# Properties Complementary to Program Self-Reference*

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**Abstract.** In computability theory, program self-reference is formalized by the *not-necessarily-constructive* form of Kleene's Recursion Theorem (**krt**). In a programming system in which **krt** holds, for any preassigned, algorithmic task, there exists a program that, in a sense, creates a copy of itself, and then performs that task on the self-copy. Herein, properties *complementary* to **krt** are considered. Of particular interest are those properties involving the implementation of *control structures*. One main result is that *no* property involving the implementation of *denotational* control structures is complementary to **krt**. This is in contrast to a result of Royer, which showed that implementation of **if-then-else** — a denotational control structure — *is* complementary to the *constructive* form of Kleene's Recursion Theorem. Examples of *non*-denotational control structures whose implementation *is* complementary to **krt** are then given. Some such control structures so nearly resemble denotational control structures that they might be called *quasi-denotational*.

Keywords: Computability Theory, Programming Language Semantics, Self-Reference.

## 1 Introduction

Let $\mathbb{N} = \{0, 1, 2, ...\}$, let $\langle \cdot, \cdot \rangle : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be any 1-1, onto, computable function, and let $\psi$ be an *effective programming system* (eps)[1] [20, 16]. For all $p \in \mathbb{N}$, let $\psi_p = \psi(\langle p, \cdot \rangle)$. One can think of $\psi$ as a *programming language*, and of $\psi_p$ as the partial function (mapping $\mathbb{N}$ to $\mathbb{N}$) computed by the $\psi$-program with numerical name $p$. The *not-necessarily-constructive form of Kleene's Recursion Theorem (***krt***)* [20, page 214, problem 11-4] *holds in* $\psi \stackrel{\text{def}}{\Leftrightarrow}$

$$(\forall p)(\exists e)(\forall x)[\psi_e(x) = \psi_p(\langle e, x \rangle)]. \tag{1}$$

(1) may be interpreted as follows. $\psi$-program $p$ represents an arbitrary, preassigned, algorithmic task to perform with a self-copy; $e$ represents a $\psi$-program that

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[1] That is, $\psi$ is a partial computable function mapping $\mathbb{N}$ to $\mathbb{N}$ such that, for every partial computable function $\alpha$ mapping $\mathbb{N}$ to $\mathbb{N}$, $(\exists p \in \mathbb{N})[\psi(\langle p, \cdot \rangle) = \alpha]$.

1. creates a copy of itself, external to itself, and, *then*,
2. runs the preassigned task $p$ on the pair consisting of this self-copy and $e$'s input $x$.

The '$e$' on the right-hand side of the equation in (1) *is* the self-copy of the original '$e$' on the left-hand side of this equation. The way in which $e$ uses this self-copy is according to how the preassigned task $p$ says to. Thus, in an important sense, $e$ creates a *usable self-model*; it has usable self-knowledge [7].

The form of *recursion* provided by **krt** is *more general* than that built into most programming languages, i.e., that treated by denotational semantics [17, 23]. The '$e$' on the right-hand side of the equation in (1) is $e$'s own syntactic code-script, wiring/flow diagram, etc. Thus, $e$'s I/O behavior can depend, not *only* upon $e$'s *denotational* characteristics,[2] but upon $e$'s *connotational* characteristics (e.g., $e$'s number of symbols or run-time complexity) as well [21].

Self-referential programs were first introduced by Kleene in [14], where he used such programs to prove properties of ordinal notations. His theorem, and its variants [20, 6],[3] have since found widespread application. Interesting examples can be found in [2, 3, 20, 7, 22, 11, 4].[4]

Our ultimate goal is to characterize insightfully the power of program self-reference, as formalized by **krt**.[5] In this paper, we examine the subject somewhat indirectly, by studying properties *complementary* to **krt**, as described below.

An eps $\psi$ is *acceptable* $\overset{\text{def}}{\Leftrightarrow}$ $(\forall \text{ eps } \xi)(\exists \text{ computable } t : \mathbb{N} \to \mathbb{N})\ (\forall p)[\psi_{t(p)} = \xi_p]$ [20, 16, 18, 19, 21]. Thus, the acceptable epses are exactly those epses into which every eps can be compiled. Any eps corresponding to a real-world, general purpose programming language (e.g., C++, Java, Haskell) is acceptable. A characterization of the acceptable epses due to the first author[6] is that they are exactly those epses having an implementation of *every* control structure (see Definitions 1 (§3.1) and 4 (§4.1), herein).[7]

In [21], Royer showed that **if-then-else** (Example 1 in §3.1, herein) and the *constructive* form of Kleene's Recursion Theorem (**KRT**) (equation (6) in §3.2, herein) are *complementary*, in the sense that: for each, there is an eps having

[2] Since $\psi$ is, itself, a partial computable function mapping $\mathbb{N}$ to $\mathbb{N}$, there exists a *universal simulator* or *self-interpreter* for $\psi$, i.e., a $u$ such that $\psi_u = \psi$ [13]. $e$ *can* inquire about its own denotational characteristics by running $u$ on $e$, i.e., for all $x \in \mathbb{N}$, $\psi_u(\langle e, x\rangle) = \psi(\langle e, x\rangle) = \psi_e(x)$.

[3] Rogers' *Fixed-Point Recursion Theorem* (**fprt**) is a variant of **krt** that is often attributed to Kleene [20, page 180]. For an eps $\psi$, **fprt** *holds in* $\psi$ $\overset{\text{def}}{\Leftrightarrow}$ for all computable $f : \mathbb{N} \to \mathbb{N}$, $(\exists e)[\psi_e = \psi_{f(e)}]$. **fprt** and **krt** should *not* be confused, however, as **fprt** is *strictly weaker* than **krt** [18, Theorems 5.1 and 5.3].

[4] In the work of Bongard, *et al.* [5], robots employ self-modeling to recover locomotion after injury. See also [1, 10].

[5] For recent work in this direction, see [8].

[6] This result appears in [18, 19].

[7] The term *control structure* is given a formal definition in the literature [18, 19, 21], and includes more than what is indicated in Definitions 1 (§3.1) and 4 (§4.1). The first author's result holds for all control structures covered by this formal definition.

that one, and *not* the other; *but*, any eps having *both* is acceptable [21, Theorem 4.1.12] (Theorem 2 in §3.2, herein).[8] Given the first author's characterization of the acceptable epses, one way of interpreting Royer's result is: **if-then-else** and **KRT** are independent notions that, together, yield all control structures.

The proof of Royer's result employs, quite essentially, the *constructivity* of **KRT**. Many other *similar* results concerning **KRT** (not described herein) have the same undesirable quality; that is, the constructivity of **KRT** is *all mixed up* with the program self-modeling.

**krt** is the focus of the present paper, as it embodies *pure* self-modeling with*out* the *additional* constructivity of **KRT**. Specifically, the interest herein is in properties complementary to **krt**, similar to the way in which implementation of **if-then-else** is complementary to **KRT** (see Definition 2 in §3.3, herein).

One main result (Corollary 1 in §3.3, herein) is that **krt** is *not* complementary to the implementation of *any class of denotational control structures* [21, 23] (Definition 1 in §3.1, herein) — a type of control structure that includes **if-then-else**. This says, in part, that the constructivity of **KRT** is *essential* to establishing Royer's result.

Despite this outcome, there *do* exist reasonable *non*-denotational control structures whose implementation *is* complementary to **krt**. We give examples of such control structures in Section 4. Some such control structures so nearly resemble denotational control structures that they might be called *quasi-denotational*.

Section 2, just below, covers notation and preliminaries.

Due to space constraints, nearly all proofs appear in the appendix.

## 2 Notation and Preliminaries

Computability-theoretic concepts not explained below are treated in [20].

Lowercase Roman letters, with or without decorations, range over elements of $\mathbb{N}$, unless stated otherwise. Uppercase Roman letters, with or without decorations, range over subsets of $\mathbb{N}$, unless stated otherwise.

The pairing function $\langle\cdot,\cdot\rangle$ was introduced in Section 1. For all $x$, $\langle x\rangle \stackrel{\text{def}}{=} x$. For all $x_1,...,x_n$, where $n > 2$, $\langle x_1,...,x_n\rangle \stackrel{\text{def}}{=} \big\langle x_1, \langle x_2,...,x_n\rangle\big\rangle$. For all $n$, all $i \in \{1,...,n\}$, and all $x_1,...,x_n$, $\pi_i^n(\langle x_1,...,x_n\rangle) \stackrel{\text{def}}{=} x_i$.

$\mathcal{P}$ denotes the collection of all partial functions mapping $\mathbb{N}$ to $\mathbb{N}$. $\alpha$, $\beta$, $\gamma$, $\delta$, $\xi$, and $\psi$, with or without decorations, range over elements of $\mathcal{P}$. For all $\alpha$ and $p$, $\alpha_p \stackrel{\text{def}}{=} \alpha(\langle p,\cdot\rangle)$. We use Church's lambda-notation [20] to name partial functions, including total functions and predicates, as is standard in many programming languages.[9]

For all $\alpha$ and $x$, $\alpha(x){\downarrow}$ denotes that $\alpha(x)$ converges; $\alpha(x){\uparrow}$ denotes that $\alpha(x)$ diverges.[10] We identify a partial function with its graph, e.g., we identify $\alpha$

[8] Note that, in [21], **if-then-else** is called **conditional**.

[9] For example, $\lambda x\,\boldsymbol{.}\,(x+1)$ denotes the successor function.

[10] For all $\alpha$ and $x$, $\alpha(x)$ *converges* iff there exists $y$ such that $\alpha(x) = y$; $\alpha(x)$ *diverges* iff there is *no* $y$ such that $\alpha(x) = y$. If $\alpha$ is partial computable, and $x$ is such that

with the set $\{(x,y) : \alpha(x) = y\}$. We use $\uparrow$ to denote the value of a divergent computation.

As per footnote 1, $\psi$ is an eps $\overset{\text{def}}{\Leftrightarrow}$ $\psi$ is partial computable, *and*, $(\forall$ partial computable $\alpha)(\exists p)[\psi_p = \alpha]$ [20, 16]. $\mathcal{EPS}$ denotes the collection of all epses. $\varphi$ denotes a fixed, acceptable eps [20, 16, 18, 19, 21].

Intuitively, a mapping $\Gamma : \mathbb{N}^m \times \mathcal{P}^n \to \mathcal{P}$, where $m + n > 0$, is a *computable operator* iff there exists an algorithm for listing the *graph of* the partial function $\Gamma(x_1, ..., x_m, \alpha_1, ..., \alpha_n)$ from $x_1, ..., x_m$ and listings of the *graphs of* the partial functions $\alpha_1, ..., \alpha_n$ — independently of the enumeration order chosen for each of $\alpha_1, ..., \alpha_n$ [20, §9.8]. Let $\Gamma_0, \Gamma_1, ...$ be any standard, algorithmic enumeration of the computable operators of all types $\mathbb{N}^m \times \mathcal{P}^n \to \mathcal{P}$, where $m + n > 0$.[11] Let $\Theta_0, \Theta_1, ...$ be a similar enumeration of the computable operators of all types $\mathbb{N}^m \times \mathcal{P}^{n+1} \to \mathcal{P}$, where $m + n > 0$. The $\Theta_i$ are those computable operators of a type suitable to determine a recursive denotational control structure (Definition 1(b) in §3.1, herein). For ease of presentation, we shall use the $\Theta_i$ exclusively for the purpose of describing recursive denotational control structures. The $\Gamma_i$, being of (more or less) arbitrary type, will be used to describe *non*recursive (denotational and *non*-denotational) control structures (Definition 1(a) in §3.1, and Definition 4(a) in §4.1, herein, respectively).

Let $\mu : \mathbb{N} \to \mathbb{N}$ be a computable function such that, for all $i$, $\Gamma_{\mu(i)}$ is the *least fixed point* of $\Theta_i$ w.r.t. the last argument of $\Theta_i$ [20, §11.5] (see also [17, 23]). That is, if $i$, $m$, and $n$ are such that $\Theta_i : \mathbb{N}^m \times \mathcal{P}^{n+1} \to \mathcal{P}$, then $\Gamma_{\mu(i)} : \mathbb{N}^m \times \mathcal{P}^n \to \mathcal{P}$ is such that, for all $x_1, ..., x_m$, $\alpha_1, ..., \alpha_n$, and $\beta$, $\Gamma_{\mu(i)}(x_1, ..., x_m, \alpha_1, ..., \alpha_n) = \beta$ implies (i) and (ii) below.

