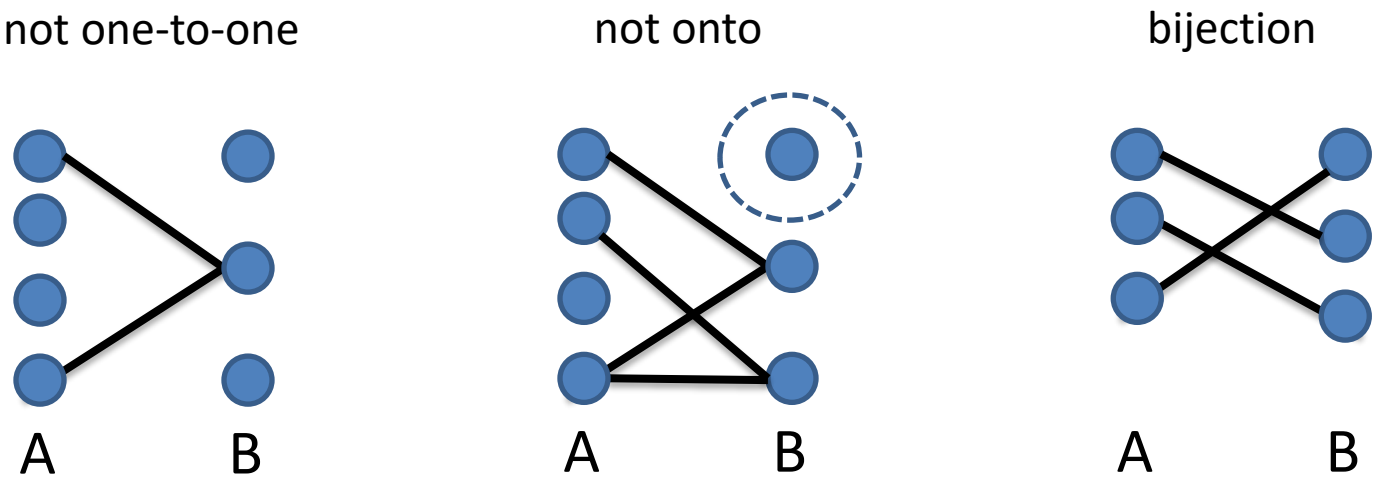
# Chapter 7

# *Undecidable Problems*

Is there a problem for which no algorithm can produce a correct answer for every input?

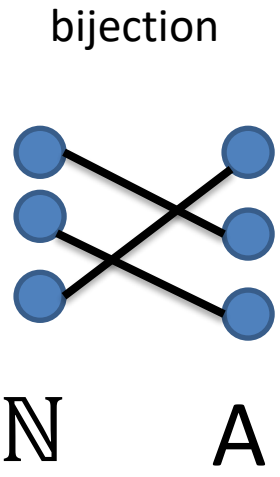
## Definition (bijection)

- A function $f : A \to B$ is *one-to-one* if $f$ never assigns the same value to two different elements of its domain.
- It is *onto* if its range is the entire set $B$.
- A function from $A$ to $B$ that is both one-to-one and onto is called a bijection from $A$ to $B$.



**Definition** A set $A$ is *countably infinite* (the same size as $\mathbb{N}$) if there is a bijection $f : \mathbb{N} \to A$, or a list $a_0, a_1, \ldots$ of elements of $A$ such that every element of $A$ appears exactly once in the list. $A$ is countable if $A$ is either finite or countably infinite.

**Theorem** Every infinite set has a countably infinite subset, and every subset of a countable set is countable.




Question: Is N countable? Answer: Yes. A corresponding bijection from N to N is f(x)=x.

