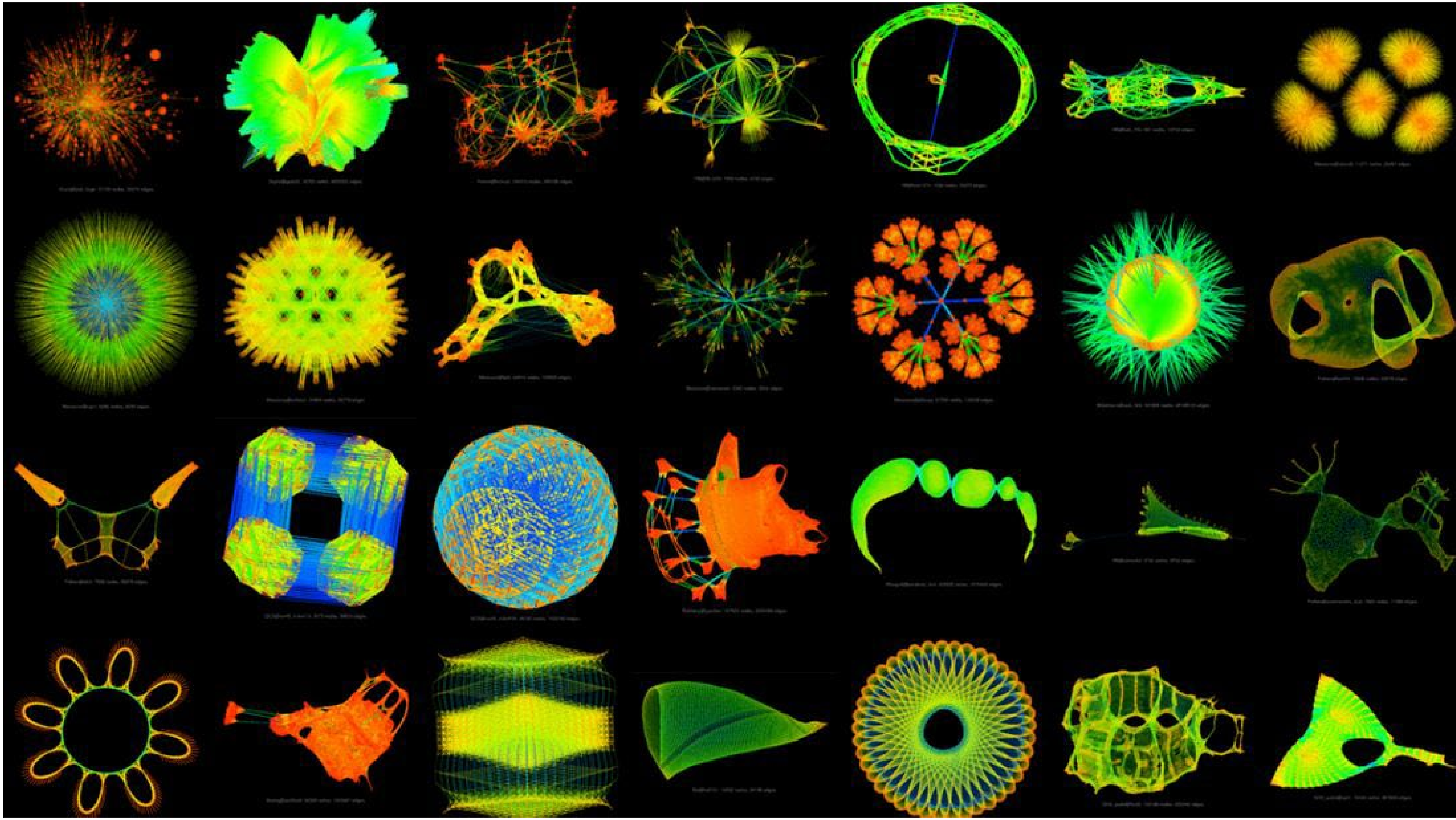


The repulsive force calculation resembles the *n*-body problem in physics, which is well studied. One of the widely used techniques to calculate the repulsive forces in $O(n \log n)$ time with good accuracy, but without ignoring long-range forces, is to treat groups of faraway vertices as supernodes, using a suitable data Structure.

