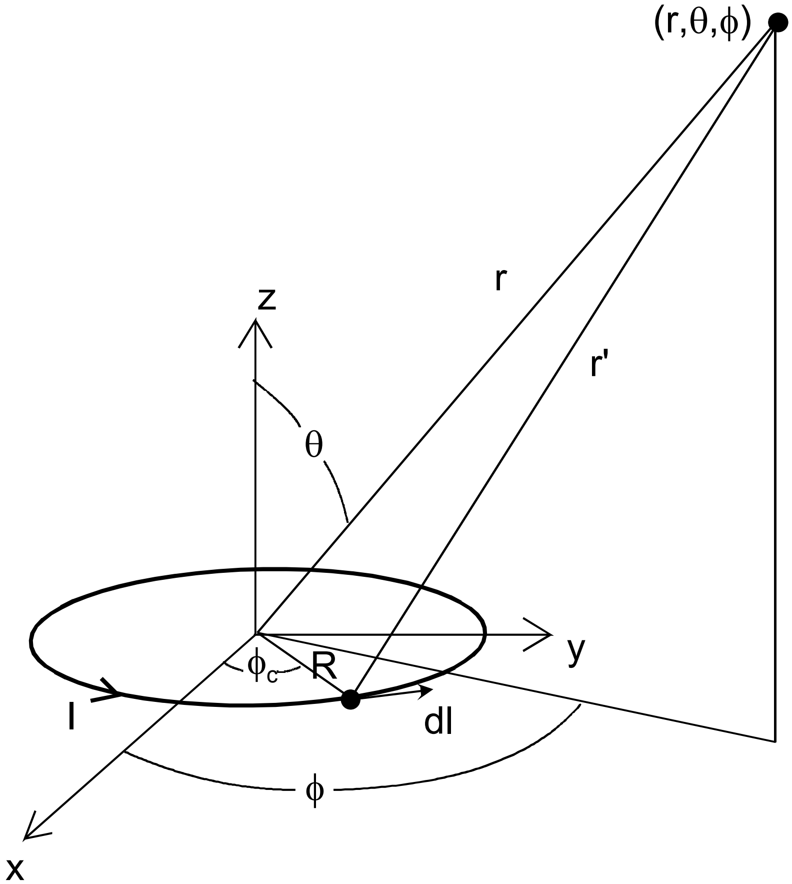
ELEG240- Spring, 2005
Homework 5
Due 3/22, noon

1. For the circular current ring shown below, for the current element shown, what is **dA**(r,θ,φ)? Hint: you cannot assume r=r' for this problem. Hint 2: write the answer in Cartesian coordinates, that is, actually give me **dA**(x,y,z) in terms of x, y, z, R and ϕ_c.



ANSWER:
From the text,

$$\mathbf{dA} = \left(\frac{\mu_0 I}{4\pi}\right)\frac{\mathbf{dl}}{r'}.$$

For an arc of a circle,

$$|\mathbf{dl}| = R d\phi_c.$$

To get the direction of **dl**, use Cartesian coordinates:

$$\mathbf{dl} = R d\phi_c(\_\_\_\ \mathbf{a_x} + \_\_\_\ \mathbf{a_y}),$$

and to get the blanks in the equation, think about what happens at various angles ϕ_c. For zero angle, **dl** points in the y-direction. For 90 degrees angle, **dl** points in the –x direction. Thus,

$$\mathbf{dl} = rd\phi_c(-\sin\phi_c\mathbf{a_x} + \cos\phi_c\mathbf{a_y}),$$

$$\mathbf{dA} = (\frac{\mu_0 I}{4\pi})\frac{Rd\phi_c(-\sin\phi_c\mathbf{a_x} + \cos\phi_c\mathbf{a_y})}{r'}.$$

Now, we must find r'. Use Cartesian coordinates to get the length between the point at space we are measuring the field at and the current element:

$$r' = \sqrt{(x-x_c)^2 + (y-y_c)^2 + z^2},$$

$$x_c = R\cos\phi_c,$$

$$y_c = R\sin\phi_c, \text{ so}$$

$$r' = \sqrt{(x-R\cos\phi_c)^2 + (y-R\sin\phi_c)^2 + z^2}, \text{ and}$$

$$\mathbf{dA} = (\frac{\mu_0 I}{4\pi})\frac{Rd\phi_c(-\sin\phi_c\mathbf{a_x} + \cos\phi_c\mathbf{a_y})}{\sqrt{(x-R\cos\phi_c)^2 + (y-R\sin\phi_c)^2 + z^2}}.$$