(i) $\Theta_i(x_1, ..., x_m, \alpha_1, ..., \alpha_n, \beta) = \beta$.
(ii) $(\forall \gamma)[\Theta_i(x_1, ..., x_m, \alpha_1, ..., \alpha_n, \gamma) = \gamma \Rightarrow \beta \subseteq \gamma]$.

The *non*recursive denotational control structure determined $\Gamma_{\mu(i)}$ is the recursive denotational control structure determined by $\Theta_i$ under *least fixed point semantics* [17, 21, 23] (see Definition 1 in §3.1, herein).

In several places, we make reference to the following result.

**Theorem 1 (Machtey, *et al.* [15, Theorem 3.2]).** For all epses $\psi$, $\psi$ is acceptable $\Leftrightarrow$ $(\exists$ computable $f : \mathbb{N} \to \mathbb{N})(\forall a, b)[\psi_{f(\langle a,b \rangle)} = \psi_a \circ \psi_b]$.

## 3 Denotational Control Structures and krt

In this section, we show that there is *no* class of denotational control structures whose implementation is complementary to **krt** (Corollary 1). Theorem 3, from which the preceding result follows, is proven by a finite injury priority argument.[12]

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$\alpha(x)$ diverges, then one can imagine that a program associated with $\alpha$ *goes into an infinite loop* on input $x$.

[11] A formal definition of the computable operators (called *recursive operators* in [20]) as well as a construction of the $\Gamma_i$'s can be found in [20, §9.8] and in our appendix.

[12] Rogers [20] explains priority arguments. We should also mention that our proof of Theorem 3 makes essential use of Royer's [21, Theorem 4.2.15].

We begin with a brief introduction to denotational control structures in the context of epses. Then, we formally state Royer's result (Theorem 2), that implementation of **if-then-else** (Example 1, below) — a denotational control structure — *is* complementary to the *constructive* form of Kleene's Recursion Theorem (**KRT**) (equation (6), below).

**3.1.** In the context of epses, an instance of a control structure [18, 19, 21, 12, 9] is a means of forming a composite program from given constituent programs and/or data. An instance of a *denotational* control structure, more specifically, is one for which the I/O behavior of a composite program can depend *only* upon the I/O behavior of the constituent programs and upon the data. So, for example, the I/O behavior of such a composite program can*not* depend upon the connotational characteristics of its constituent programs, e.g., their number of symbols or run-time complexity.

*Recursive* denotational control structures differ from *non*recursive ones in that, for the former, the composite program is, in a sense, one of the constituent programs. For such a control structure, the I/O behavior of a composite program can*not* depend upon the connotational characteristics of the composite program, itself, just as it can*not* depend upon those of the other constituent programs.

In the following definition, $x_1, ..., x_m$ represent data, $x_{m+1}, ..., x_{m+n}$ represent constituent programs, and $f(\langle x_1, ..., x_{m+n}\rangle)$ represents the composite program formed from $x_1, ..., x_{m+n}$.

**Definition 1.** For all epses $\psi$, and all $f : \mathbb{N} \to \mathbb{N}$, (a) and (b) below.

(a) Suppose $i$, $m$, and $n$ are such that $\Gamma_i : \mathbb{N}^m \times \mathcal{P}^n \to \mathcal{P}$. Then, $f$ *is effective instance in* $\psi$ *of the* non*recursive denotational control structure determined by* $\Gamma_i \Leftrightarrow f$ is computable and, for all $x_1, ..., x_{m+n}$, $\psi_{f(\langle x_1,...,x_{m+n}\rangle)} = \Gamma_i(x_1, ..., x_m, \psi_{x_{m+1}}, ..., \psi_{x_{m+n}})$.
(b) Suppose $i$, $m$, and $n$ are such that $\Theta_i : \mathbb{N}^m \times \mathcal{P}^{n+1} \to \mathcal{P}$. Then, $f$ *is an effective instance in* $\psi$ *of the recursive denotational control structure determined by* $\Theta_i \Leftrightarrow f$ is computable and, for all $x_1, ..., x_{m+n}$, $\psi_{f(\langle x_1,...,x_{m+n}\rangle)} = \Theta_i(x_1, ..., x_m, \psi_{x_{m+1}}, ..., \psi_{x_{m+n}}, \psi_{f(\langle x_1,...,x_{m+n}\rangle)})$.

For the remainder of the present subsection (3.1), let $\psi$ be any fixed eps.

*Example 1.* Choose $i_{\text{ite}}$ such that $\Gamma_{i_{\text{ite}}} : \mathcal{P}^3 \to \mathcal{P}$, and, for all $\alpha$, $\beta$, and $\gamma$,

$$\Gamma_{i_{\text{ite}}}(\alpha, \beta, \gamma)(x) = \begin{cases} \beta(x), \text{ if } [\alpha(x)\!\downarrow \wedge\ \alpha(x) > 0]; \\ \gamma(x), \text{ if } [\alpha(x)\!\downarrow \wedge\ \alpha(x) = 0]; \\ \uparrow, \quad\ \text{otherwise.}^{13} \end{cases} \tag{2}$$

Then, the *nonrecursive denotational control structure determined by* $\Gamma_{i_{\text{ite}}}$ is **if-then-else** [18, 19]. Furthermore, an effective instance in $\psi$ of **if-then-else** is any computable function $f : \mathbb{N} \to \mathbb{N}$, such that, for all $a$, $b$, $c$, and $x$,

$$\psi_{f(\langle a,b,c\rangle)}(x) = \begin{cases} \psi_b(x), \text{ if } [\psi_a(x)\!\downarrow \wedge\ \psi_a(x) > 0]; \\ \psi_c(x), \text{ if } [\psi_a(x)\!\downarrow \wedge\ \psi_a(x) = 0]; \\ \uparrow, \quad\ \text{otherwise.} \end{cases} \tag{3}$$

[13] Note that the choice of $i_{\text{ite}}$ is *not* unique.

*Example 2.* Choose $i'_{\rm ite}$ such that $\Theta_{i'_{\rm ite}} : \mathcal{P}^4 \to \mathcal{P}$, and, for all $\alpha$, $\beta$, $\gamma$, and $\delta$,

$$\Theta_{i'_{\rm ite}}(\alpha,\beta,\gamma,\delta)(x) = \begin{cases} \beta(x), \text{ if } & [\alpha(x)\downarrow \wedge\ \alpha(x) > 0] \\ & \vee [\quad \beta(x)\downarrow \wedge\ \gamma(x)\downarrow \wedge\ \delta(x)\downarrow \\ & \quad \wedge\ \beta(x) = \gamma(x) = \delta(x) \quad ]; \\ \gamma(x), \text{ if } [\alpha(x)\downarrow \wedge\ \alpha(x) = 0]; \\ \uparrow, \quad \text{otherwise.} \end{cases} \tag{4}$$

Then, an $f$ as in (3) above is an effective instance in $\psi$ of the recursive denotational control structure determined by $\Theta_{i'_{\rm ite}}$. Furthermore, if $g : \mathbb{N} \to \mathbb{N}$ is a computable function such that, for all $a$, $b$, $c$, and $x$,

$$\psi_{g(\langle a,b,c\rangle)}(x) = \begin{cases} \psi_b(x), \text{ if } & [\psi_a(x)\downarrow \wedge\ \psi_a(x) > 0] \\ & \vee [\psi_b(x)\downarrow \wedge\ \psi_c(x)\downarrow \wedge\ \psi_b(x) = \psi_c(x)]; \\ \psi_c(x), \text{ if } [\psi_a(x)\downarrow \wedge\ \psi_a(x) = 0]; \\ \uparrow, \quad \text{otherwise;} \end{cases} \tag{5}$$

then, $g$ is *also* such an effective instance.[14]

Intuitively, in Example 2, $g$ is an instance of a *hasty* variant of **if-then-else**. That is, $g(\langle a,b,c\rangle)$ runs $\psi$-programs $a$, $b$, and $c$, on $x$, in parallel. If both $b$ and $c$ halt on $x$, and yield the same result, then $g(\langle a,b,c\rangle)$ does not wait to see if $a$ halts on $x$.

As the preceding examples demonstrate, recursive denotational control structures are *more general* than *non*recursive ones, in that they allow greater versatility among their implementations.

Henceforth, when convenient, we will abbreviate *effective instance*, *nonrecursive denotational control structure*, and *recursive denotational control structure*, by ei, ndcs, and rdcs, respectively.

**3.2.** For any eps $\psi$, the *constructive form of Kleene's Recursion Theorem (***KRT***) holds in* $\psi \overset{\rm def}{\Leftrightarrow}$

$$(\exists \text{ computable } r : \mathbb{N} \to \mathbb{N})(\forall p, x)[\psi_{r(p)}(x) = \psi_p(\langle r(p), x\rangle)]. \tag{6}$$

In (6), $r(p)$ plays the role of the self-referential $e$ in (1). Thus, in an eps in which **KRT** holds, self-referential programs can be found *algorithmically* from a program for the preassigned task.

The following is the formal statement of Royer's result.

**Theorem 2 (Royer [21, Theorem 4.1.12]).** For all epses $\psi$, let $P(\psi) \Leftrightarrow$ there is an ei of **if-then-else** in $\psi$. Then, (a)-(c) below.

(a) $(\exists$ eps $\psi)[$**KRT** *holds* in $\psi \wedge \neg P(\psi)]$.
(b) $(\exists$ eps $\psi)[$**KRT** does *not* hold in $\psi \wedge P(\psi)]$.
(c) $(\forall$ eps $\psi)\big[[$**KRT** holds in $\psi \wedge P(\psi)] \Leftrightarrow \psi$ is acceptable$\big]$.

[14] It can be seen that $\Gamma_{i_{\rm ite}} = \Gamma_{\mu(i'_{\rm ite})}$. Thus, $f$ provides a minimal fixed-point solution of (4); whereas, $g$ provides a *non*-minimal fixed-point solution of (4) [17, 21, 23].

**3.3.** The following definition makes precise what it means for a *property* of an eps to be complementary to **krt**.[15]

**Definition 2.** For all $P \subseteq \mathcal{EPS}$, $P$ *is complementary to* **krt** $\Leftrightarrow$ (a)-(c) below.

(a) $(\exists$ eps $\psi)[$**krt** *holds* in $\psi \wedge \neg P(\psi)]$.
(b) $(\exists$ eps $\psi)[$**krt** does *not* hold in $\psi \wedge P(\psi)]$.
(c) $(\forall$ eps $\psi)\big[[$**krt** holds in $\psi \wedge P(\psi)] \Leftrightarrow \psi$ is acceptable$\big]$.

The following notion is used in the proofs of Theorem 3 and Corollary 1.

**Definition 3.** For all $I$, $\{\Theta_i : i \in I\}$ is *recursively denotationally omnipotent* $\Leftrightarrow$ $(\forall$ eps $\psi)[\ (\forall i \in I)[$there is an ei in $\psi$ of the rdcs determined by $\Theta_i] \Rightarrow \psi$ is acceptable$]$.

Thus, a class of recursive operators is recursively denotationally omnipotent iff it is *so powerful* that: having an effective instance of each of the control structures that it determines causes an eps to be acceptable.[16]

Corollary 1, our main result of this section, follows from the next theorem.[17]

**Theorem 3.** Let $I$ be such that $\{\Theta_i : i \in I\}$ is *not* recursively denotationally omnipotent. Then, there exists an eps $\psi$ such that (a)-(c) below.

(a) **krt** holds in $\psi$.
(b) For each $i \in I$, there is an ei in $\psi$ of the ndcs determined by $\Gamma_{\mu(i)}$.
(c) $\psi$ is *not* acceptable.

The proof of Theorem 3 appears in the appendix.

**Corollary 1.** There is *no* $I$ such that $\lambda\psi \in \mathcal{EPS}\,\boldsymbol{.}(\forall i \in I)[$there is an ei in $\psi$ of the rdcs determined by $\Theta_i]$ is complementary to **krt**.

