A1 **Logic** (25 points)

Using resolution or another proof technique of your *stated* choice, establish each of the following.

a. (6.25 points)
$\models (\exists x)(\forall y)P(x, y, f(x, y)) \supset (\exists x)(\forall y)(\exists z)P(x, y, z)).$

b. (6.25 points)
$\Gamma \models (\forall x)P(x, x),$ *where* Γ contains the sentences

$$(\forall x)(\exists y)P(x, y),$$
$$(\forall x)(\forall y)(\forall z)[P(x, y) \supset [P(y, z) \supset P(x, z)]],$$
$$(\forall x)(\forall y)[P(x, y) \supset P(y, x)].$$

c. (6.25 points)
$\{[\neg P(x), \neg P(f(a)), Q(y)], [P(y)], [\neg P(g(b, x)), \neg Q(b)]\}$ is *un*satisfiable.

d. (6.25 points)
Any set containing the following clauses is *un*satisfiable:

$$[\neg P(x), Q(x), R(x, f(x))],$$
$$[T(a)],$$
$$[P(a)],$$
$$[\neg R(a, z), T(z)],$$
$$[\neg T(x), \neg Q(x)],$$
$$[\neg T(y), \neg S(y)],$$
$$[\neg P(x), Q(x), S(f(x))].$$

A2 **Logic** (25 points)

a. (6.25 points)

Consider the following interpretation (U, φ) of a fo language ℓ with 1-ary predicate symbols F, S, N and 2-ary predicate symbol O, *where* we do not care about the initial interpretation of the variables.

Let U = the set of past, present, and future animals, including humans.

Let $\varphi(F) = \{u \in U \mid u \text{ is a goldfish}\}$.

Let $\varphi(S) = \{u \in U \mid u \text{ is sad}\}$.

Let $\varphi(O) = \{(u, v) \in U^2 \mid u \text{ owns } v\}$.

Let $\varphi(N) = \{u \in U \mid u \text{ has no fishfood}\}$.

Employing the fo language ℓ above and its interpretation also above, provide a translation of the following English sentence into a cwff of ℓ.

`Some goldfish is sad if all its owners have no fishfood.` (1)

Hint: You may want to translate (1) above clause by clause and then put the clauses together appropriately. This may also help with obtaining partial credit. Pronouns, like 'its' in (1) above, nicely translate as variables.

b. (18.75 points @ 6.25 points/argument)

Express each of the following arguments in predicate logic using the notation '$\models$' for *logical consequence.*

N.B. *Be sure to specify clearly the meaning of all your predicate symbols, constant symbols, etc. E.g., Frank and Mary just below should each be named by a distinct constant symbol.*

b1. (6.25 points)
Frank is a boy who does not own a car. Mary dates only boys who own cars. Therefore, Mary does not date Frank.

b2. (6.25 points)
Every member of the policy committee is either a Democrat or a Republican. Some members of the policy committee are wealthy. Mary is not a Democrat, but she is wealthy. Therefore, if Mary is a member of the policy committee, she is a Republican.

b3. (6.25 points)
Some scientific subjects are not interesting, but all scientific subjects are edifying. Therefore, some edifying things are not interesting.

A3 **Logic** (25 points)

For each of the following cwffs, provide an explicit interpretation witnessing it to be *in*valid.

a. (6.25 points)
$(\forall x)(\exists y)R(x, y) \supset (\exists z)R(z, z).$

b. (6.25 points)
$(\forall x)(\exists y)R(x, y) \supset (\exists y)(\forall x)R(x, y).$

c. (6.25 points)
$(\forall x)[R(x) \ \vee \ S(x)] \supset [(\forall x)R(x) \ \vee \ (\forall x)S(x)].$

d. (6.25 points)
$[(\exists x)R(x) \ \wedge \ (\exists x)S(x)] \supset [(\exists x)[R(x) \ \wedge \ S(x)].$

A4 **Logic** (25 points)

For *this* problem, A4, we provide *considerable* information, much of which is to *motivate* the problem (which is to prove a surprising result). *Then*, near the end and on the next page, we state what is to be done for this problem and subsequently give a hint making it not so difficult. (◡̈)