function MultilevelLayout (G^i, tol)

- Coarsest graph layout
  - if (n^{i+1} < MinSize or $n^{i+1} / n^i > \rho$) {
    - x^i = random initial layout
    - x^i = ForceDirectedAlgorithm(G^i, x^i, tol)
    - return x^i
  - }
- The coarsening phase:
  - set up the $n^i \times n^{i+1}$ prolongation matrix P^i
  - $G^{i+1} = {P^i}^T \; G^i \, P^i$
  - x^{i+1} = MultilevelLayout(G^{i+1}, tol)
- The prolongation and refinement phase:
  - prolongate to get initial layout: $x^i = P^i \, x^{i+1}$
  - refinement: x^i = ForceDirectAlgorithm(G^i, x^i, tol)
  - return x^i

Algorithm 2. A multilevel force-directed algorithm.



https://sparse.tamu.edu/

High-dimensional Embedding

see Koren, Harel "Graph Drawing by High-Dimensional Embedding"

Algorithm

- Choose m pivots $\{p_1, ..., p_m\}$, each $p_i \in V$
- Each $v \in V$ is associated with m coordinates

$$\{X^i(v)\}_{i=1}^{m}, \text{ where } X^i(v) = d(p_i, v)$$

- Project m-dimensional coordinates into 2- or 3-dimensional space

How to choose p_i

- choose p_1 at random
- For $j = 2, ..., m$ choose p_j that maximizes the shortest distance from $\{p_k\}_{k=1}^{j-1}$

Similar to the k-center problem where the goal is to minimize the distance from V to k centers.

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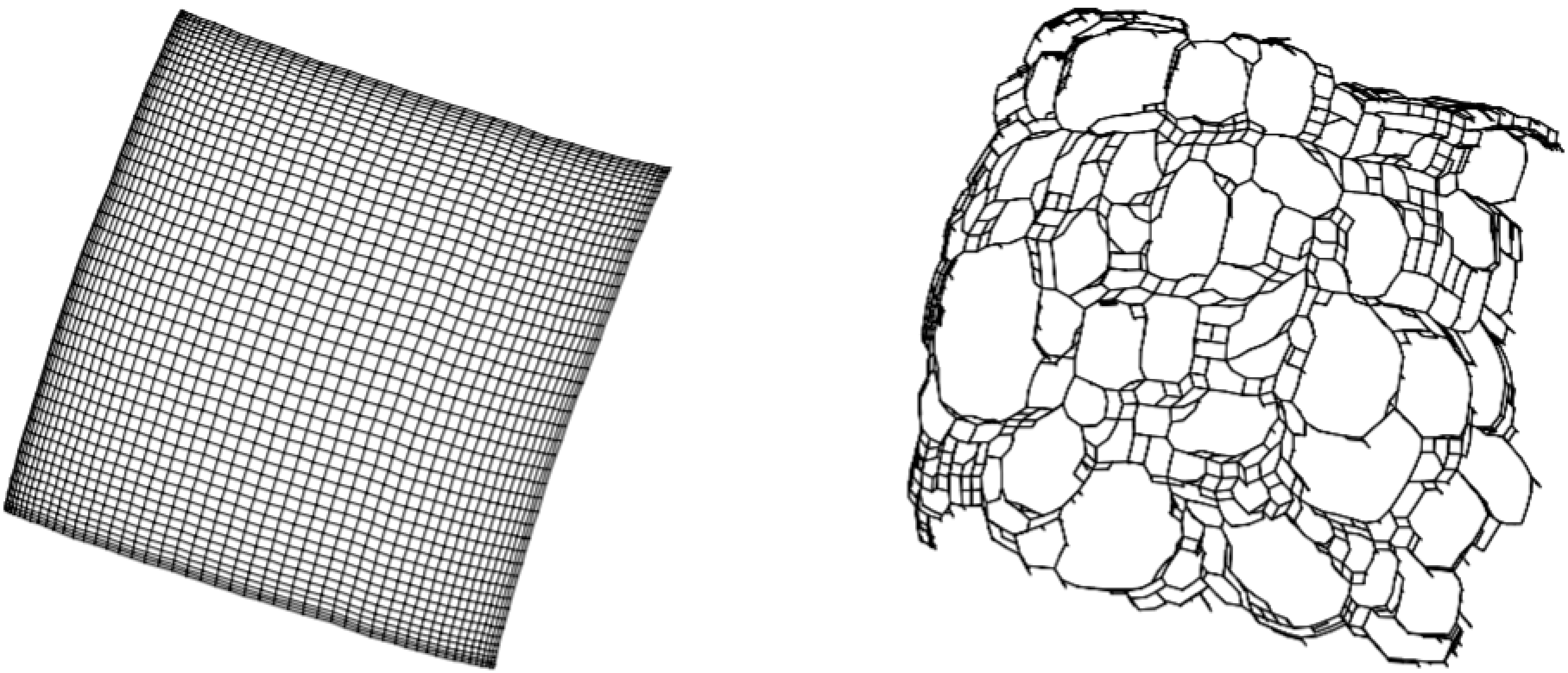