*Proof of Corollary.* Suppose, by way of contradiction, that such an $I$ exists.
Case $\{\Theta_i : i \in I\}$ is recursively denotationally omnipotent. Then, clearly, the stated property does *not* satisfy Definition 2(b) — a contradiction.
Case $\{\Theta_i : i \in I\}$ is *not* recursively denotationally omnipotent. Then, clearly, by Theorem 3, the stated property does *not* satisfy ($\Rightarrow$) of Definition 2(c) — a contradiction. □ (*Corollary 1*)

[15] It is relatively straightfoward to show that *no* single property *characterizes* the complement of **krt**, e.g., there exist $P \subseteq \mathcal{EPS}$ and $Q \subseteq \mathcal{EPS}$ such that both $P$ and $Q$ are complementary to **krt**, but $P \not\subseteq Q$ and $Q \not\subseteq P$.

[16] For example, choose $i_{\text{comp}}$ such that $\Theta_{i_{\text{comp}}} : \mathcal{P}^3 \to \mathcal{P}$, and, for all $\alpha$, $\beta$, and $\gamma$, $\Theta_{i_{\text{comp}}}(\alpha,\beta,\gamma) = \alpha\circ\beta$. Then, the recursive denotational control structure determined by $\Theta_{i_{\text{comp}}}$ is ordinary composition, and, by Theorem 1, the class consisting of just $\{\Theta_{i_{\text{comp}}}\}$ is recursively denotationally omnipotent.

[17] The statement of Theorem 3 is slightly stronger than needed. To establish Corollary 1, it would suffice that, for each $i \in I$, there exist an ei in $\psi$ of the rdcs determined by $\Theta_i$. As is, the theorem has an additional corollary. For all $I$, we say that $\{\Gamma_i : i \in I\}$ is *nonrecursively denotationally omnipotent* $\Leftrightarrow$ $(\forall$ eps $\psi)[(\forall i \in I)[$there is an ei in $\psi$ of the ndcs determined by $\Gamma_i] \Rightarrow \psi$ is acceptable$]$. Then, since the eps $\psi$ constructed in Theorem 3 is *not* acceptable, one has that, for all $I$, $\{\Theta_i : i \in I\}$ is recursively denotationally omnipotent $\Leftrightarrow$ $\{\Gamma_{\mu(i)} : i \in I\}$ is *non*recursively denotationally omnipotent. This result is of some independent interest.

## 4 Control Structures Complementary to krt

In this section, we give examples of control structures whose implementation *is* complementary to **krt**. Each of our examples is drawn from a class of control structures that we call *coded composition* (**CC**). Although, the control structures in this class can, in general, be *non*-denotational, they are still quite reasonable, in that they look much like control structures with which one could actually program.

Note that Definition 2, in the preceding section, formalized what it means for a property of an eps to be complementary to **krt**.

**4.1.** The following definition introduces the notion of *non*recursive control structures, generally.

**Definition 4.** Suppose $i$ and $m$ are such that $\varGamma_i : \mathbb{N}^m \times \mathcal{P} \to \mathcal{P}$. Then, for all epses $\psi$, and all $f : \mathbb{N} \to \mathbb{N}$, $f$ *is an effective instance in* $\psi$ *of the* non*recursive control structure determined by* $\varGamma_i \Leftrightarrow f$ is computable and, for all $x_1, ..., x_m$, $\psi_{f(\langle x_1,...,x_m\rangle)} = \varGamma_i(x_1, ..., x_m, \psi)$.

Henceforth, we will abbreviate *nonrecursive control structure* by ncs.

**Definition 5.**

(a) Suppose that $(f^{\mathrm{L}}, g^{\mathrm{L}}, f^{\mathrm{R}}, g^{\mathrm{R}}) : (\mathbb{N} \to \mathbb{N})^4$ is such that (i)-(iii) below.
   (i) Each of $f^{\mathrm{L}}$, $g^{\mathrm{L}}$, $f^{\mathrm{R}}$, and $g^{\mathrm{R}}$ is computable.
   (ii) For all $a$, $f^{\mathrm{L}}_a$ and $f^{\mathrm{R}}_a$ are onto.[18]
   (iii) For all $a$, $g^{\mathrm{L}}_a$ and $g^{\mathrm{R}}_a$ are 1-1.
   Then, $(f^{\mathrm{L}}, g^{\mathrm{L}}, f^{\mathrm{R}}, g^{\mathrm{R}})$-**CC** $: \mathbb{N}^2 \times \mathcal{P} \to \mathcal{P}$ is the computable operator, such that, for all $a$, $b$, and $\psi$,

$$(f^{\mathrm{L}}, g^{\mathrm{L}}, f^{\mathrm{R}}, g^{\mathrm{R}})\text{-}\mathbf{CC}(a, b, \psi) = f^{\mathrm{L}}_a \circ \psi_a \circ g^{\mathrm{L}}_a \circ f^{\mathrm{R}}_b \circ \psi_b \circ g^{\mathrm{R}}_b. \tag{7}$$

(b) Suppose that $(f^{\mathrm{L}}, g^{\mathrm{L}}, f^{\mathrm{R}}, g^{\mathrm{R}}) : (\mathbb{N} \to \mathbb{N})^4$ is as in (a) above. Then, for all epses $\psi$, $(f^{\mathrm{L}}, g^{\mathrm{L}}, f^{\mathrm{R}}, g^{\mathrm{R}})$-**CC** *holds in* $\psi \Leftrightarrow$ there is an ei in $\psi$ of the ncs determined by $(f^{\mathrm{L}}, g^{\mathrm{L}}, f^{\mathrm{R}}, g^{\mathrm{R}})$-**CC**.

(c) For all epses $\psi$, **CC** *holds in* $\psi \Leftrightarrow$ there exists $(f^{\mathrm{L}}, g^{\mathrm{L}}, f^{\mathrm{R}}, g^{\mathrm{R}}) : (\mathbb{N} \to \mathbb{N})^4$ as in (a) above such that $(f^{\mathrm{L}}, g^{\mathrm{L}}, f^{\mathrm{R}}, g^{\mathrm{R}})$-**CC** holds in $\psi$.

(L and R are mnemonic for *left* and *right*, respectively. **CC** is mnemonic for *coded composition*.) Thus, if $\psi$ is an eps, and $\diamond : \mathbb{N} \to \mathbb{N}$ is an ei in $\psi$ of the ncs determined by $(f^{\mathrm{L}}, g^{\mathrm{L}}, f^{\mathrm{R}}, g^{\mathrm{R}})$-**CC**, then, for all $a$ and $b$,

$$\psi_{a \diamond b} = f^{\mathrm{L}}_a \circ \psi_a \circ g^{\mathrm{L}}_a \circ f^{\mathrm{R}}_b \circ \psi_b \circ g^{\mathrm{R}}_b, \tag{8}$$

where, in (8), $\diamond$ is written using infix notation.

**CC** may be thought of as a *collection of* control structures, one for each choice of $(f^{\mathrm{L}}, g^{\mathrm{L}}, f^{\mathrm{R}}, g^{\mathrm{R}})$. As the next theorem shows, the property of having an effective instance of *some* control structure in this collection, is complementary to **krt**.

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[18] Recall that $f_a = f(\langle a, \cdot\rangle)$.

**Theorem 4.** $\lambda\psi \in \mathcal{EPS}\,.\,[\mathbf{CC}$ holds in $\psi]$ is complementary to **krt**.

The proof of Theorem 4 appears in the appendix. The proof employs a trick similar to that used in the proof of Theorem 1 to show that: if $\psi$ is an eps in which both **krt** and **CC** hold, then $\psi$ is acceptable. Intuitively, if **krt** holds in $\psi$, then $\psi$-program $b$ as in (8) can *know* its own program number. Thus, $b$ can *decode* its input as *encoded* by the 1-1 function $g^{\mathrm{R}}_b$. Similarly, $b$ can *pre-encode* its output, so that the onto function $f^{\mathrm{R}}_b$ sends this output to the value that $b$ would *actually* like to produce. The situation is similar for $\psi$-program $a$.[19]

**4.2.** As mentioned above, **CC** may be thought of as a *collection of* control structures, one for each choice of $(f^{\mathrm{L}}, g^{\mathrm{L}}, f^{\mathrm{R}}, g^{\mathrm{R}})$. However, it is *not* the case that, for *each* choice of $(f^{\mathrm{L}}, g^{\mathrm{L}}, f^{\mathrm{R}}, g^{\mathrm{R}})$, the property $\lambda\psi \in \mathcal{EPS}\,.\,[(f^{\mathrm{L}}, g^{\mathrm{L}}, f^{\mathrm{R}}, g^{\mathrm{R}})$-$\mathbf{CC}$ holds in $\psi]$ is complementary to **krt**.[20] An obvious question is: *which* choices of $(f^{\mathrm{L}}, g^{\mathrm{L}}, f^{\mathrm{R}}, g^{\mathrm{R}})$ yield properties complementary to **krt**, and *which* do *not*? As the following, somewhat curious result shows, the answer is intimately tied to the choice of $f^{\mathrm{R}}$, specifically.

**Theorem 5.**

(a) There exists a computable $f^{\mathrm{R}} : \mathbb{N} \to \mathbb{N}$ and an eps $\psi$ such that

$$(\forall a)(\exists y)(\forall x)\left[f^{\mathrm{R}}_a(x) = \begin{cases} y, \text{ if } x = 0; \\ 0, \text{ if } x = y; \\ x, \text{otherwise} \end{cases}\right]; \tag{10}$$

$(\pi^2_2, \pi^2_2, f^{\mathrm{R}}, \pi^2_2)$-$\mathbf{CC}$ holds in $\psi$, and $\psi$ *not* acceptable.[21]

(b) Suppose that $(f^{\mathrm{L}}, g^{\mathrm{L}}, f^{\mathrm{R}}, g^{\mathrm{R}}) : (\mathbb{N} \to \mathbb{N})^4$ is as in Definition 5(a), and that $f^{\mathrm{R}} = \pi^2_2$. Then, any eps in which $(f^{\mathrm{L}}, g^{\mathrm{L}}, f^{\mathrm{R}}, g^{\mathrm{R}})$-$\mathbf{CC}$ holds is acceptable.

The proof of Theorem 5 appears in the appendix.[22]

Note that, in Theorem 5(a), for all $a$, $f^{\mathrm{R}}_a$ is a recursive permutation that acts like the identity on all but at most two values. Arguably, a control structure such as that in Theorem 5(a) so nearly resembles a *denotational* control structure that it might be called *quasi-denotational*.[23]

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[19] The details of the proof, however, are rather involved. One complication arises from the fact that, if $\diamond$ is as in (8), then the $f^{\mathrm{L}}$, $g^{\mathrm{L}}$, $f^{\mathrm{R}}$, and $g^{\mathrm{R}}$ *stack up* when one tries to iterate $\diamond$. This can be seen, for example, from the underlined terms in the following calculation.

$$\begin{aligned} \psi_{a\diamond(b\diamond c)} &= f^{\mathrm{L}}_a \circ \psi_a \circ g^{\mathrm{L}}_a \circ f^{\mathrm{R}}_{b\diamond c} \circ \psi_{b\diamond c} \circ g^{\mathrm{R}}_{b\diamond c} \\ &= f^{\mathrm{L}}_a \circ \psi_a \circ g^{\mathrm{L}}_a \circ \underline{f^{\mathrm{R}}_{b\diamond c}} \circ f^{\mathrm{L}}_b \circ \psi_b \circ g^{\mathrm{L}}_b \circ f^{\mathrm{R}}_c \circ \psi_c \circ g^{\mathrm{R}}_c \circ \underline{g^{\mathrm{R}}_{b\diamond c}}. \end{aligned} \tag{9}$$

[20] For example, the control structure determined by $(\pi^2_2, \pi^2_2, \pi^2_2, \pi^2_2)$-$\mathbf{CC}$ is ordinary composition, which, by Theorem 1, causes an eps to be acceptable.

[21] Thus, by Theorem 4 (and $(\Rightarrow)$ of Definition 2(c)), **krt** does *not* hold in $\psi$.

[22] The eps used in the proof of Theorem 5(a) is that constructed in [20, page 42, problem 2-11].

[23] We *do* know of control structures that are *not* forms of coded composition and whose implementation is complementary to **krt**, but *none* so nearly resemble denotational control structures.