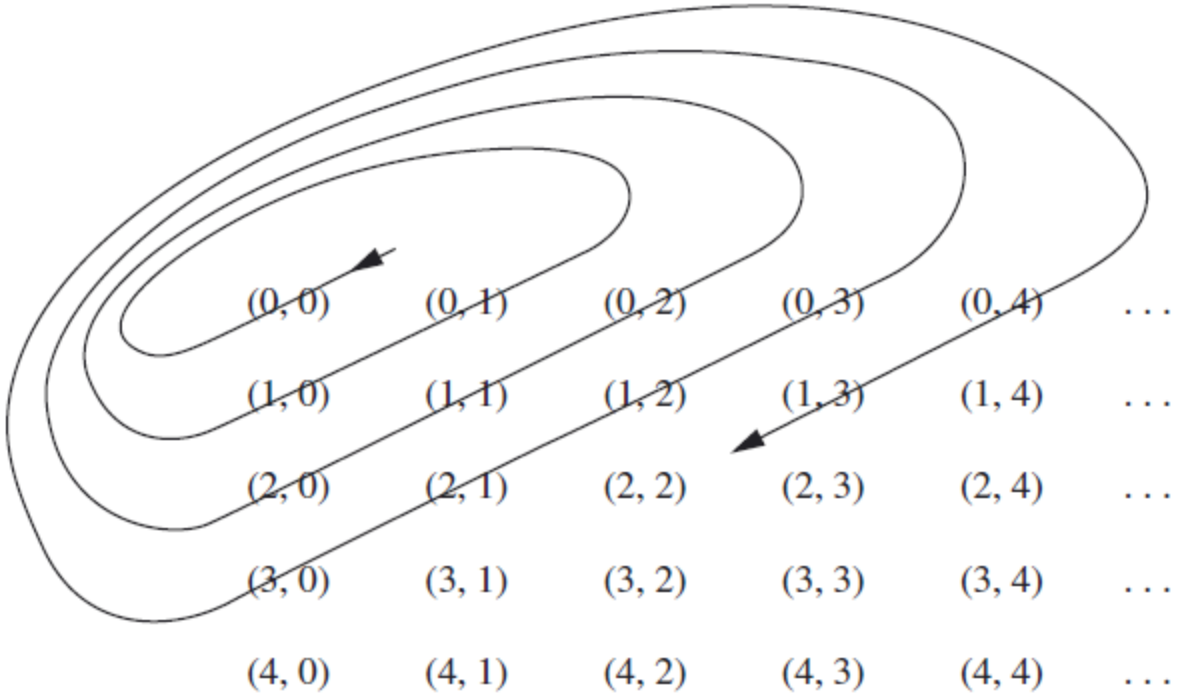
## Example

The set $\mathbb{N} \times \mathbb{N}$ is countable.

We can describe the set by drawing a two-dimensional array:

| | | | | |
|---|---|---|---|---|
| (0,0) | (0,1) | (0,2) | (0,3) | ... |
| (1,0) | (1,1) | (1,2) | (1,3) | ... |
| (2,0) | (2,1) | (2,2) | (2,3) | ... |
| (3,0) | (3,1) | (3,2) | (3,3) | ... |
| ... | | | | |

How to count it? In other words, how to find a bijection from N to NxN?



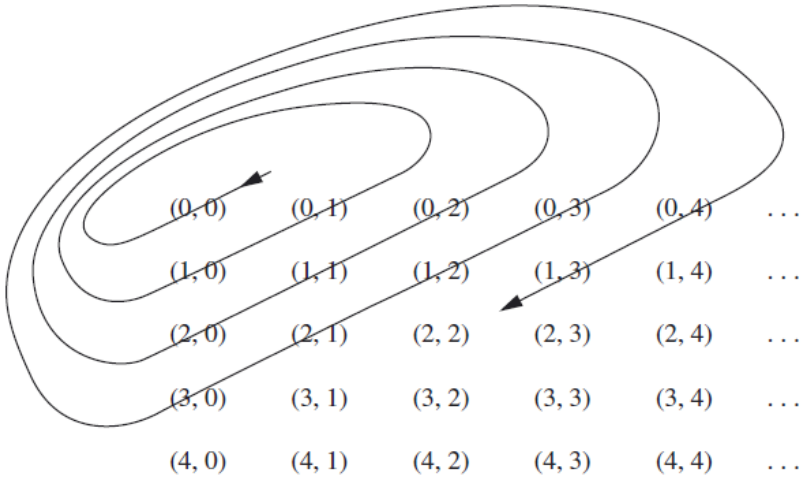
## Examples

- The countable union of countable sets is countable. If $S_i$ is countable for every $i \in \mathbb{N}$ then

$$S = \cup_{i=1}^{\infty} S_i$$

is countable.

Proof: This is a generalization of the previous example. This time the ordered pair $(i, j)$ in the figure stands for the $j$th element of $S_i$, so that the $i$th row of the two-dimensional array represents the elements of $S_i$.



## Examples

- For a finite alphabet $\Sigma$ (such as $\{a, b\}$), the set $\Sigma^*$ of all strings over $\Sigma$ is countable.

Proof: This follows from previous example, because

$$\Sigma^* = \cup_{i=0}^{\infty} \Sigma^i$$

and each of the sets $\Sigma^i$ is countable.