(a) (b)

Figure 3: Layouts of: (a) A 50×50 grid; (b) A 50×50 grid with $\frac{1}{3}$ of the edges omitted at random; (c) A 100×100 grid with opposite corners connected; (d) A a 100×100 torus; (e) The Crack graph; (f) The 3elt graph

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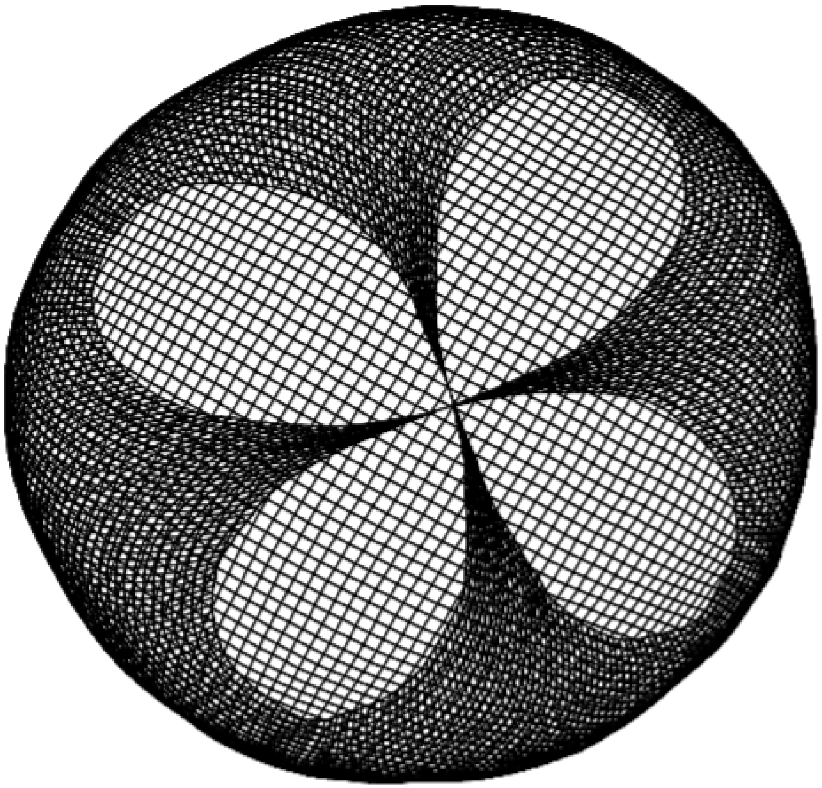
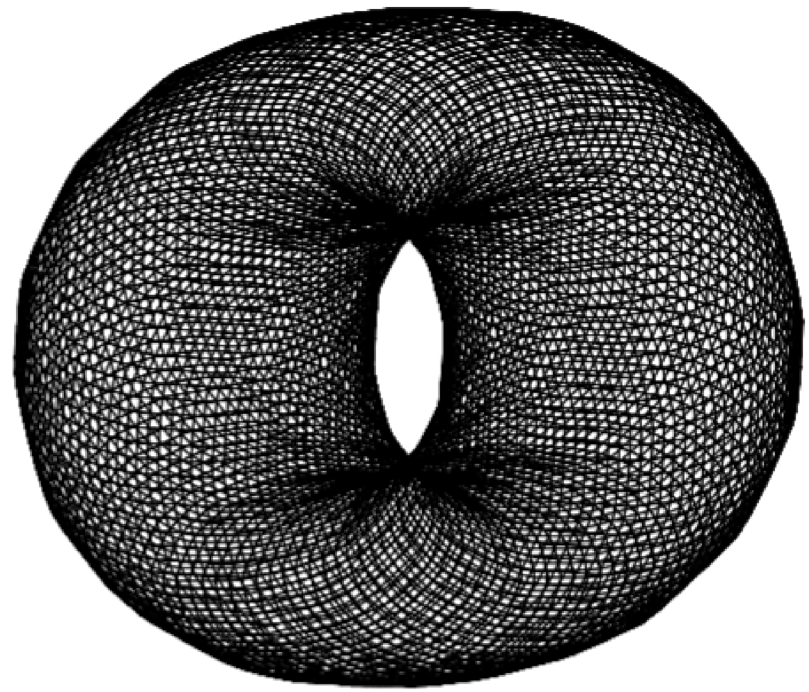