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## Appendix

Henceforth, let $\langle \cdot, \cdot \rangle$ be Cantor's pairing function [20], which, when iterated, satisfies: for all $n$, all $i \in \{1, ..., n\}$, and all $x_1, ..., x_n$, $x_i \leq \langle x_1, ..., x_n \rangle$. id : $\mathbb{N} \to \mathbb{N}$ denotes the identity function. For all $\alpha$, $\mathrm{dom}(\alpha) \stackrel{\mathrm{def}}{=} \{x : \alpha(x)\downarrow\}$ and $\mathrm{rng}(\alpha) \stackrel{\mathrm{def}}{=} \{y : (\exists x)[\alpha(x) = y]\}$. For all epses $\xi$ and $\psi$, $\xi \leq_R \psi$ (pronounced: $\xi$ *is Rogers reducible to* $\psi$) $\stackrel{\mathrm{def}}{\Leftrightarrow}$ ($\exists$ computable $t : \mathbb{N} \to \mathbb{N})(\forall p)[\psi_{t(p)} = \xi_p]$ [21]. $\Phi$ denotes a fixed Blum complexity measure for $\varphi$ [2].[24] For all $p$ and $t$,

$$\varphi_p^t \stackrel{\mathrm{def}}{=} \{(x, y) : x \leq t \wedge \Phi_p(x) \leq t \wedge \varphi_p(x) = y\}. \tag{11}$$

For all $p$, $x$, and $t$,

$$W_p^0 \stackrel{\mathrm{def}}{=} \emptyset; \tag{12}$$

$$W_p^{\langle x,t \rangle + 1} \stackrel{\mathrm{def}}{=} W_p^{\langle x,t \rangle} \cup \begin{cases} \{x\}, \text{ if } \Phi_p(x) \leq t; \\ \emptyset, \quad \text{otherwise.} \end{cases} \tag{13}$$

Note that, for all $p$ and $u$, $|W_p^u| \leq |W_p^{u+1}| \leq |W_p^u| + 1$. This helps to ensure that the $\Gamma_i$'s in (14) and (15) below are single valued.

$F_0, F_1, ...$ denotes a fixed, canonical enumeration of all finite (partial) functions mapping $\mathbb{N}$ to $\mathbb{N}$ [20, 16].

For the purposes of this appendix, we fix a particular enumeration of the computable operators, rather than assume any standard one, as we did in Section 2. For all $i$, the mappings $\Gamma_i^0, \Gamma_i^1, ...$, each of type $\mathbb{N}^m \times \mathcal{P}^n \to \mathcal{P}$, where $m = \pi_2^3(i)$ and $n = \max\{\pi_2^3(i) + \pi_3^3(i), 1\}$,[25] are defined recursively thus. For all $u$, $x_1, ..., x_m$, $\alpha_1, ..., \alpha_n$, and $y$,

$$\Gamma_i^0(x_1, ..., x_m, \alpha_1, ..., \alpha_n)(y) = \uparrow; \tag{14}$$

$$\begin{aligned} &\Gamma_i^{u+1}(x_1, ..., x_m, \alpha_1, ..., \alpha_n)(y) \\ &= \begin{cases} z, \text{ where } j_1, ..., j_n \text{ and } z \text{ are such that} \\ \qquad \langle x_1, ..., x_m, j_1, ..., j_n, y, z \rangle \in W_{\pi_1^3(i)}^u \\ \quad \wedge\ F_{j_1} \subseteq \alpha_1 \ \wedge \ \cdots \ \wedge \ F_{j_n} \subseteq \alpha_n \\ \quad \wedge\ [\quad \Gamma_i^u(x_1, ..., x_m, \alpha_1, ..., \alpha_n)(y)\uparrow \\ \qquad \vee\ \Gamma_i^u(x_1, ..., x_m, \alpha_1, ..., \alpha_n)(y) = z], \\ \quad \text{if such } j_1, ..., j_n, \text{ and } z \text{ exist;} \\ \uparrow, \text{ otherwise.} \end{cases} \end{aligned} \tag{15}$$

For all $i$, $\Gamma_i \stackrel{\mathrm{def}}{=} \lim_{u \to \infty} \Gamma_i^u$. For all $p$, $m$, $n$, and $u$, $\Theta_{\langle p,m,n \rangle}^u \stackrel{\mathrm{def}}{=} \Gamma_{\langle p,m,n+1 \rangle}^u$ and $\Theta_{\langle p,m,n \rangle} \stackrel{\mathrm{def}}{=} \Gamma_{\langle p,m,n+1 \rangle}$. Clearly, there exists an algorithm to determine $k$ from $i$, $x_1, ..., x_m$, $j_1, ..., j_n$, and $u$ such that, if $\Gamma_i : \mathbb{N}^m \times \mathcal{P}^n \to \mathcal{P}$, then $F_k = \Gamma_i^u(x_1, ..., x_m, F_{j_1}, ..., F_{j_n})$. Let $\mu$ be as defined in Section 2, but for *this* $\Gamma$ and $\Theta$.

[24] For any partial computable function, e.g., an eps, many such measures exist.

[25] This requirement on $n$ ensures that $m + n > 0$, i.e., that $\Gamma_i^u$ takes at least one argument.

**Lemma 1.** Suppose that $\psi$ is partial computable, and that $\xi$ is an $\mathsf{eps}$. Further suppose that $A \subseteq \mathbb{N}$ is such that, for all $p$, there exists $a \in A$ such that

$$\psi_a = \xi_p(\langle a, \cdot \rangle). \tag{16}$$

Then, (a) and (b) below.

(a) For all partial computable $\alpha$, there exists $a \in A$ such that $\psi_a = \alpha$.
(b) $\psi$ is an $\mathsf{eps}$ in which **krt** holds.

*Proof.* Suppose the hypotheses. To see (a), let partial computable $\alpha$ be fixed, and let $p$ be such that, for all $a$ and $x$,

$$\xi_p(\langle a, x \rangle) = \alpha(x). \tag{17}$$

Let $a \in A$ be as in (16) for $p$. Then, for all $x$,

$$\begin{aligned} \psi_a(x) &= \xi_p(\langle a, x \rangle) \ \{\text{by (16)}\} \\ &= \alpha(x) \qquad\quad \{\text{by (17)}\}. \end{aligned}$$

To see (b), note that (a) implies that $\psi$ is an $\mathsf{eps}$. Next, let $\psi$-program $b$ be fixed, and let $p$ be such that

$$\xi_p = \psi_b. \tag{18}$$

Let $a \in A$ be as in (16) for $p$. Then, for all $x$,

$$\begin{aligned} \psi_a(x) &= \xi_p(\langle a, x \rangle) \ \{\text{by (16)}\} \\ &= \psi_b(\langle a, x \rangle) \ \{\text{by (18)}\}. \end{aligned}$$

□ (*Lemma 1*)

**Lemma 2.** Suppose that $i$ is fixed, and that $m$ and $n$ are such that $\Theta_i : \mathbb{N}^m \times \mathcal{P}^{n+1} \to \mathcal{P}$. Further suppose that $\psi$ is partial computable, and that $\Psi$ is a Blum complexity measure for $\psi$. For all $a$ and $u$, let

$$\psi_a^u = \{(x, y) : x \le u \ \wedge \ \Psi_a(x) \le u \ \wedge \ \psi_a(x) = y\}. \tag{19}$$

Then, for all $g : \mathbb{N} \to \mathbb{N}$, (a) and (b) below are equivalent.

(a) For all $x = \langle x_1, ..., x_{m+n} \rangle$,

$$\psi_{g(x)} = \Theta_i(x_1, ..., x_m, \psi_{x_{m+1}}, ..., \psi_{x_{m+n}}, \psi_{g(x)}). \tag{20}$$

(b) For all $t$, there exists $u$ such that, for all $x = \langle x_1, ..., x_{m+n} \rangle < t$, (i) and (ii) below.
   (i) $\psi_{g(x)}^t \subseteq \Theta_i^u(x_1, ..., x_m, \psi_{x_{m+1}}^u, ..., \psi_{x_{m+n}}^u, \psi_{g(x)}^u)$.
   (ii) $\Theta_i^t(x_1, ..., x_m, \psi_{x_{m+1}}^t, ..., \psi_{x_{m+n}}^t, \psi_{g(x)}^t) \subseteq \psi_{g(x)}^u$.

*Proof.* A straightforward argument using the monotonicity and continuity properties of $\Theta_i$ [20, page 147]. □ (*Lemma 2*)

**Corollary 2.** For all $\mathsf{eps}$es $\psi$ and all $g : \mathbb{N} \to \mathbb{N}$,

$$\begin{array}{c}(\forall w)[\psi_{g(w)} = \varphi_w] \\ \Leftrightarrow \\ (\forall t)(\exists u)(\forall w < t)[\psi^t_{g(w)} \subseteq \varphi^u_w \ \wedge \ \varphi^t_w \subseteq \psi^u_{g(w)}].\end{array} \tag{21}$$

*Proof of Corollary.* Immediate by Lemma 2 with $i$ chosen such that $\Theta_i = \lambda w, \alpha \centerdot \varphi_w$. □ (*Corollary 2*)

**Lemma 3 (Royer [21, Theorem 4.2.15]).** For all partial computable $\xi$, $\xi$ is an acceptable $\mathsf{eps}$ $\Leftrightarrow$ there exists an acceptable $\mathsf{eps}$ $\psi$ such that $\psi \subseteq \xi$.

**Corollary 3.** Suppose that $\psi$ is an acceptable $\mathsf{eps}$ and that $\xi$ is partial computable. Further suppose that $t : \mathbb{N} \to \mathbb{N}$ is a computable function such that, for all $p$, $\psi_p \subseteq \xi_{t(p)}$. Then, $\xi$ is an acceptable $\mathsf{eps}$.

*Proof of Corollary.* Let $\psi$, $\xi$, and $t$ be as stated. For all $q$, $a$ and $i$, let $t'$, $\psi'$, and $\xi'$ be as follows.

$$t'(p) = \langle t(p), i \rangle, \text{ where } i = |\{p' < p : t(p') = t(p)\}|. \tag{22}$$

$$\psi'_{\langle a,i \rangle} = \begin{cases} \psi_p, & \text{where } p \text{ is such that } t'(p) = \langle a, i \rangle, \\ & \quad \text{if such a } p \text{ exists;} \\ \lambda x \centerdot \uparrow, & \text{otherwise.} \end{cases} \tag{23}$$

$$\xi'_{\langle a,i \rangle} = \xi_a. \tag{24}$$

Clearly, $t'$ is 1-1, and, thus, $\psi'$ is well-defined. Clearly, $\psi \leq_R \psi'$, as witnessed by $t'$, and, thus, $\psi'$ is an acceptable $\mathsf{eps}$. Clearly, $\xi' \equiv_R \xi$. Thus, to show that $\xi$ is an acceptable $\mathsf{eps}$, by Lemma 3, it suffices to show that $\psi' \subseteq \xi'$. For all $a$ and $i$, consider the following cases.

Case $(\exists p)[t'(p) = \langle a, i \rangle]$. Let $p$ be as asserted to exist by the case. Clearly, by (22),

$$a = t(p). \tag{25}$$

Thus,

$$\begin{aligned} \psi'_{\langle a,i \rangle} &= \psi_p && \{\text{by the case, the choice of } p\text{, and (23)}\} \\ &\subseteq \xi_{t(p)} && \{\text{by assumption}\} \\ &= \xi_a && \{\text{by (25)}\} \\ &= \xi'_{\langle a,i \rangle} && \{\text{by (24)}\}. \end{aligned}$$

Case $(\forall p)[t'(p) \neq \langle a, i \rangle]$. Then, $\psi'_{\langle a,i \rangle} = \lambda x \centerdot \uparrow$, and, thus, $\psi'_{\langle a,i \rangle} \subseteq \xi'_{\langle a,i \rangle}$.

□ (*Corollary 3*)

**Theorem 3.** Let $I$ be such that $\{\Theta_i : i \in I\}$ is *not* recursively denotationally omnipotent. Then, there exists an $\mathsf{eps}$ $\psi$ such that (a)-(c) below.

(a) **krt** holds in $\psi$.
(b) For each $i \in I$, there is an $\mathsf{ei}$ in $\psi$ of the $\mathsf{ndcs}$ determined by $\Gamma_{\mu(i)}$.
(c) $\psi$ is *not* acceptable.

*Proof.* Let $I$ be as stated. Since $I$ is *not* recursively denotationally omnipotent, there exists a *non*-acceptable $\mathsf{eps}$ $\xi$ such that, for each $i \in I$, there is an $\mathsf{ei}$ in $\xi$ of the $\mathsf{rdcs}$ determined by $\Theta_i$.