1. order alphabetically
2. each of them is countable

**Corollary 1: Languages are countable sets**

## Examples

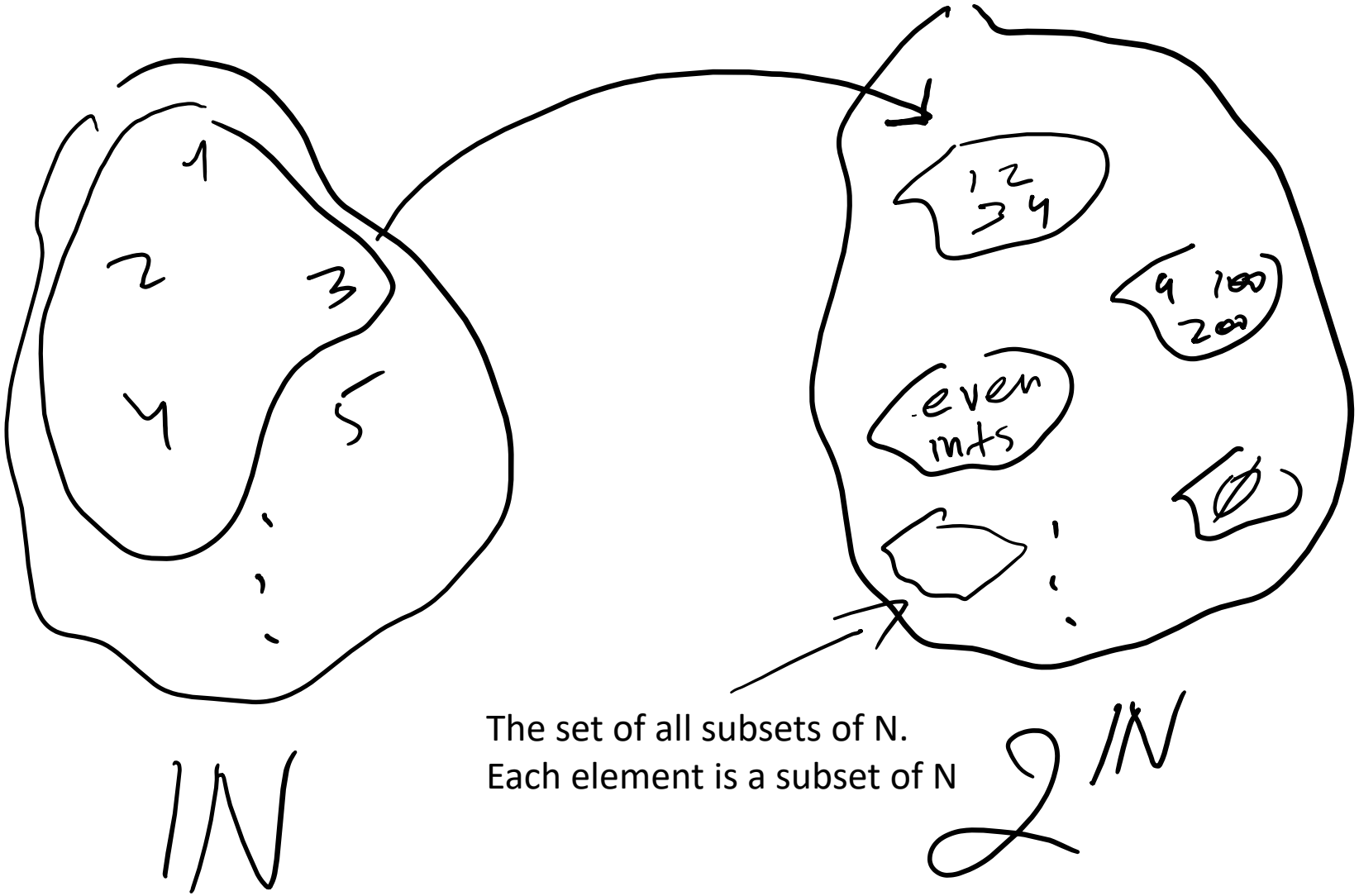
- The set of Turing machines is countable.

Let $\mathbb{T}$ represent the set of Turing machines.

- A TM $T$ can be represented by the string $e(T) \in \{0,1\}^*$, and a string can represent at most one TM.
- Therefore, the resulting function $e$ is one-to-one, and we may think of it as a bijection from $\mathbb{T}$ to a subset of $\{0,1\}^*$.
- Because $\{0,1\}^*$ is countable, every subset is, and we can conclude that $\mathbb{T}$ is countable.

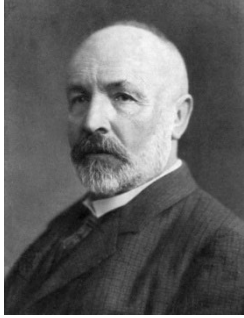
# Examples

- What about the set $2^{\mathbb{N}}$?




The set of all subsets of N.
Each element is a subset of N

**Theorem**



Georg Cantor
1845-1918

- The set $2^{\mathbb{N}}$ is uncountable.

  We wish to show that there can be no list of subsets of $\mathbb{N}$ containing every subset of $\mathbb{N}$ that can be enumerated as $\mathbb{N}$. In other words, every list $A_0, A_1, A_2, \ldots$ of subsets of $\mathbb{N}$ must leave out at least one.