(c)



(d)

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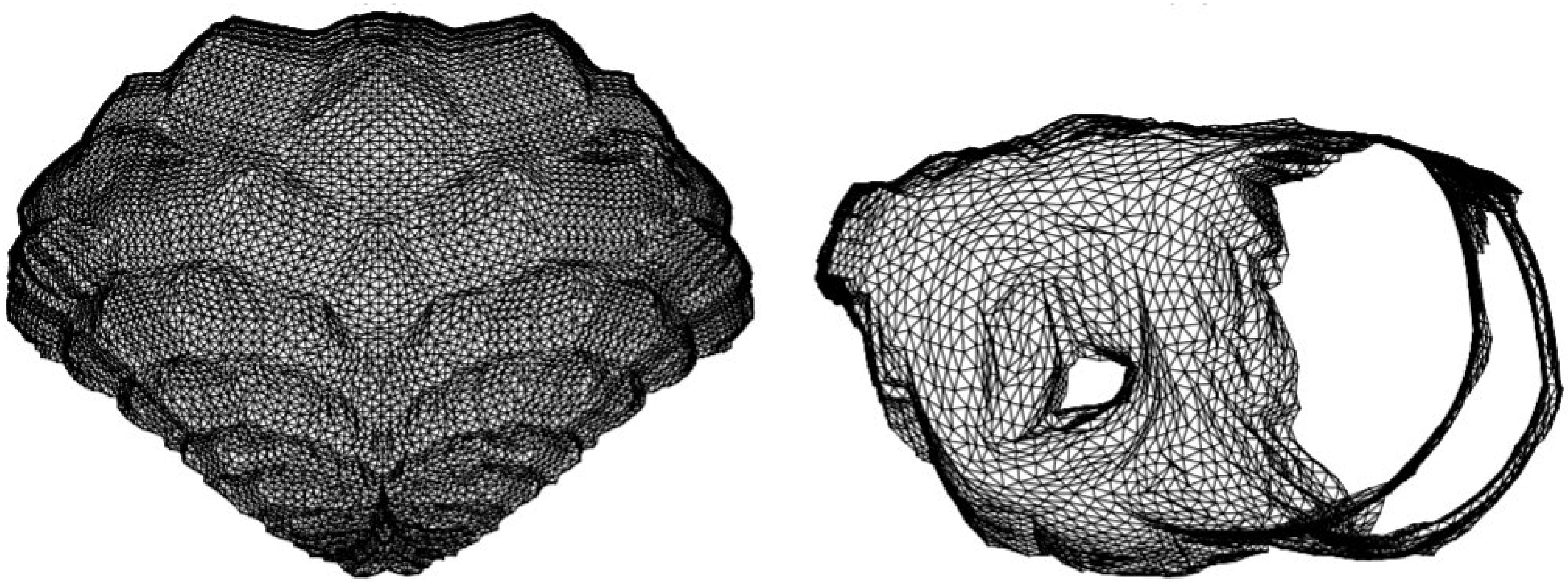