$\psi$ is constructed via a finite injury priority argument. For all $p$, $q$, and $i$, the requirements, $R_p$, $S_{\langle q,i\rangle}$, and $T$, are as follows.

$$\begin{aligned} R_p \quad &\Leftrightarrow (\exists b)[\psi_b = \varphi_p(\langle b, \cdot\rangle)].\\ S_{\langle q,i\rangle} &\Leftrightarrow \big[[\varphi_q \text{ is an } \mathsf{ei} \text{ in } \xi \text{ of the } \mathsf{rdcs} \text{ determined by } \Theta_i]\\ &\qquad\quad \Rightarrow [\text{there exists an } \mathsf{ei} \text{ in } \psi \text{ of the } \mathsf{ndcs} \text{ determined by } \Gamma_{\mu(i)}]\big].\\ T \quad &\Leftrightarrow (\forall \text{ computable } g : \mathbb{N} \to \mathbb{N})(\exists w)[\psi_{g(w)} \neq \varphi_w]. \end{aligned}$$

The requirements, in order of *decreasing* priority, are: $T, S_0, R_0, S_1, R_1, \ldots$ . The satisfaction of $R_p$, for all $p$, ensures that $\psi$ is an $\mathsf{eps}$ in which **krt** holds. The satisfaction of $S_{\langle q,i\rangle}$, for all $q$ and $i$, ensures that, for each $i \in I$, there is an $\mathsf{ei}$ in $\psi$ of the $\mathsf{ndcs}$ determined by $\Gamma_{\mu(i)}$. The satisfaction of $T$ ensures that $\psi$ is *not* acceptable.

Our strategy for satisfying $T$ is as follows. We construct $\psi$ so that: for all computable $g : \mathbb{N} \to \mathbb{N}$, if $g$ were a *counterexample* to $T$, then there would exist a partial computable $\tau'$ such that $\big(\forall b \in \mathrm{rng}(g)\big)[\tau'(b){\downarrow} \ \wedge \ \psi_b \subseteq \xi_{\tau'(b)}]$. If $\psi$ has this property, then the existence of such a counterexample would imply that $\xi$ is acceptable, which would be a contradiction.

$\psi$ is constructed in stages. For all $u$ and $b$, $\psi_b^u$ denotes $\psi_b$ at the beginning of stage $u$. For all $b$, $\psi_b^0 = \lambda x\boldsymbol{.}{\uparrow}$. For all $u$, $b$, and $x$, $\psi_b^{u+1}(x) = \psi_b^u(x)$, unless stated otherwise.

Two (total) functions, $\lambda u, a\boldsymbol{.}d^u(a)$ and $\lambda u, a\boldsymbol{.}e^u(a)$, and one partial function, $\lambda u, b\boldsymbol{.}\tau^u(b)$, are constructed in conjunction with $\psi$. $d$ and $\tau$ are used to help satisfy the $T$ requirement; $e$ is used to help satisfy the $S$ requirements. For all $a$, $d^0(a) = e^0(a) = 0$. $\tau^0 = \lambda b\boldsymbol{.}{\uparrow}$. The values of $d$, $e$, $\tau$ remain the same from one stage to the next, unless stated otherwise.

The following will be clear, by construction.

$$(\forall u)[\mathrm{dom}(\tau^u) \text{ is finite}]. \tag{26}$$

Moreover,

$$\begin{array}{c}\text{there exists an algorithm to determine}\\ i \text{ from } u \text{ such that } D_i = \mathrm{dom}(\tau^u).\end{array} \tag{27}$$

Let $\lambda u, p\boldsymbol{.}r^u(p)$ be such that, for all $u$ and $p$,

$$r^u(p) = \begin{cases} \langle p, j\rangle, \text{ where } j \text{ is } \textit{least} \text{ such that } (\forall x)[\tau^u(\langle\langle p,j\rangle, x\rangle){\uparrow}] \\ \quad \text{and } (\forall p' < p)[\langle p, j\rangle > r^u(p')]. \end{cases} \tag{28}$$

$r$ is used to help satisfy the $R$ and $S$ requirements. It can be shown, by a straightforward induction, that, if (26) holds as claimed, then, for all $u$, $\lambda p \centerdot r^u(p)$ is total and monotonically increasing. Furthermore, if (27) holds as claimed, then $\lambda u, p \centerdot r^u(p)$ is computable.

For all $p$ and $u$, it can be seen that $R_p$ is injured in stage $u$ whenever $r^{u+1}(p) \neq r^u(p)$. There are two ways that this can occur. The first is when, for some $x$, $[\tau^u(\langle r^u(p), x\rangle)\uparrow \ \wedge \ \tau^{u+1}(\langle r^u(p), x\rangle)\downarrow]$, or, equivalently, $\tau(\langle r^u(p), x\rangle)$ becomes defined in stage $u+1$. The second is when, for some $p' < p$, $[r^u(p') < r^u(p) \ \wedge \ r^{u+1}(p') \geq r^u(p)]$. In this latter case, $R_p$ is injured as a result of a *cascading effect*. Clearly, either condition causes $r^{u+1}(p) \neq r^u(p)$.

It can be seen that the $S$ requirements are injured in a similar fashion.

For all $b$, let $f$ be as follows.

$$D_{f(b)} = \{b\} \cup \begin{cases} \emptyset, & \text{if } \pi_2^2(b) = 0; \\ D_{f(x_{m+1})} \cup & \text{otherwise, where} \\ \quad \cdots \cup D_{f(x_{m+n})}, & \quad b = \langle\langle\langle q, i\rangle, j\rangle, x+1\rangle, \\ & \quad \Theta_i : \mathbb{N}^m \times \mathcal{P}^{n+1} \to \mathcal{P}, \\ & \quad \text{and } x = \langle x_1, .., x_{m+n}\rangle. \end{cases} \tag{29}$$

Let $\Xi$ be a Blum complexity measure for $\xi$. For all $c$ and $u$, let

$$\xi_c^u = \{(x, y) : x \leq u \ \wedge \ \Xi_c(x) \leq u \ \wedge \ \xi_c(x) = y\}. \tag{30}$$

Construct $\psi$, $d$, $e$, and $\tau$ by executing stages $u = 0, 1, ...$, successively, as in Figure 1.

*Claim 1.* For all $b$, $u$, and $x$, if $\psi_b^u(x)\downarrow$, then $x \leq u$.

*Proof of Claim.* Clear by the construction of $\psi$. □ (*Claim 1*)

*Claim 2.* Let $\Psi$ be such that, for all $b$ and $x$,

$$\Psi_b(x) = \begin{cases} u, \text{ where } u \text{ is } \textit{least} \text{ such that } \psi_b^u(x)\downarrow, \\ \qquad \text{if such a } u \text{ exists;} \\ \uparrow, \text{ otherwise.} \end{cases} \tag{31}$$

Then, $\Psi$ is a Blum complexity measure for $\psi$. Moreover, for all $b$ and $u$,

$$\psi_b^u = \{(x, y) : x \leq u \ \wedge \ \Psi_b(x) \leq u \ \wedge \ \psi_b(x) = y\}. \tag{32}$$

*Proof of Claim.* Follows from the construction of $\psi$ and Claim 1. □ (*Claim 2*)

*Claim 3.* For all $b \in \text{dom}(\tau)$, $\psi_b \subseteq \xi_{\tau(b)}$.

*Proof of Claim.* Clear by the construction of $\psi$ and $\tau$. □ (*Claim 3*)

STAGE $u = \langle p, k\rangle = \langle\langle q, i\rangle, k\rangle$. Let $a = r^u(p)$ $(= r^u(\langle q, i\rangle))$, and perform steps (1)-(3) below.

(1) Let $t = d^u(a)$, and determine whether conditions (a) and (b) below are satisfied.
   (a) $\psi^u_{\langle a,0\rangle}(t)\downarrow$.
   (b) $(\forall w < t)(\forall b)\big[\psi^t_{\langle a,0\rangle}(w) = b \;\Rightarrow\; [\psi^t_b \subseteq \varphi^u_w \;\wedge\; \varphi^t_w \subseteq \psi^u_b]\big]$.
   If conditions (a) and (b) above are satisfied, then perform substeps $(*)$ and $(**)$ below.
   $(*)$ For all $b \in D_{(f\circ\psi^u_{\langle a,0\rangle})(t)}$ such that $[\pi^2_1(b) > a \;\wedge\; \tau^u(b)\uparrow]$, find any $c$ such that $\psi^u_b \subseteq \xi_c$, and set $\tau^{u+1}(b) = c$.
   $(**)$ Set $d^{u+1}(a) = t + 1$.
(2) For all $x \leq u + 1$ and all $y$ such that $[\psi^u_{\langle a,0\rangle}(x)\uparrow \;\wedge\; \varphi^u_p(\langle\langle a, 0\rangle, x\rangle) = y]$, set $\psi^{u+1}_{\langle a,0\rangle}(x) = y$.
(3) Let $t = e^u(a)$, let $m$ and $n$ be such that $\Theta_i : \mathbb{N}^m \times \mathcal{P}^{n+1} \to \mathcal{P}$, and determine whether conditions (a) and (b) below are satisfied.
   (a) $\varphi^u_q(t)\downarrow$.
   (b) For all $x = \langle x_1, ..., x_{m+n}\rangle < t$, and all $c$ such that $\varphi^u_q(x) = c$, (i) and (ii) below.
      (i) $\xi^t_c \subseteq \Theta^u_i(x_1, ..., x_m, \xi^u_{x_{m+1}}, ..., \xi^u_{x_{m+n}}, \xi^u_c)$.
      (ii) $\Theta^t_i(x_1, ..., x_m, \xi^t_{x_{m+1}}, ..., \xi^t_{x_{m+n}}, \xi^t_c) \subseteq \xi^u_c$.
   If conditions (a) and (b) above are satisfied, then perform steps $(*)$ and $(**)$ below.
   $(*)$ For all $x = \langle x_1, ..., x_{m+n}\rangle < t$, all $y \leq u + 1$, and all $z$ such that $[\psi^u_{\langle a,x+1\rangle}(y)\uparrow \;\wedge\; \Gamma^u_{\mu(i)}(x_1, ..., x_m, \psi^u_{x_{m+1}}, ..., \psi^u_{x_{m+n}})(y) = z]$, set $\psi^{u+1}_{\langle a,x+1\rangle}(y) = z$.
   $(**)$ Set $e^{u+1}(a) = t + 1$.

**Fig. 1.** The construction of $\psi$, $d$, $e$, and $\tau$ in the proof of Theorem 3.

*Claim 4.* Let $a = \langle\langle q, i\rangle, j\rangle$ be fixed, and let $m$ and $n$ be such that $\Theta_i : \mathbb{N}^m \times \mathcal{P}^{n+1} \to \mathcal{P}$. For all $t$ and $u$ such that $e^u(a) = t$ and $e^{u+1}(a) = t + 1$, (a)-(c) below.

(a) For all $x \leq t$, $\varphi_q^u(x)\downarrow$.
(b) For all $x = \langle x_1, ..., x_{m+n}\rangle < t$, and all $c$ such that $\varphi_q^u(x) = c$, (i) and (ii) below.
  (i) $\xi_c^t \subseteq \Theta_i^u(x_1, ..., x_m, \xi_{x_{m+1}}^u, ..., \xi_{x_{m+n}}^u, \xi_c^u)$.
  (ii) $\Theta_i^t(x_1, ..., x_m, \xi_{x_{m+1}}^t, ..., \xi_{x_{m+n}}^t, \xi_c^t) \subseteq \xi_c^u$.
(c) For all $x = \langle x_1, ..., x_{m+n}\rangle < t$, all $y \leq u + 1$, and all $z$ such that $\Gamma_{\mu(i)}^u(x_1, ..., x_m, \psi_{x_{m+1}}^u, ..., \psi_{x_{m+n}}^u)(y) = z$, $\psi_{\langle a,x+1\rangle}^{u+1}(y) = z$.

*Proof of Claim.* (a) and (c) are each proven by a straightforward induction. (b) is clear by the construction of $\psi$ and $e$. □ (*Claim 4*)

*Claim 5.* Let $a = \langle\langle q, i\rangle, j\rangle$ be fixed, and let $m$ and $n$ be such that $\Theta_i : \mathbb{N}^m \times \mathcal{P}^{n+1} \to \mathcal{P}$. If $\lambda u \centerdot e^u(a)$ is *un*bounded, then (a)-(c) below.