  A *diagonal argument* and construction are provided by Cantor. Here is a subset, constructed from the ones in the list, that cannot possibly be in the list:

$$A = \{i \in \mathbb{N} \mid i \notin A_i\}$$

  The reason that $A$ must be different from $A_i$ for every $i$ is that $A$ and $A_i$ differ because of the number $i$, which is in one but not both of the two sets: if $i \in A_i$ , then by definition of $A$, $i$ does not satisfy the defining condition of $A$, and so $i \notin A$; and if $i \notin A_i$ , then (by definition of $A$) $i \in A$.

Given $S = \{1, 2, 3, 4, 5, 6\}$ and its subsets $A_1 = \{1, 3, 4, 5\}$, $A_2 = \{2, 4, 5, 6\}$, $A_3 = \{1, 2, 3, 4\}$, and $A_4 = \{5\}$.

| S | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Characteristic function of $A_1$ | 1 | 0 | 1 | 1 | 1 | 0 |
| Characteristic function of $A_2$ | 0 | 1 | 0 | 1 | 1 | 1 |
| Characteristic function of $A_3$ | 1 | 1 | 1 | 1 | 0 | 0 |
| Characteristic function of $A_4$ | 0 | 0 | 0 | 0 | 1 | 0 |

Flip the diagonal indicators

| S | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Characteristic function of $A_1$ | 0 | 0 | 1 | 1 | 1 | 0 |
| Characteristic function of $A_2$ | 0 | 0 | 0 | 1 | 1 | 1 |
| Characteristic function of $A_3$ | 1 | 1 | 0 | 1 | 0 | 0 |
| Characteristic function of $A_4$ | 0 | 0 | 0 | 1 | 1 | 0 |

The new set is $A_5 = \{4\}$

Wait … maybe this subset is in the list?
Let's check what happens in this case …

Given $S = \{1, 2, 3, 4, 5, 6\}$ and its subsets $A_1 = \{1, 3, 4, 5\}$, $A_2 = \{2, 4, 5, 6\}$, $A_3 = \{1, 2, 3, 4\}$, and $A_4 = \{5\}$.

| S | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Characteristic function of $A_1$ | 1 | 0 | 1 | 1 | 1 | 0 |
| Characteristic function of $A_2$ | 0 | 1 | 0 | 1 | 1 | 1 |
| Characteristic function of $A_3$ | 1 | 1 | 1 | 1 | 0 | 0 |
| Characteristic function of $A_4$ | 0 | 0 | 0 | 0 | 1 | 0 |
| New characteristic function | 0 | 0 | 0 | 1 | 0 | 0 |

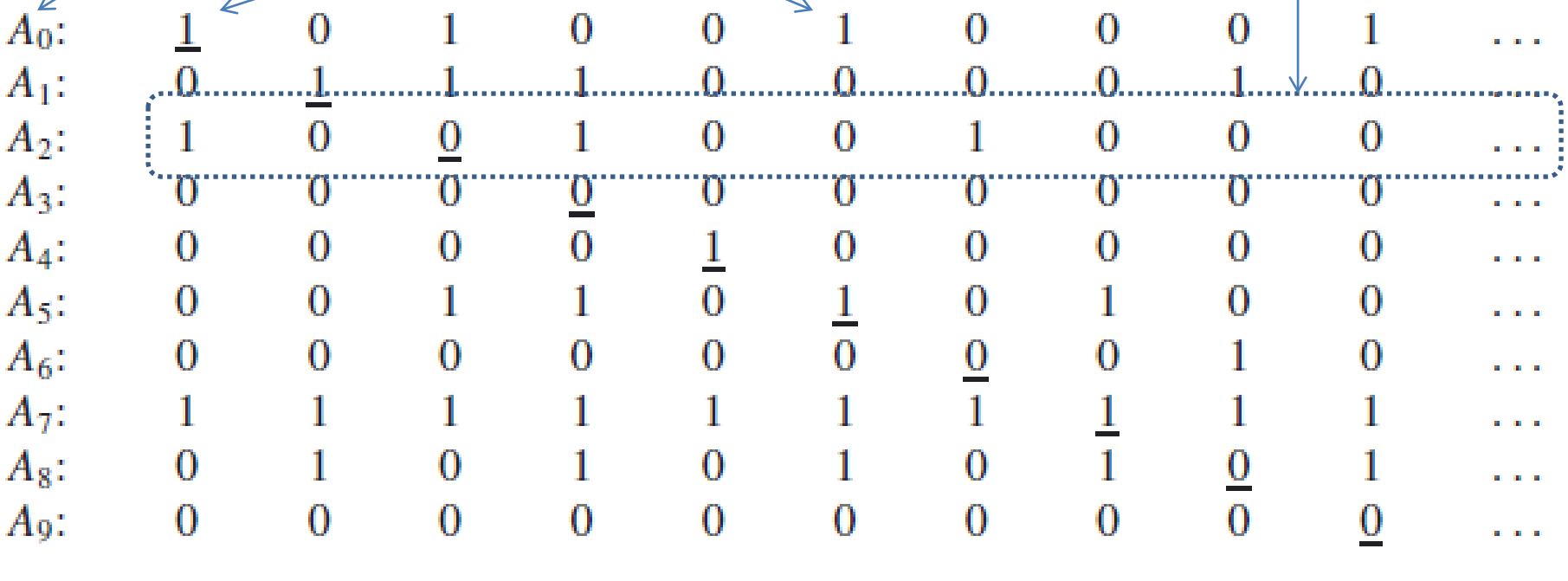
Flip the diagonal once again. The missing subset is $\{4, 5\}$.