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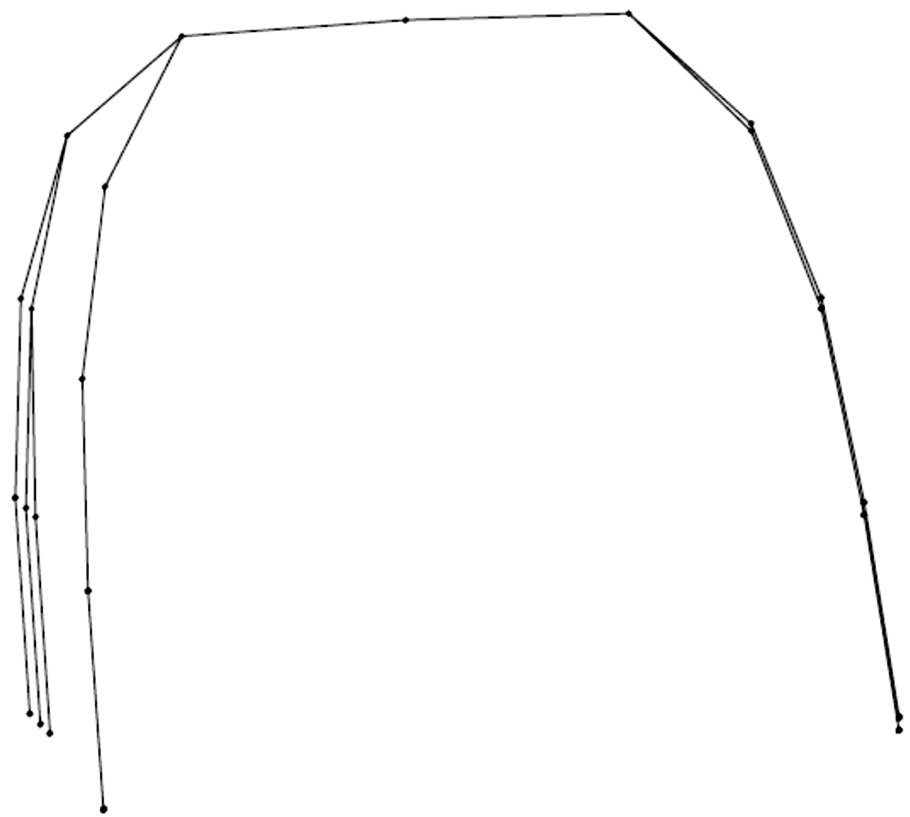
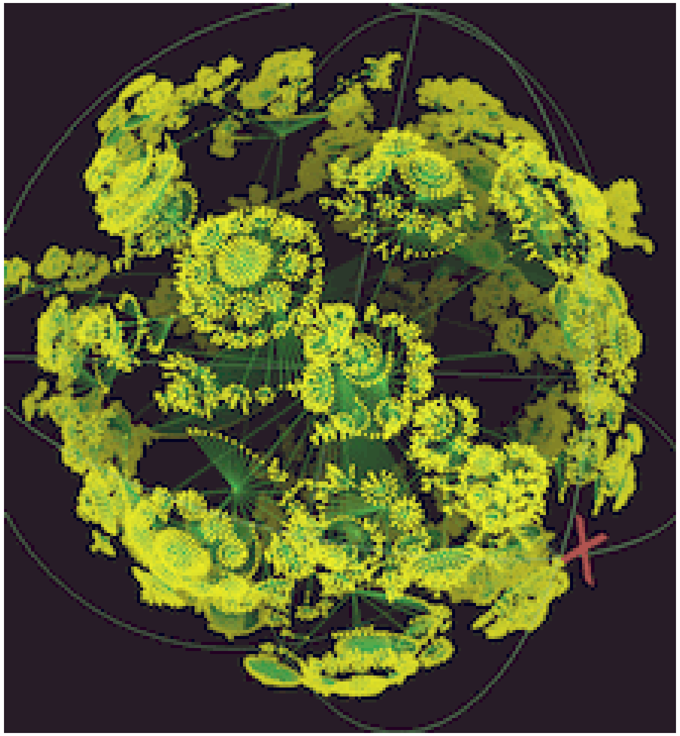
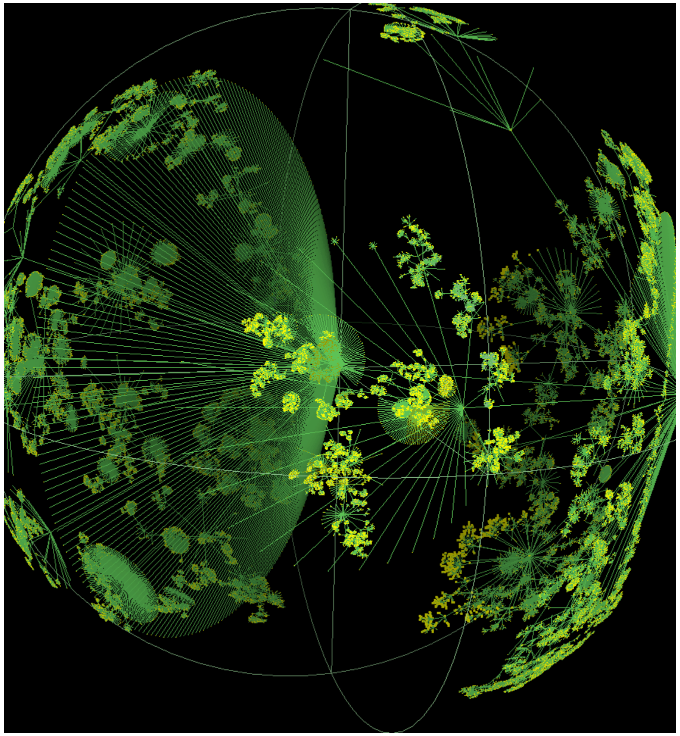