Definition PA *(called:* the f.o. theory of Peano arithmetic*) has binary predicate symbol* $=$*, binary function symbols* $+, \cdot$*, constant symbols* $0, 1, 2, \ldots$*, and the equality axioms together with the infinite, algorithmically decidable set of the following axioms (in infix notation):*

$$(\forall v_1) v_1 + 0 = v_1; \tag{2}$$

$$(\forall v_1)(\forall v_2) v_1 + (v_2 + 1) = (v_1 + v_2) + 1; \tag{3}$$

$$(\forall v_1) v_1 \cdot 0 = 0; \tag{4}$$

$$(\forall v_1)(\forall v_2) v_1 \cdot (v_2 + 1) = (v_1 \cdot v_2) + v_1; \tag{5}$$

$$0 + 1 = 1; \tag{6}$$

for each positive $n \in N$*,*

$$\underbrace{1 + \ldots + 1}_{n\ 1\text{'s}} = n, \tag{7}$$

where the n *on the right-hand side of (7) is understood to be the constant symbol for representing* n *and where it is again understood that the expression* $1 + \ldots + 1$ *is parenthesized with association to the left;*

$$(\forall v_1)(\forall v_2)[v_1 + 1 = v_2 + 1 \supset v_1 = v_2]; \tag{8}$$

$$(\forall v_1) 0 \neq v_1 + 1; \tag{9}$$

and all cwffs (closed well formed formulas) of the the underlying language of the form

$$\text{closure}([[\{A\}[x/0] \ \wedge \ (x)[A \supset \{A\}[x/(x+1)]] \supset (x)A]), \tag{10}$$

where A *is an arbitrary wff (well formed formula) of PA,* where *closure*(B) *is* $(\forall x_1) \cdots (\forall x_m) B$ *for the distinct free variables* $x_1, \ldots, x_m$ *of wff* B*,* and, where $\{A\}[x/t]$ *is the result of simultaneously substituting for each free occurrence of* x *in the wff* A*, the term* t*.*

PA has a decidable language.

The first four axioms constitute the recursive definitions of $+$ and $\cdot$. The axioms of the form (10) all together constitute a statement of the principle of mathematical induction.

PA suffices to prove all the theorems in *elementary* number theory books! It has as its *standard normal model* the obvious one for the arithmetic of non-negative integers.[1] It *was* originally *intended* to be complete[2] (but, thanks to Gödel's First Incompleteness Theorem from 1931, we now know it is not — proof omitted).

PA is not decidable (proof omitted), but its theorems can be algorithmically listed.

(Problem A4 continues onto the next page.)

[1]A *normal* model is (by definition) one in which the interpretation of the equality *symbol* is equality on the model's universe/domain of discourse.

In PA's standard normal model, the universe/domain of discourse is the (countable) set of non-negative integers, each constant symbol n is interpreted as the non-negative integer n, $+$ is interpreted as addition of the non-negative integers, … .

[2]This is since it is and was supposed to be an (algorithmically decidable) axiomatization *of a fixed standard normal model.* By contrast the first order theory of groups (not detailed here) was intended to axiomatize a whole giant, interesting collection of rather disparate normal models.

(This page is the continuation of Problem A4.)

Definition T_{PA} *(called:* the f.o. theory of arithmetical truth*) has the same language as* PA *above and its axioms are the entire set of cwffs true in the standard normal model of* PA*.*

Trivially, T_{PA} is complete. However, it is not decidable and has no (algorithmically decidable) axiomatization (proofs omitted).

What is to be done for this problem: *Follow the hint just below to employ the Compactness Theorem as well as a suitable form of the Skolem-Löwenheim Theorem to prove* in*formally that T_{PA} also has a* countable non-*standard normal model, i.e., a countable one* not *isomorphic to its countable standard normal model.*

Hint: Form an extension $\mathrm{T}'_{\mathrm{PA}}$ of T_{PA} by adding a *new* constant symbol $\mathbf{i}$ and the *additional* infinite set of axioms

$$I = \{\mathbf{i} > 0, \mathbf{i} > 1, \mathbf{i} > 2, \ldots\}, \tag{11}$$

where, for terms t_1,t_2 of $\mathrm{T}'_{\mathrm{PA}}$,

$$t_1 > t_2 \tag{12}$$

is an abbreviation for the wff, also of $\mathrm{T}'_{\mathrm{PA}}$,

$$(\exists v_1)[t_1 = t_2 + (v_1 + 1)]. \tag{13}$$

Show informally that each *finite* subset of the axioms of $\mathrm{T}'_{\mathrm{PA}}$ has a normal model.

Explain how to conclude from Compactness that $\mathrm{T}'_{\mathrm{PA}}$ itself has a normal model.

Explain how to conclude from a suitable form of the Skolem-Löwenheim Theorem that $\mathrm{T}'_{\mathrm{PA}}$ has a *countable* normal model.

Show informally this new, countable normal model *restricted to the language of just* T_{PA} can*not* be isomorphic to the (countable) standard normal model of T_{PA}. To do this, show that, in the new normal model itself, the constant symbol $\mathbf{i}$ has a interpretation very unlike the *standard* meaning of any of the constant symbols *of* PA. N.B. The restricted version of the new normal model still has the same universe/domain of discourse; therefore, the object which is the *interpretation of* $\mathbf{i}$ from the unrestricted version *is still in there.*