(a) $\varphi_q$ is total.
(b) For all $x = \langle x_1, ..., x_{m+n}\rangle$, $\xi_{\varphi_q(x)} = \Theta_i(x_1, ..., x_m, \xi_{x_{m+1}}, ..., \xi_{x_{m+n}}, \xi_{\varphi_q(x)})$.
(c) For all $x = \langle x_1, ..., x_{m+n}\rangle$, $\psi_{\langle a,x+1\rangle} = \Gamma_{\mu(i)}(x_1, ..., x_m, \psi_{x_{m+1}}, ..., \psi_{x_{m+n}})$.

*Proof of Claim.* (a) follows immediately from (a) of Claim 4. (b) follows from Claim 2, (b) of Claim 4, and the right-to-left direction of Lemma 2. (c) follows from (c) of Claim 4, and from the fact that no *other* pairs are ever inserted into the graph of $\psi_{\langle a,x+1\rangle}$. Note that (a) and (b), together, imply that $\varphi_q$ is an ei in $\xi$ of the rdcs determined by $\Theta_i$. Also, note that (c) implies that $\lambda x \centerdot \langle a, x + 1\rangle$ is an ei in $\psi$ of the ndcs determined by $\Gamma_{\mu(i)}$. □ (*Claim 5*)

*Claim 6.* If $a$ is such that $\lambda u \centerdot e^u(a)$ is bounded, then, for all $x$ such that $(\forall u)[e^u(a) < x]$, $\psi_{\langle a,x+1\rangle} = \lambda x \centerdot \uparrow$.

*Proof of Claim.* Let $a$ and $x$ be as stated. Then, clearly, by the construction of $\psi$, *no* pairs are ever inserted into the graph of $\psi_{\langle a,x+1\rangle}$. □ (*Claim 6*)

*Claim 7.* For all $a$, $t$, and $u$, if $d^u(a) = t$ and $d^{u+1}(a) = t+1$, then (a)-(c) below.

(a) For all $w \leq t$, $\psi_{\langle a,0\rangle}^u(w)\downarrow$.
(b) For all $w < t$, and all $b$, if $\psi_{\langle a,0\rangle}^u(w) = b$, then $[\psi_b^t \subseteq \varphi_w^u \ \wedge\ \varphi_w^t \subseteq \psi_b^u]$.
(c) For all $w \leq t$, and all $b \in D_{(f \circ \psi_{\langle a,0\rangle}^u)(w)}$ such that $\pi_1^2(b) > a$, $\tau^{u+1}(b)\downarrow$.

*Proof of Claim.* (a) and (c) are each proven by a straightforward induction. (b) is clear by the construction of $\psi$ and $d$. □ (*Claim 7*)

*Claim 8.* For all $a$, if $\lambda u \centerdot d^u(a)$ is *un*bounded, then (a)-(c) below.

(a) $\psi_{\langle a,0\rangle}$ is total.
(b) For all $w$, $\psi_{\psi_{\langle a,0\rangle}(w)} = \varphi_w$.
(c) For all $b \in \bigcup_{w \in \mathbb{N}} D_{(f \circ \psi_{\langle a,0\rangle})(w)}$ such that $\pi_1^2(b) > a$, $\tau(b)\downarrow$.

*Proof of Claim.* (a) and (c) follow immediately from (a) and (c), respectively, of Claim 7. (b) follows from Claim 2, (b) of Claim 7, and (⇐) of Corollary 2.

□ (*Claim 8*)

*Claim 9.* For all $a$, there exists $u$ such that, for all $v > u$, $d^v(a) = d^u(a)$.

*Proof of Claim.* By way of contradiction, let $a$ be such that $\lambda u \centerdot d^u(a)$ is *un*bounded.

Let $\tau'$ be such that, for all $p$, $q$, $i$, and $j$, all $m$ and $n$ such that $\Theta_i : \mathbb{N}^m \times P^{n+1} \to P$, and all $x = \langle x_1, ..., x_{m+n} \rangle$,

$$\tau'(\langle\langle p,j\rangle,0\rangle) = \begin{cases} \tau(\langle\langle p,j\rangle,0\rangle), & \text{if } \langle p,j\rangle > a; \\ \textit{any } c \text{ such that } \psi_{\langle\langle p,j\rangle,0\rangle} \subseteq \xi_c, & \text{otherwise;} \end{cases} \tag{33}$$

$$\begin{aligned} &\tau'(\langle\langle\langle q,i\rangle,j\rangle,x+1\rangle) \\ &= \begin{cases} \tau(\langle\langle\langle q,i\rangle,j\rangle,x+1\rangle), & \text{if (i) } \langle\langle q,i\rangle,j\rangle > a; \\ \textit{any } c \text{ such that} & \text{if } \neg\text{(i) and } \lambda u \centerdot e^u(\langle\langle q,i\rangle,j\rangle) \text{ is bounded,} \\ \quad \psi_{\langle\langle\langle q,i\rangle,j\rangle,x+1\rangle} \subseteq \xi_c, & \quad \text{but } (\exists u)[e^u(\langle\langle q,i\rangle,j\rangle) \geq x]; \\ 0, & \text{if } \neg\text{(i) and } (\forall u)[e^u(\langle\langle q,i\rangle,j\rangle) < x]; \\ \varphi_q\big(x_1,...,x_m,\tau'(x_{m+1}), & \text{if } \neg\text{(i) and } \lambda u \centerdot e^u(\langle\langle q,i\rangle,j\rangle) \text{ is} \\ \quad ...,\tau'(x_{m+n})\big), & \quad \textit{un}\text{bounded.} \end{cases} \end{aligned} \tag{34}$$

Note that there are only finitely many $\langle p,j\rangle \leq a$. Thus, the otherwise case in (33) applies to only finitely many of the inputs of $\tau'$. Similarly, there are only finitely many $\langle\langle q,i\rangle,j\rangle \leq a$. For those for which $\lambda u \centerdot e^u(\langle\langle q,i\rangle,j\rangle)$ is bounded, there are only finitely many $x$ such that $(\exists u)[e^u(\langle\langle q,i\rangle,j\rangle) \geq x]$. Clearly, then, $\tau'$ is partial computable.

If follows from Claim 5(a) that, for all $b$,

$$\pi_1^2(b) \leq a \;\Rightarrow\; \tau'(b)\downarrow. \tag{35}$$

Let $B$ be such that

$$B = \bigcup_{w \,\in\, \mathbb{N}} D_{(f\circ\psi_{\langle a,0\rangle})(w)}. \tag{36}$$

Then, it follows from (35) and Claim 8(c) that, for all $b \in B$, $\tau'(b)\downarrow$.

The remainder of the proof of the claim is to show that, for all $b \in B$,

$$\psi_b \subseteq \xi_{\tau'(b)}. \tag{37}$$

Recall that, by (a) and (b) of Claim 8, for all $w$, $\psi_{\psi_{\langle a,0\rangle}(w)} = \varphi_w$. Thus, since $\mathrm{rng}(\psi_{\langle a,0\rangle}) \subseteq B$, establishing (37) would imply that, for all $w$,

$$\varphi_w = \psi_{\psi_{\langle a,0\rangle}(w)} \subseteq \xi_{(\tau'\circ\psi_{\langle a,0\rangle})(w)}, \tag{38}$$

which, by Corollary 3, would contradict the fact that $\xi$ is *not* acceptable.

For all $b \in B$, consider the following cases, the last of which is inductive.

Case $b = \langle\langle p,j\rangle,0\rangle$, where $\langle p,j\rangle > a$. Then,

$$\begin{aligned} \psi_b &\subseteq \xi_{\tau(b)} \;\; \{\text{by Claim 3}\} \\ &= \xi_{\tau'(b)} \;\{\text{by (33)}\}. \end{aligned}$$

Case $b = \langle\langle\langle q, i\rangle, j\rangle, x + 1\rangle$, where $\langle\langle q, i\rangle, j\rangle > a$. Similar to the previous case.
Case $b = \langle\langle p, j\rangle, 0\rangle$, where $\langle p, j\rangle \leq a$. Then, by (33),

$$\psi_b \subseteq \xi_c = \xi_{\tau'(b)}, \tag{39}$$

for some $c$.
Case $b = \langle\langle\langle q, i\rangle, j\rangle, x + 1\rangle$, where $\lambda u . e^u(\langle\langle q, i\rangle, j\rangle)$ is bounded, but $(\exists u)[e^u(\langle\langle q, i\rangle, j\rangle) \geq x]$. Similar to the previous case.
Case $b = \langle\langle\langle q, i\rangle, j\rangle, x + 1\rangle$, where $(\forall u)[e^u(\langle\langle q, i\rangle, j\rangle) < x]$. Then,

$$\begin{aligned}\psi_b &= \lambda x . \uparrow && \{\text{by Claim 6}\}\\ &\subseteq \xi_0 && \{\text{regardless of the value of } \xi_0\}\\ &= \xi_{\tau'(b)} && \{\text{by (33)}\}.\end{aligned}$$

Case $b = \langle\langle\langle q, i\rangle, j\rangle, x + 1\rangle$, where $\lambda u . e^u(\langle\langle q, i\rangle, j\rangle)$ is *un*bounded. Let $m$ and $n$ be such that $\Theta_i : \mathbb{N}^m \times \mathcal{P}^{n+1} \to \mathcal{P}$, and let $x = \langle x_1, ..., x_{m+n}\rangle$. Since $\{x_{m+1}, ..., x_{m+n}\} \subseteq B$, suppose, inductively, that

$$\psi_{x_{m+1}} \subseteq \xi_{\tau'(x_{m+1})} \ \wedge \ \cdots \ \wedge \ \psi_{x_{m+n}} \subseteq \xi_{\tau'(x_{m+n})}. \tag{40}$$

By (a) and (b) of Claim 5, $\varphi_q$ is an ei in $\xi$ of the rdcs determined by $\Theta_i$. Furthermore, by (c) of Claim 5, $\lambda x . \langle\langle\langle q, i\rangle, j\rangle, x+1\rangle$ is an ei in $\psi$ of the ndcs determined by $\Gamma_{\mu(i)}$. Thus,

$$\begin{aligned}\psi_b &= \Gamma_{\mu(i)}(x_1, ..., x_m, \psi_{x_{m+1}}, ..., \psi_{x_{m+n}}) && \{\text{by (c) of Claim 5}\}\\ &\subseteq \Gamma_{\mu(i)}(x_1, ..., x_m, \xi_{\tau'(x_{m+1})}, ..., \xi_{\tau'(x_{m+n})}) && \{\text{by (40) and the monotonicity of } \Gamma_{\mu(i)}\}\\ &\subseteq \Theta_i(x_1, ..., x_m, \xi_{\tau'(x_{m+1})}, ..., \xi_{\tau'(x_{m+n})},\\ &\qquad \xi_{\varphi_q(x_1,...,x_m,\tau'(x_{m+1}),...,\tau'(x_{m+n}))}) && \{\text{by (a),(b) of Claim 5 and properties of } \mu\}\\ &= \xi_{\varphi_q(x_1,...,x_m,\tau'(x_{m+1}),...,\tau'(x_{m+n}))} && \{\text{by (a),(b) of Claim 5}\}\\ &= \xi_{\tau'(b)} && \{\text{by (34)}\}.\end{aligned}$$

□ (*Claim 9*)

*Claim 10.* For all $p$, there exists $u$ such that, for all $v > u$, $r^v(p) = r^u(p)$.

*Proof of Claim.* By way of contradiction, let $p$ be *least* such that, for infinitely many $u$, $r^{u+1}(p) \neq r^u(p)$. By the choice of $p$, there exists $u$ such that, for all $p' < p$ and $v \geq u$, $r^v(p') = r^u(p')$. By Claim 9, there exists $v \geq u$ such that, for all $p' < p$ and $w \geq v$, $(d^w \circ r^u)(p') = (d^v \circ r^u)(p')$. Clearly, for all $p' < p$ and $w \geq v$,

$$(d^{w+1} \circ r^w)(p') = (d^{w+1} \circ r^u)(p') = (d^w \circ r^u)(p') = (d^w \circ r^w)(p'). \tag{41}$$

Let $w \geq v$ be such that $r^{w+1}(p) \neq r^w(p)$. Clearly, by the construction of $\psi$ and $d$, there must exist $a < r^w(p)$ such that $d^{w+1}(a) \neq d^w(a)$. Furthermore, there must exist $p'$ such that $r^w(p') = a$. Finally, since $r^w$ is monotonically increasing and $r^w(p') = a < r^w(p)$, it must be the case that $p' < p$. To summarize: there exists $p' < p$ and $w \geq v$ such that

$$(d^{w+1} \circ r^w)(p') \neq (d^w \circ r^w)(p'). \tag{42}$$

But this contradicts (41). □ (*Claim 10*)

*Claim 11.* For all $p$, there exists $a \in \text{rng}(r)$ such that $\psi_{\langle a,0\rangle} = \varphi_p(\langle\langle a,0\rangle,\cdot\rangle)$, i.e., $R_p$ is satisfied.