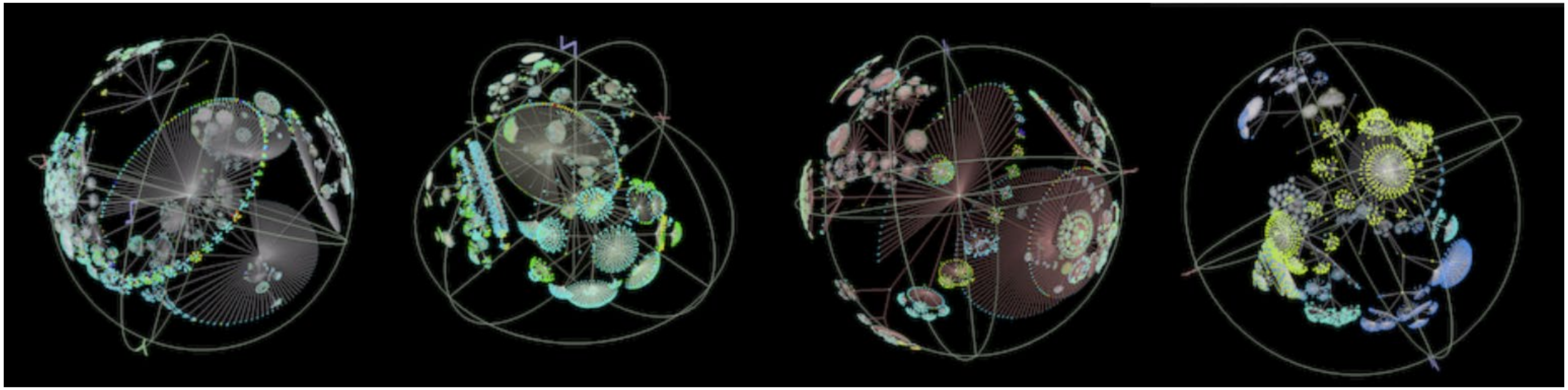
Figure 8: Drawing of a depth 5 full binary tree

Embedding in 3D using hyperbolic geometry

see Kriukov et al. "Hyperbolic geometry of complex networks"
github.com/CAIDA/walrus

