1. For the circular current ring shown below, for the current element shown, what is $dA(r, \theta, \phi)$? Hint: you cannot assume $r=r'$ for this problem. Hint 2: write the answer in Cartesian coordinates, that is, actually give me $dA(x, y, z)$ in terms of $x, y, z, R$ and $\phi_c$.

\[ I \, dl = \frac{\mu_0 I}{4\pi} \frac{dl}{r'}. \]

For an arc of a circle,

\[ |dl| = Rd\phi_c. \]

To get the direction of $dl$, use Cartesian coordinates:

\[ dl = Rd\phi_c (\_\_ a_x + \_\_ a_y), \]
and to get the blanks in the equation, think about what happens at various angles $\phi$. For zero angle, $dl$ points in the y-direction. For 90 degrees angle, $dl$ points in the $-x$ direction. Thus,

$$dl = rd\phi_r(-\sin\phi_r a_x + \cos\phi_r a_y),$$

$$dA = \left(\frac{\mu_o I}{4\pi}\right) \frac{Rd\phi_r(-\sin\phi_r a_x + \cos\phi_r a_y)}{r'}.$$

Now, we must find $r'$. Use Cartesian coordinates to get the length between the point at space we are measuring the field at and the current element:

$$r' = \sqrt{(x-x_c)^2 + (y-y_c)^2 + z^2},$$

$$x_c = R \cos\phi_c,,$$

$$y_c = R \sin\phi_c,$$

so

$$r' = \sqrt{(x-R \cos\phi_c)^2 + (y-R \sin\phi_c)^2 + z^2},$$

and

$$dA = \left(\frac{\mu_o I}{4\pi}\right) \frac{Rd\phi_r(-\sin\phi_r a_x + \cos\phi_r a_y)}{\sqrt{(x-R \cos\phi_c)^2 + (y-R \sin\phi_c)^2 + z^2}}.$$

2. Now, find $A$. Hint: do you have to find $A$ everywhere to know what it is everywhere? Hint 2: after using this hint, convert $r'$ back to spherical coordinates and then approximate $r\gg R$. Hint 3: you will need $(1-x)^{-1/2} \simeq 1 + x/2$.

ANSWER:
No, we don’t have to find it everywhere, since by symmetry it will be the same at any $\phi$, or anywhere around the $z$-axis. So, we can find it on the $y$-$z$ plane, setting $x=0$. Then,

$$r' = \sqrt{R^2 \cos^2\phi_c + (y-R \sin\phi_c)^2 + z^2} = \sqrt{R^2 \cos^2\phi_c + y^2 - 2yR \sin\phi_c + R^2 \sin^2\phi_c + z^2}$$

$$= \sqrt{R^2 + y^2 - 2yR \sin\phi_c + z^2}.$$

If we are on the $y$-$z$ plane, then

$$y^2 + z^2 = r^2,$$ and
\[ y = r \sin \theta . \] Plugging in,
\[
r' = \sqrt{R^2 + r^2 - 2Rr \sin \theta \sin \phi_c}.
\]
Approximating that \( r >> R \), we have that
\[
r' \approx \sqrt{r^2 - 2Rr \sin \theta \sin \phi_c} = r[1 - \frac{2R}{r} \sin \theta \sin \phi_c]^{1/2}.
\]
Plugging this into \( dA \),
\[
dA = \left( \frac{\mu_0 I}{4\pi r} \right) Rd\phi_c (-\sin \phi_c a_x + \cos \phi_c a_y) = \frac{\mu_0 I}{4\pi r} Rd\phi_c (-\sin \phi_c a_x + \cos \phi_c a_y)[1 - \frac{2R}{r} \sin \theta \sin \phi_c]^{-1/2},
\]
which by the binomial expansion \([ (1-x)^{-1/2} \approx 1 + x/2 \] is approximately
\[
dA \approx \left( \frac{\mu_0 I}{4\pi r} \right) Rd\phi_c (-\sin \phi_c a_x + \cos \phi_c a_y)[1 + \frac{R}{r} \sin \theta \sin \phi_c].
\]
Now,
\[
A = \int_0^{2\pi} dA = \int_0^{2\pi} \left( \frac{\mu_0 I}{4\pi r} \right) Rd\phi_c (-\sin \phi_c a_x + \cos \phi_c a_y)[1 + \frac{R}{r} \sin \theta \sin \phi_c].
\]
Note that for this integral, terms with just \( \sin \phi_c \) or \( \cos \phi_c \) will go to zero because those functions go through their full cycle over \( 2\pi \). Eliminating those terms,
\[
A = \int_0^{2\pi} \left( \frac{\mu_0 I}{4\pi r} \right) Rd\phi_c (-\sin \phi_c a_x + \cos \phi_c a_y)[\frac{R}{r} \sin \theta \sin \phi_c].
\]
Furthermore, note that since \( \sin \phi_c \cos \phi_c = \frac{1}{2} \sin 2\phi_c \), the term with \( \sin \phi_c \cos \phi_c \) will integrate to zero also. Thus the integral simplifies to
\[
A = \int_0^{2\pi} \left( \frac{\mu_0 I}{4\pi r} \right) Rd\phi_c (-\sin \phi_c a_y)[\frac{R}{r} \sin \theta \sin \phi_c] = \left( \frac{\mu_0 I R^2}{4\pi r^2} \right) \int_0^{2\pi} d\phi_c \sin^2 \phi_c.
\]
This integral is easily solved yielding
\[ A = (-\frac{\mu_0 IR^2 \sin \theta a_x}{4\pi r^2}) \pi = -\frac{\mu_0 IR^2 \sin \theta}{4r^2} a_x. \]

Now, we found this on the y-z plane. Note that on that plane, \(-a_x = a_\phi\).

Thus, generally,

\[ A = \frac{\mu_0 IR^2 \sin \theta}{4r^2} a_\phi. \]

Note that the field has the same dependence on \(\theta\) as for the short current dipole, that is the EM radiation is primarily broadcast perpendicular to the z-axis.