*Proof of Claim.* Let $p$ be fixed. By Claim 10, there exists $a = r(p)$. Note that, for infinitely many $u$, $\psi^{u+1}_{\langle a,0\rangle}(x)$ is set equal to $\varphi^u_p(\langle\langle a,0\rangle,x\rangle)$ for each $x \leq u+1$ such that $[\psi^u_{\langle a,0\rangle}(x)\uparrow \wedge \varphi^u_p(\langle\langle a,0\rangle,x\rangle)\downarrow]$. Furthermore, no *other* pairs are ever inserted into the graph of $\psi_{\langle a,0\rangle}$. Clearly, then, in the limit, $\psi_{\langle a,0\rangle} = \varphi_p(\langle\langle a,0\rangle,\cdot\rangle)$. □ (*Claim 11*)

*Claim 12.*

(a) For all $\alpha$, there exists $a \in \text{rng}(r)$ such that $\psi_{\langle a,0\rangle} = \alpha$.
(b) $\psi$ is an eps in which **krt** holds.

*Proof of Claim.* Immediate by Claim 11 and Lemma 1. □ (*Claim 12*)

*Claim 13.* For all $q$ and $i$, if $\varphi_q$ is an ei in $\xi$ of the rdcs determined by $\Theta_i$, then there exists $a$ such that $\lambda x.\langle a, x+1\rangle$ is an ei in $\psi$ of the ndcs determined by $\Gamma_{\mu(i)}$, i.e., $S_{\langle q,i\rangle}$ is satisfied.

*Proof of Claim.* Let $q$ and $i$ be such that $\varphi_q$ is an ei in $\xi$ of the rdcs determined by $\Theta_i$. Let $m$ and $n$ be such that $\Theta_i : \mathbb{N}^m \times \mathcal{P}^{n+1} \to \mathcal{P}$. By Claim 10, there exists $a = r(\langle q,i\rangle)$. Since $\varphi_q$ is total, for all $t$, there exists $u$ such that $\varphi^u_q(t)\downarrow$. Furthermore, by the Claim 2 and the left-to-right direction of Lemma 2, for all $t$, there exists $u$, such that, for all $x = \langle x_1, ..., x_{m+n}\rangle < t$, (i) and (ii) below.

(i) $\xi^t_{\varphi_q(x)} \subseteq \Theta^u_i(x_1, ..., x_m, \xi^u_{x_{m+1}}, ..., \xi^u_{x_{m+n}}, \xi^u_{\varphi_q(x)})$.
(ii) $\Theta^t_i(x_1, ..., x_m, \xi^t_{x_{m+1}}, ..., \xi^t_{x_{m+n}}, \xi^t_{\varphi_q(x)}) \subseteq \xi^u_{\varphi_q(x)}$.

Thus, it must be the case that $\lambda u.e^u(a)$ is *un*bounded. Therefore, by Claim 5(c), $\lambda x.\langle a, x+1\rangle$ is an ei in $\psi$ of the ndcs determined by $\Gamma_{\mu(i)}$. □ (*Claim 13*)

*Claim 14.* For all computable $g : \mathbb{N} \to \mathbb{N}$, there exists $x$ such that $\psi_{g(w)} \neq \varphi_w$, i.e., $T$ is satisfied.

*Proof of Claim.* By way of contradiction, let $g : \mathbb{N} \to \mathbb{N}$ be a computable function such that, for all $w$,

$$\psi_{g(w)} = \varphi_w. \tag{43}$$

Let $a \in \text{rng}(r)$ be such that $\psi_{\langle a,0\rangle} = g$. By (a) of Claim 12, such an $a$ exists. Since $\psi_{\langle a,0\rangle}$ is total, for all $t$, there exists $u$ such that $\psi^u_{\langle a,0\rangle}(t)\downarrow$. Furthermore, by Claim 2 and ($\Rightarrow$) of Corollary 2, for all $t$, there exists $u$, such that

$$(\forall w < t)(\forall b)\big[\psi_{\langle a,0\rangle}(x) = b \;\Rightarrow\; [\psi^t_b \subseteq \varphi^u_w \;\wedge\; \varphi^t_w \subseteq \psi^u_b]\big]. \tag{44}$$

Thus, it must be the case that, for infinitely many $u$, $d^{u+1}(a) \neq d^u(a)$. But this contradicts Claim 9. $\square$ (*Claim 14*)

$\square$ (*Theorem 3*)

**Corollary 4.** For all $I$, $\{\Theta_i : i \in I\}$ is recursively denotationally omnipotent $\Leftrightarrow$ $\{\Gamma_{\mu(i)} : i \in I\}$ is *non*recursively denotationally omnipotent.

**Theorem 4.** $\lambda\psi \in \mathcal{EPS}\,\boldsymbol{.}[\mathbf{CC}$ holds in $\psi]$ is complementary to **krt**.

*Proof.* Let $P$ be the stated property. That $P$ satisfies (a)-(c) of Definition 2 is shown as follows. (Recall: Definition 2 (§3.3) formalized what it means for a property of an eps to be complementary to **krt**.)

That $P$ satisfies (a): Riccardi [18, 19] showed that there exist *non*-acceptable epses in which **krt** holds. Thus, it suffices to show that $P$ satisfies ($\Rightarrow$) of Definition 2(c), which fact is shown below.

That $P$ satisfies (b): Follows from Theorem 5(a), and the fact that $P$ satisfies ($\Rightarrow$) of Definition 2(c), both of which are shown below.

That $P$ satisfies (c): To show that $P$ satisfies ($\Leftarrow$) of Definition 2(c), note that, in any acceptable eps, **krt** and ordinary composition hold. It remains to show that $P$ satisfies ($\Rightarrow$) of Definition 2(c). Let $\psi$ be any eps in which **krt** and **CC** hold. Let $(f^{\mathrm{L}}, g^{\mathrm{L}}, f^{\mathrm{R}}, g^{\mathrm{R}}) : (\mathbb{N} \to \mathbb{N})^4$ be such that there exists an ei in $\psi$ of the ncs determined by $(f^{\mathrm{L}}, g^{\mathrm{L}}, f^{\mathrm{R}}, g^{\mathrm{R}})$-**CC**, and let $\lambda\langle a, b\rangle\,\boldsymbol{.}(a \diamond b) : \mathbb{N} \to \mathbb{N}$ be such an effective instance. To reduce the need for parentheses, assume that $\diamond$ associates to the right, i.e., assume that, for all $a$, $b$, and $c$, $a \diamond b \diamond c = a \diamond (b \diamond c)$. For all $a$, $b$, and $p$, let

$$a^p \diamond b = \overbrace{a \diamond \ldots \diamond a}^{p \text{ occurrences}} \diamond b. \tag{45}$$

Let $f^{-\mathrm{L}}$ and $f^{-\mathrm{R}}$ be computable functions, and let $g^{-\mathrm{L}}$ and $g^{-\mathrm{R}}$ be partial computable functions, such that, for all $a$,

$$f_a^{\mathrm{L}} \circ f_a^{-\mathrm{L}} = g_a^{-\mathrm{L}} \circ g_a^{\mathrm{L}} = f_a^{\mathrm{R}} \circ f_a^{-\mathrm{R}} = g_a^{-\mathrm{R}} \circ g_a^{\mathrm{R}} = \mathrm{id}. \tag{46}$$

Clearly, these exist. By **krt** in $\psi$, there exist $b$, $c$, and $d$, which, for all $x$ and $z$, behave as follows.

$$\psi_b(z) = (f_b^{-\mathrm{L}} \circ \varphi_q)(y), \text{ where } \langle q, y\rangle = g_b^{-\mathrm{L}}(z). \tag{47}$$

$$\psi_c(z) = (f_c^{-\mathrm{L}} \circ f_{c^{q+1} \diamond d}^{-\mathrm{R}})(\langle q+1, y\rangle), \text{ where } \langle q, y\rangle = g_c^{-\mathrm{L}}(z). \tag{48}$$

$$\psi_d(x) = f_d^{-\mathrm{R}}(\langle 0, x\rangle). \tag{49}$$

For all $q$ and $x$, let $k$ and $k^{-1}$ be as follows.

$$k(\langle q, x\rangle) = \begin{cases} x, & \text{if } q = 0; \\ (g_d^{\mathrm{R}} \circ g_{c^1 \diamond d}^{\mathrm{R}} \circ \ldots \circ g_{c^{q-1} \diamond d}^{\mathrm{R}})(x), & \text{otherwise.} \end{cases} \tag{50}$$

$$k^{-1}(\langle q, x\rangle) = \begin{cases} x, & \text{if } q = 0; \\ (g_{c^{q-1} \diamond d}^{-\mathrm{R}} \circ \ldots \circ g_{c^1 \diamond d}^{-\mathrm{R}} \circ g_d^{-\mathrm{R}})(x), & \text{otherwise.} \end{cases} \tag{51}$$

Clearly, $k$ is computable, and, for all $q$, $k_q$ is 1-1. Clearly, $k^{-1}$ is partial computable. Note that the $-1$ in $k^{-1}$ is to emphasize that, for all $q$, $k_q^{-1}$ is a *left* inverse of $k_q$, i.e., for all $q$,

$$k_q^{-1} \circ k_q = \text{id}. \tag{52}$$

*Claim 1.* For all $q$ and $x$, $\psi_{c^q \diamond d}(x) = f^{-\mathrm{R}}_{c^q \diamond d}(\langle q, k_q(x)\rangle)$.

*Proof of Claim.* By induction on $q$. In the case when $q = 0$,

$$\psi_{c^q \diamond d}(x) = \psi_d(x) = f^{-\mathrm{R}}_d(\langle 0, x\rangle) = f^{-\mathrm{R}}_{c^q \diamond d}(\langle q, k_q(x)\rangle).$$

So suppose, inductively, that the claim holds for $q$. Then, for all $x$,

$$\begin{aligned}
&\psi_{c^{q+1} \diamond d}(x) \\
&= \psi_{c \diamond c^q \diamond d}(x) && \{\text{by (45)}\} \\
&= (f^{\mathrm{L}}_c \circ \psi_c \circ g^{\mathrm{L}}_c \circ f^{\mathrm{R}}_{c^q \diamond d} \circ \psi_{c^q \diamond d} \circ g^{\mathrm{R}}_{c^q \diamond d})(x) && \{\text{by the choice of } \diamond\} \\
&= (f^{\mathrm{L}}_c \circ \psi_c \circ g^{\mathrm{L}}_c \circ f^{\mathrm{R}}_{c^q \diamond d} \circ f^{-\mathrm{R}}_{c^q \diamond d})(\langle q, (k_q \circ g^{\mathrm{R}}_{c^q \diamond d})(x)\rangle) && \{\text{by the i.h.}\} \\
&= (f^{\mathrm{L}}_c \circ \psi_c \circ g^{\mathrm{L}}_c \circ f^{\mathrm{R}}_{c^q \diamond d} \circ f^{-\mathrm{R}}_{c^q \diamond d})(\langle q, k_{q+1}(x)\rangle) && \{\text{by (50)}\} \\
&= (f^{\mathrm{L}}_c \circ \psi_c \circ g^{\mathrm{L}}_c)(\langle q, k_{q+1}(x)\rangle) && \{\text{by (46)}\} \\
&= \big(f^{\mathrm{L}}_c \circ f^{-\mathrm{L}}_c \circ f^{-\mathrm{R}}_{c^{q+1} \diamond d})(\langle q+1, k_{q+1}(x)\rangle) && \{\text{by (48) and (46)}\} \\
&= f^{-\mathrm{R}}_{c^{q+1} \diamond d}(\langle q+1, k_{q+1}(x)\rangle) && \{\text{by (46)}\}.
\end{aligned}$$

□ (*Claim 1*)

*Claim 2.* For all $q$ and $x$, $\psi_{b \diamond c^q \diamond d}(x) = (\varphi_q \circ k_{q+1})(x)$.