| S | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Characteristic function of $A_1$ | 0 | 0 | 1 | 1 | 1 | 0 |
| Characteristic function of $A_2$ | 0 | 0 | 0 | 1 | 1 | 1 |
| Characteristic function of $A_3$ | 1 | 1 | 0 | 1 | 0 | 0 |
| Characteristic function of $A_4$ | 0 | 0 | 0 | 1 | 1 | 0 |
| New characteristic function | 0 | 0 | 0 | 1 | 1 | 0 |

Let us visualize the subsets, and the selection process of missing *A*

This subset contains 0, 2, 5, 9 …

each row is a characteristic (or indicator)function of $A_i$




we obtain the sequence corresponding to the set *A* by reversing diagonal: 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, . . . Each entry we encounter as we make our way down the diagonal allows us to distinguish the set *A* from one more of the sets $A_i$ . The missing subset *A* = {2, 3, 6, 8, 9, . . .}.

Same argument works if we want to find a language that … cannot be accepted by a Turing machine - in other words, a language that is different from *L(T )*, for every Turing machine *T* with input alphabet {0, 1}, i.e.,

**there are languages that are not accepted by Turing machines!**
**Set of TMs is countable. Set of all languages is uncountable.**

List of TMs

A language of Turing machine $A_0$

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $A_0$: | **1** | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | … |
| $A_1$: | 0 | **1** | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | … |
| $A_2$: | 1 | 0 | **0** | 1 | 0 | 0 | 1 | 0 | 0 | 0 | … |
| $A_3$: | 0 | 0 | 0 | **0** | 0 | 0 | 0 | 0 | 0 | 0 | … |
| $A_4$: | 0 | 0 | 0 | 0 | **1** | 0 | 0 | 0 | 0 | 0 | … |
| $A_5$: | 0 | 0 | 1 | 1 | 0 | **1** | 0 | 1 | 0 | 0 | … |
| $A_6$: | 0 | 0 | 0 | 0 | 0 | 0 | **0** | 0 | 1 | 0 | … |
| $A_7$: | 1 | 1 | 1 | 1 | 1 | 1 | 1 | **1** | 1 | 1 | … |
| $A_8$: | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | **0** | 1 | … |
| $A_9$: | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **0** | … |
| … | | | | | | | | | | | |

Another conclusion:

There are uncountably many languages because we know that S={0,1}* is equivalent to N and each subset of S is a language.

# Recursive and recursively enumerable languages

| A TM T with input alphabet $\Sigma$ … | |
|---|---|
| *accepts* | *decides* |
| a language $L \subseteq \Sigma^*$ if it accepts the strings in L and no others. | a language $L \subseteq \Sigma^*$ if T computes the characteristic function $\chi_L : \Sigma^* \rightarrow \{0,1\}$ that returns 1 on strings in L and 0 otherwise. |
| In both cases, the issue is whether the input string is an element of *L*. However, the second approach may be more informative, because a TM *accepting L* may not return an answer if the string is not in *L* | |
| *Recursively enumerable* (RE) languages are those that can be *accepted* by a TM | *Recursive* languages are those that can be *decided* by a TM |

Only in the decision case there is a guaranteed answer to the question: Given a string *x*, is *x* an element of the language?

Theorem: Every recursive language is recursively enumerable.

Theorem: If $L \subseteq \Sigma^*$ is accepted by a TM $T$ that halts on every input string, then $L$ is recursive.

Theorem: If $L_1$ and $L_2$ are both recursively enumerable languages over $\Sigma$, then $L_1 \cup L_2$ and $L_1 \cap L_2$ are also recursively enumerable.

Theorem: If $L_1$ and $L_2$ are both recursive languages over $\Sigma$, then $L_1 \cup L_2$ and $L_1 \cap L_2$ are also recursive.

Theorem: If $L$ is a recursive language over $\Sigma$, then its complement is also recursive.

Theorem: If $L$ is a recursively enumerable language, and its complement is also recursively enumerable, then $L$ is recursive.

Theorem: *Not all languages are recursively enumerable. In fact, the set of languages over* {0,1} *that are not recursively enumerable is uncountable*.