2. Now, find **A**. Hint: do you have to find **A** everywhere to know what it is everywhere? Hint 2: after using this hint, convert r' back to spherical coordinates and then approximate r>>R. Hint 3: you will need $(1-x)^{-1/2} \cong 1 + x/2$.
ANSWER:
No, we don't have to find it everywhere, since by symmetry it will be the same at any ϕ, or anywhere around the z-axis. So, we can find it on the y-z plane, setting x=0. Then,

$$r' = \sqrt{R^2\cos^2\phi_c + (y-R\sin\phi_c)^2 + z^2} = \sqrt{R^2\cos^2\phi_c + y^2 - 2yR\sin\phi_c + R^2\sin^2\phi_c + z^2}$$
$$= \sqrt{R^2 + y^2 - 2yR\sin\phi_c + z^2}$$

.

If we are on the y-z plane, then

$$y^2 + z^2 = r^2, \text{ and}$$

$y = r\sin\theta$. Plugging in,

$$r' = \sqrt{R^2 + r^2 - 2Rr\sin\theta\sin\phi_c} .$$

Approximating that r>>R, we have that

$r' \cong \sqrt{r^2 - 2Rr\sin\theta\sin\phi_c} = r[1 - \frac{2R}{r}\sin\theta\sin\phi_c]^{1/2}$. Plugging this into **dA**,

$$\mathbf{dA} = (\frac{\mu_0 I}{4\pi})\frac{Rd\phi_c(-\sin\phi_c\mathbf{a_x} + \cos\phi_c\mathbf{a_y})}{r[1 - \frac{2R}{r}\sin\theta\sin\phi_c]^{1/2}} = ,$$
$$(\frac{\mu_0 I}{4\pi r})Rd\phi_c(-\sin\phi_c\mathbf{a_x} + \cos\phi_c\mathbf{a_y})[1 - \frac{2R}{r}\sin\theta\sin\phi_c]^{-1/2}$$

which by the binomial expansion [$(1-x)^{-1/2} \cong 1 + x/2$] is approximately

$$\mathbf{dA} \cong (\frac{\mu_0 I}{4\pi r})Rd\phi_c(-\sin\phi_c\mathbf{a_x} + \cos\phi_c\mathbf{a_y})[1 + \frac{R}{r}\sin\theta\sin\phi_c].$$

Now,

$$\mathbf{A} = \int_0^{2\pi}\mathbf{dA} = \int_0^{2\pi}(\frac{\mu_0 I}{4\pi r})Rd\phi_c(-\sin\phi_c\mathbf{a_x} + \cos\phi_c\mathbf{a_y})[1 + \frac{R}{r}\sin\theta\sin\phi_c].$$

Note that for this integral, terms with just $\sin\phi_c$ or $\cos\phi_c$ will go to zero because those functions go through their full cycle over 2π. Eliminating those terms,

$$\mathbf{A} = \int_0^{2\pi}(\frac{\mu_0 I}{4\pi r})Rd\phi_c(-\sin\phi_c\mathbf{a_x} + \cos\phi_c\mathbf{a_y})[\frac{R}{r}\sin\theta\sin\phi_c].$$

Furthermore, note that since $\sin\phi_c\cos\phi_c = \frac{1}{2}\sin 2\phi_c$, the term with $\sin\phi_c\cos\phi_c$ will integrate to zero also. Thus the integral simplifies to

$$\mathbf{A} = \int_0^{2\pi}(\frac{\mu_0 I}{4\pi r})Rd\phi_c(-\sin\phi_c\mathbf{a_x})[\frac{R}{r}\sin\theta\sin\phi_c] = (-\frac{\mu_0 IR^2\sin\theta\mathbf{a_x}}{4\pi r^2})\int_0^{2\pi}d\phi_c\sin^2\phi_c .$$

This integral is easily solved yielding

$$\mathbf{A} = \left(-\frac{\mu_0 I R^2 \sin\theta \mathbf{a_x}}{4\pi r^2}\right)\pi = -\frac{\mu_0 I R^2 \sin\theta}{4r^2}\mathbf{a_x}.$$

Now, we found this on the y-z plane. Note that on that plane, $-\mathbf{a_x} = \mathbf{a_\varphi}$. Thus, generally,

$$\mathbf{A} = \frac{\mu_0 I R^2 \sin\theta}{4r^2}\mathbf{a_\varphi}.$$

Note that the field has the same dependence on θ as for the short current dipole, that is the EM radiation is primarily broadcast perpendicular to the z-axis.