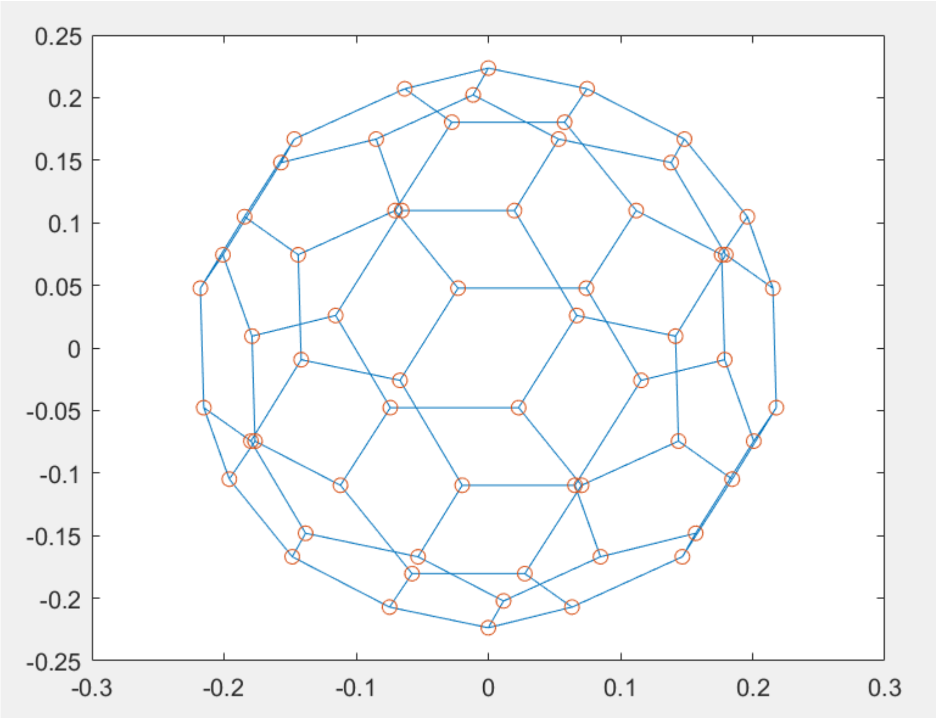






Spectral graph drawing

```
A = full(bucky);
D = diag(sum(A));
L = D - A;
[v, e] = eig(L);
gplot(A, v(:, [2 3]))
hold on;
gplot(A,v(:, [2 3]), 'o')
```



Eigenvectors and energy

For a nonzero $x \in \mathbb{R}^n$ and $M \in \mathbb{R}^{n\times n}$ the Raleigh quotient is defined

$$R(x) = \frac{x^T M x}{x^T x}$$

Courant-Fischer Theorem. Let $M \in \mathbb{R}^{n\times n}$ be symmetric with eigenvalues $\lambda_0 \leq ... \leq \lambda_{n-1}$. Let X^k be a k-dim subspace of $\mathbb{R}^n$ and $x \perp X^k$. Then

$$\lambda_i = \min_{X^{n-i-1}} (\max_{x \perp X^{n-i-1}, x \neq = 0} R(x)) = \max_{X^i}(\min_{x \perp X^i, x \neq 0} R(x))$$

Fiedler Theorem.

$$\lambda_2(L) = n \min_{x \in \mathbb{R}^n} \left(\frac{\sum_{ij \in E}(x_i - x_j)^2}{\sum_{ij \in \binom{V}{2}}(x_i - x_j)^2}\right) \text{ same for } \lambda_n \text{ and } \max$$

A symmetric minor of A is a submatrix B obtained by deleting some rows and the corresponding columns.

Theorem (Interlacing eigenvalues). Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix with eigenvalues $\lambda_1 \leq ... \leq \lambda_n$. Let $B \in \mathbb{R}^{(n-k)\times(n-k)}$ be a symmetric minor of A with eigenvalues $\mu_1 \leq ... \leq \mu_{n-k}$. Then

$$\lambda_i \leq \mu_i \leq \lambda_{i+k}.$$