Proof:
We know that $2^N$ is uncountable and we observed that because {0,1}* is the same size as N, it follows that the set of languages over {0,1} is uncountable.

We know that the set of RE languages over {0,1} is countable (because the set of TM is countable).

If *T* is any countable subset of an uncountable set S then *S*-*T* is uncountable.

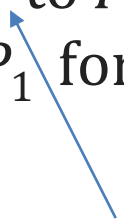
# A Language That Can't Be Accepted, and a Problem That Can't Be Decided

- Definition: Let
  - $NSA = \{e(T) \mid T \text{ is a TM and } e(T) \notin L(T)\}$
  - $SA = \{e(T) \mid T \text{ is a TM and } e(T) \in L(T)\}$
    - ("non-self-accepting" and "self-accepting")
- **Theorem**:
  - The language $NSA$ is not recursively enumerable
  - The language $SA$ is recursively enumerable but not recursive

- The statement of the theorem says that there is no algorithm to determine whether a given string represents a TM that accepts its own encoding
  - It might seem that for a TM $T$, deciding whether $T$ accepts the string $e(T)$ is particularly difficult, but this is not the right interpretation
  - All we needed for the diagonal argument was a string associated with $T$; we chose $e(T)$, but we could just as easily have used something else
- The more correct conclusion is that it's hard to answer questions about TMs and the languages they accept

# Reductions

- We can often solve problems by reducing them to other, simpler ones
- We will reduce one decision problem $P_1$ to another $P_2$
- The two crucial features in a reduction $F$ are:
  - For every instance $I$ of $P_1$ we must be able to obtain an instance $F(I)$ of $P_2$ algorithmically
  - The answer to $P_2$ for the instance $F(I)$ must be the same as the answer to $P_1$ for $I$

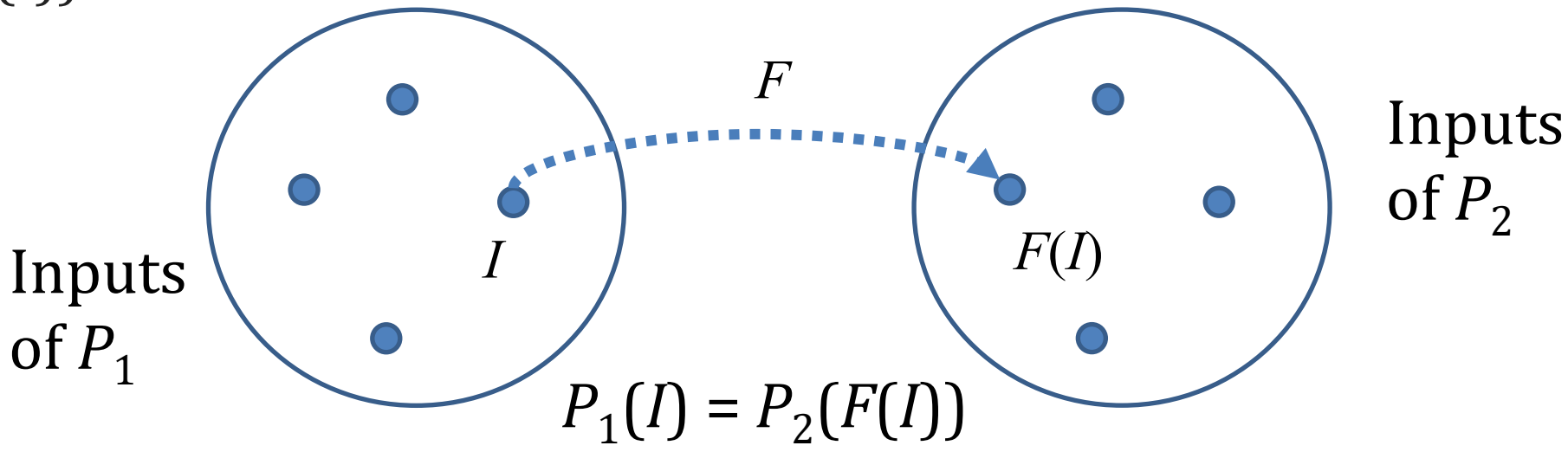


For simplicity, binary

# Reductions

**Definition**: Suppose $P_1$ and $P_2$ are decision problems.

We say $P_1$ is reducible to $P_2$ $(P_1 \leq P_2)$ if there is an algorithm that finds, for an arbitrary instance $I$ of $P_1$, an instance $F(I)$ of $P_2$ such that the **two answers are the same**, i.e., (the answer to $P_1$ for the instance $I$, and the answer to $P_2$ for the instance $F(I)$)




Idea 1: For example, I don't know how to solve $P_1$ but I can solve $P_2$ and know how to map the instances to preserve answers. Then I can solve $P_1$

# Reductions

**Definition**: Suppose $P_1$ and $P_2$ are decision problems.

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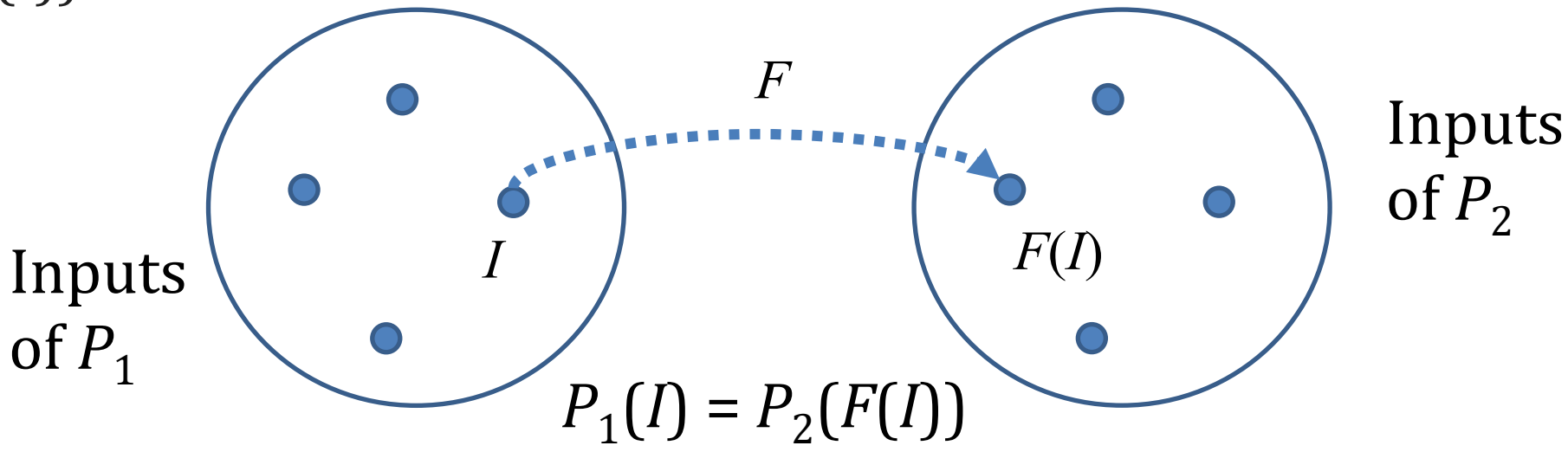


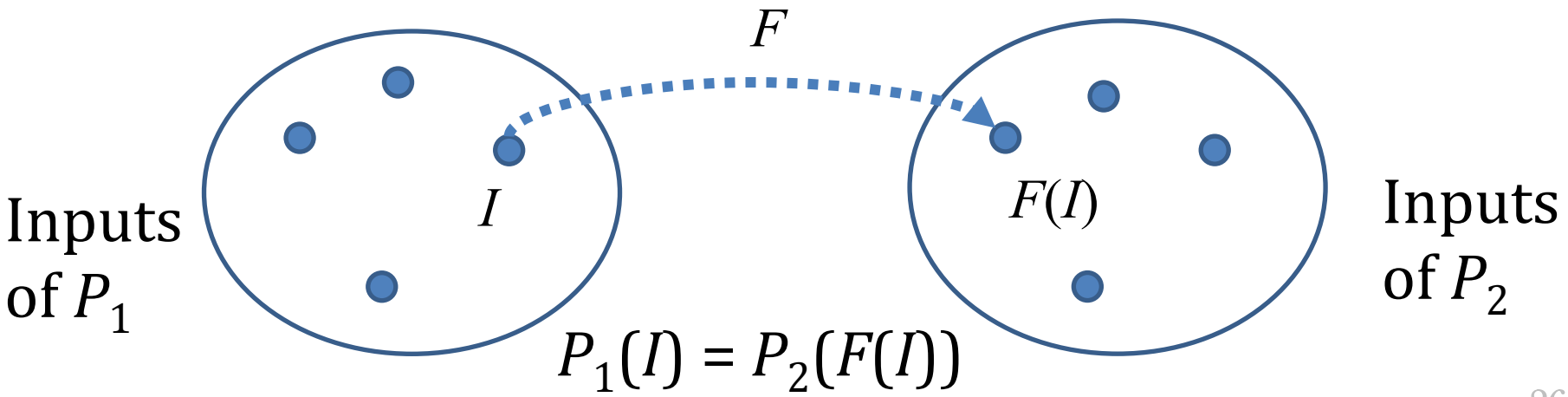

Idea 2: Say, $P_1$ is computationally difficult, and I don't know the difficulty of $P_2$. If I know how to map the instances then I can state that $P_2$ is at least as difficult as $P_1$

# Reductions

**Example**

- Problem $P_2(x, y)$ decides $x < y$ for $x, y \in \{1, ...5\}$.
- Problem $P_1(a^2, b^2)$ decides $a^2 < b^2$ for $a^2, b^2 \in \{1, ..., 25\}$.
- Imagine that $P_1$ is hard (or unknown how) to compute but $P_2$ is not, and let us assume that we can compute a positive $\sqrt{x^2}$.
- For each instance $I$ of $P_1$ we can compute an instance $F(I)$ of $P_2$, and the answers to $P_1$, and $P_2$ are the same.

→ **Now we need solve $P_2$ instead of $P_1$**



# Reductions

- Definition:
  - Suppose $P_1$ and $P_2$ are decision problems
    - We say $P_1$ is reducible to $P_2$ $(P_1 \leq P_2)$ if there is an algorithm that finds, for an arbitrary instance $I$ of $P_1$, an instance $F(I)$ of $P_2$ such that the two answers (the answer to $P_1$ for the instance $I$, and the answer to $P_2$ for the instance $F(I)$) are the same
  - If $L_1$ and $L_2$ are languages over alphabets $\Sigma_1$ and $\Sigma_2$
    - We say $L_1$ is reducible to $L_2$ $(L_1 \leq L_2)$ if there is a Turing-computable function $f: {\Sigma_1}^* \rightarrow {\Sigma_2}^*$ such that for every $x \in {\Sigma_1}^*$, $x \in L_1$ if and only if $f(x) \in L_2$

# Reductions

- Theorem:
  - Suppose $L_1 \subseteq \Sigma_1^*$, $L_2 \subseteq \Sigma_2^*$, and $L_1 \leq L_2$
    - If $L_2$ is recursive, then $L_1$ is recursive (Proof: we can decide whether a string is in $L_1$ by using the reduction and deciding whether the resulting string is in $L_2$)
  - Suppose $P_1$ and $P_2$ are decision problems, and $P_1 \leq P_2$
    - If $P_2$ is decidable, then $P_1$ is decidable (Proof: we can decide whether an instance of $P_1$ is a yes-instance by using the reduction and deciding whether the resulting instance of $P_2$ is a yes-instance)
  - We will be interested in the contrapositive statement: If $P_1$ is undecidable, then $P_2$ is also.

- Consider two decision problems:
  - *Accepts*: Given a TM $T$ and a string $w$, is $w \in L(T)$?
  - *Halts*: Given a TM $T$ and a string $w$, does $T$ halt (either by accepting or by rejecting) on input $w$? (This is called the *halting problem*)
- **Theorem**: Both *Accepts* and *Halts* are undecidable
- For the first statement, we just need to show that *Self-accepting* $\leq$ *Accepts*
  - A reduction from *Self-accepting* to *Accepts* is $F(T) = (T, e(T))$
  - We can compute this algorithmically.
- For the second statement, we can reduce *Accepts* to *Halts* (see the book for the details). *Accepts* is undecidable; therefore, *Halts* is undecidable

- Theorem: The following five decision problems are undecidable:
  1. *Accepts*-$\Lambda$: Given a TM $T$, is $\Lambda \in L(T)$?
  2. *AcceptsEverything*: Given a TM $T$ with input alphabet $\Sigma$, is $L(T) = \Sigma^*$?
  3. *Subset*: Given two TMs $T_1$ and $T_2$, is $L(T_1) \subseteq L(T_2)$?
  4. *Equivalent*: Given two TMs $T_1$ and $T_2$, is $L(T_1) = L(T_2)$?
  5. *WritesSymbol*: Given a TM $T$ and a symbol $a$ in the tape alphabet of $T$, does $T$ ever write an $a$ if it starts with an empty tape?

- *WritesNonblank* problem: Given a TM *T* with *n* nonhalting states, does *T* ever write a nonblank symbol on its tape, if it starts with a blank tape?
- Theorem:
  - The decision problem *WritesNonblank* is decidable.
- Proof sketch:
  - An algorithm to decide *WritesNonblank* is to trace *T* for *n* moves, or until it halts, whichever comes first
  - within n moves, either it halts or it enters some nonhalting state *q* for the second time
  - If by that time no nonblank symbol has been written, none ever will be

- It's now clear that it's difficult to answer questions about Turing machines and the strings they accept
- A few more undecidable problems about a TM $T$:
  1. (For some language $L$) *AcceptsL*: Given a TM $T$, is $L(T) = L$?
  2. *AcceptsSomething*: Is there at least one string in $L(T)$?
  3. *AcceptsTwoOrMore*: Does $L(T)$ have at least two elements?
  4. *AcceptsFinite*: is $L(T)$ finite?
  5. *AcceptsRecursive*: is $L(T)$ recursive?

Homework: Read Chapter 8!