*Proof of Claim 2.* For all $q$ and $x$,

$$\begin{aligned}
&\psi_{b \diamond c^q \diamond d}(x) \\
&= (f^{\mathrm{L}}_b \circ \psi_b \circ g^{\mathrm{L}}_b \circ f^{\mathrm{R}}_{c^q \diamond d} \circ \psi_{c^q \diamond d} \circ g^{\mathrm{R}}_{c^q \diamond d})(x) && \{\text{by the choice of } \diamond\} \\
&= (f^{\mathrm{L}}_b \circ \psi_b \circ g^{\mathrm{L}}_b \circ f^{\mathrm{R}}_{c^q \diamond d} \circ f^{-\mathrm{R}}_{c^q \diamond d})(\langle q, (k_q \circ g^{\mathrm{R}}_{c^q \diamond d})(x)\rangle) && \{\text{by Claim 1}\} \\
&= (f^{\mathrm{L}}_b \circ \psi_b \circ g^{\mathrm{L}}_b \circ f^{\mathrm{R}}_{c^q \diamond d} \circ f^{-\mathrm{R}}_{c^q \diamond d})(\langle q, k_{q+1}(x)\rangle) && \{\text{by (50)}\} \\
&= (f^{\mathrm{L}}_b \circ \psi_b \circ g^{\mathrm{L}}_b)(\langle q, k_{q+1}(x)\rangle) && \{\text{by (46)}\} \\
&= (f^{\mathrm{L}}_b \circ f^{-\mathrm{L}}_b \circ \varphi_q \circ k_{q+1})(x) && \{\text{by (47) and (46)}\} \\
&= (\varphi_q \circ k_{q+1})(x) && \{\text{by (46)}\}.
\end{aligned}$$

□ (*Claim 2*)

By **PKRT** in $\varphi$ [20], there exists $e_0, e_1, ...$ such that $\lambda p \centerdot e_p$ is computable, and, for all $p$,

$$\varphi_{e_p} = \varphi_p \circ k^{-1}_{e_p+1}. \tag{53}$$

*Claim 3.* For all $p$, $\psi_{b \diamond c^{e_p} \diamond d} = \varphi_p$.

*Proof of Claim.* For all $p$,

$$\begin{aligned}
\psi_{b \diamond c^{e_p} \diamond d} &= \varphi_{e_p} \circ k_{e_p+1} && \{\text{by Claim 2}\} \\
&= \varphi_p \circ k^{-1}_{e_p+1} \circ k_{e_p+1} && \{\text{by (53)}\} \\
&= \varphi_p && \{\text{by (52)}\}.
\end{aligned}$$

□ (*Claim 3*)

Thus, $\lambda p \centerdot b \diamond c^{e_p} \diamond d$ witnesses that $\psi$ is acceptable. □ (*Theorem 4*)

**Theorem 5.**

(a) There exists a computable $f^{\mathrm{R}} : \mathbb{N} \to \mathbb{N}$ and an $\mathsf{eps}$ $\psi$ such that

$$(\forall a)(\exists y)(\forall x)\left[ f_a^{\mathrm{R}}(x) = \begin{cases} y, \text{ if } x = 0; \\ 0, \text{ if } x = y; \\ x, \text{ otherwise} \end{cases} \right], \tag{54}$$

$(\pi_2^2, \pi_2^2, f^{\mathrm{R}}, \pi_2^2)$-**CC** holds in $\psi$, and $\psi$ *not* acceptable.

(b) Suppose that $(f^{\mathrm{L}}, g^{\mathrm{L}}, f^{\mathrm{R}}, g^{\mathrm{R}}) : (\mathbb{N} \to \mathbb{N})^4$ is as in Definition 5, and that $f^{\mathrm{R}} = \pi_2^2$. Then, any $\mathsf{eps}$ in which $(f^{\mathrm{L}}, g^{\mathrm{L}}, f^{\mathrm{R}}, g^{\mathrm{R}})$-**CC** holds is acceptable.

*Proof.*

(a) For all $y$, $p$, and $x$, let $\psi$ and $f^{\mathrm{R}}$ be as follows.

$$\psi_{\langle y,p \rangle}(x) = \begin{cases} \uparrow, & \text{if } y = 0 \ \wedge \ x = 0; \\ y - 1, & \text{if } y > 0 \ \wedge \ x = 0; \\ \varphi_p(x), & \text{otherwise.} \end{cases} \tag{55}$$

$$f^{\mathrm{R}}_{\langle y,p \rangle}(x) = \begin{cases} y - 1, & \text{if } y > 0 \ \wedge \ x = 0; \\ 0, & \text{if } y > 0 \ \wedge \ x = y - 1; \\ x, & \text{otherwise.} \end{cases} \tag{56}$$

It is easy to show that $\psi$ is a *non*-acceptable $\mathsf{eps}$. Clearly, $f^{\mathrm{R}}$ is computable, and, for all $a$, $f_a^{\mathrm{R}}$ is onto. Note that, for all $y$, $p$, and $q$,

$$\psi_{\langle y,p \rangle}(0) = \psi_{\langle y,q \rangle}(0). \tag{57}$$

Let $g$ be such that, for all $y$ and $z$,

$$g(y, z) = \begin{cases} 0, \text{ if } z = 0; \\ y, \text{ otherwise.} \end{cases} \tag{58}$$

Clearly, $g$ is computable. Let $h$ be a computable functions such that, for all $y$, $p$, $z$, and $q$,

$$\varphi_{h(y,p,z,q)} = \psi_{\langle y,p \rangle} \circ f_{\langle z,q \rangle} \circ \psi_{\langle z,q \rangle}. \tag{59}$$

Clearly, such an $h$ exists. For all $y$, $z$, $p$, and $q$, let $\diamond$ be as follows.

$$\langle y, p \rangle \diamond \langle z, q \rangle = \langle g(y, z), h(y, p, z, q) \rangle. \tag{60}$$

The remainder of the proof of (a) is to show that $\diamond$ is an $\mathsf{ei}$ in $\psi$ of the $\mathsf{ncs}$ determined by $(\pi_2^2, \pi_2^2, f^{\mathrm{R}}, \pi_2^2)$-**CC**. Clearly, $\diamond$ is computable. To see that, for all $y$, $p$, $z$, $q$, and $x$,

$$\psi_{\langle y,p \rangle \diamond \langle z,q \rangle}(x) = (\psi_{\langle y,p \rangle} \circ f^{\mathrm{R}}_{\langle z,q \rangle} \circ \psi_{\langle z,q \rangle})(x), \tag{61}$$

consider the following cases.

Case $z = 0 \ \wedge \ x = 0$. Then,

$$\begin{aligned}
\psi_{\langle y,p\rangle \diamond \langle z,q\rangle}(0) &= \psi_{\langle g(y,z), h(y,p,z,q)\rangle}(0) && \{\text{by } (60)\}\\
&= \psi_{\langle 0, h(y,p,z,q)\rangle}(0) && \{\text{by } (58)\}\\
&= \uparrow && \{\text{by } (55)\}\\
&= (\psi_{\langle y,p\rangle} \circ f^{\mathrm{R}}_{\langle z,q\rangle} \circ \psi_{\langle z,q\rangle})(0) && \{\text{by } (55)\}.
\end{aligned}$$

Case $z > 0 \ \wedge \ x = 0$. Then,

$$\begin{aligned}
\psi_{\langle y,p\rangle \diamond \langle z,q\rangle}(0) &= \psi_{\langle g(y,z), h(y,p,z,q)\rangle}(0) && \{\text{by } (60)\}\\
&= \psi_{\langle y, h(y,p,z,q)\rangle}(0) && \{\text{by } (58)\}\\
&= \psi_{\langle y,p\rangle}(0) && \{\text{by } (57)\}\\
&= (\psi_{\langle y,p\rangle} \circ f^{\mathrm{R}}_{\langle z,q\rangle})(z-1) && \{\text{by } (56)\}\\
&= (\psi_{\langle y,p\rangle} \circ f^{\mathrm{R}}_{\langle z,q\rangle} \circ \psi_{\langle z,q\rangle})(0) && \{\text{by } (55)\}.
\end{aligned}$$

Case $x \neq 0$. Then,

$$\begin{aligned}
\psi_{\langle y,p\rangle \diamond \langle z,q\rangle}(x) &= \psi_{\langle g(y,z), h(y,p,z,q)\rangle}(x) && \{\text{by } (60)\}\\
&= \varphi_{h(y,p,z,q)}(x) && \{\text{by } (55)\}\\
&= (\psi_{\langle y,p\rangle} \circ f^{\mathrm{R}}_{\langle z,q\rangle} \circ \psi_{\langle z,q\rangle})(x) && \{\text{by } (59)\}.
\end{aligned}$$

(b) Suppose the hypotheses. Let $\lambda\langle a,b\rangle \centerdot (a \diamond b) : \mathbb{N} \to \mathbb{N}$ be an ei in $\psi$ of the ncs determined by $(f^{\mathrm{L}}, g^{\mathrm{L}}, f^{\mathrm{R}}, g^{\mathrm{R}})$-**CC**. To reduce the need for parentheses, assume that $\diamond$ associates to the right, i.e., assume that, for all $a$, $b$, and $c$, $a \diamond b \diamond c = a \diamond (b \diamond c)$. For all $a$, $b$, and $p$, let

$$a^p \diamond b = \overbrace{a \diamond \ldots \diamond a}^{p \text{ occurrences}} \diamond b. \tag{62}$$

Let $f^{-\mathrm{L}}$ be a computable function, and let $g^{-\mathrm{R}}$ be a partial computable function, such that, for all $a$,

$$f^{\mathrm{L}}_a \circ f^{-\mathrm{L}}_a = g^{-\mathrm{R}}_a \circ g^{\mathrm{R}}_a = \mathrm{id}. \tag{63}$$

Clearly, these exist. For all $a_0$, $a_1$, $x$, and $z$, let $h$, $i$, and $i^{-1}$ be as follows.

$$h(\langle a_0, a_1, x\rangle) = \begin{cases} \langle a_0, a_1, y\rangle, \text{where } y \text{ is least such that,} \\ \quad \text{for all } w < \langle a_0, a_1, x\rangle, \\ \quad i(w) < \min\left\{ g^{\mathrm{L}}_{a_j}(\langle a_0, a_1, y\rangle) : j \in \{0,1\} \right\}. \end{cases} \tag{64}$$

$$i(\langle a_0, a_1, x\rangle) = \max\left\{ (g^{\mathrm{L}}_{a_j} \circ h)(\langle a_0, a_1, x\rangle) : j \in \{0,1\} \right\}. \tag{65}$$

$$i^{-1}(z) = \min\{w : i(w) \geq z\}. \tag{66}$$

Clearly, $h$ and $i$ are computable. Furthermore, $i$ is (strictly) monotonically increasing. Thus, $i^{-1}$ is also computable. Note that the $-1$ in $i^{-1}$ is to emphasize that $i^{-1}$ is a *left* inverse of $i$.

*Claim 1.* For all $a_0$, $a_1$, $x$, $j \in \{0,1\}$, and $w < \langle a_0, a_1, x\rangle$, $i(w) < (g^{\mathrm{L}}_{a_j} \circ h)(\langle a_0, a_1, x\rangle)$.

*Proof of Claim.* Let $a_0$, $a_1$, and $x$ be fixed. Let $y$ be least such that, for all $w < \langle a_0, a_1, x\rangle$,

$$i(w) < \min\left\{g^{\mathrm{L}}_{a_j}(\langle a_0, a_1, y\rangle) : j \in \{0,1\}\right\}. \tag{67}$$

By (64),

$$h(\langle a_0, a_1, x\rangle) = \langle a_0, a_1, y\rangle. \tag{68}$$

Thus, for all $j \in \{0,1\}$ and $w < \langle a_0, a_1, x\rangle$,

$$\begin{aligned} i(w) &< g^{\mathrm{L}}_{a_j}(\langle a_0, a_1, y\rangle) && \{\text{by (67)}\} \\ &= (g^{\mathrm{L}}_{a_j} \circ h)(\langle a_0, a_1, x\rangle) && \{\text{by (68)}\}. \end{aligned}$$

□ (*Claim 1*)

*Claim 2.* For all $a_0$, $a_1$, $x$, and $j \in \{0,1\}$, $(i^{-1} \circ g^{\mathrm{L}}_{a_j} \circ h)(\langle a_0, a_1, x\rangle) = \langle a_0, a_1, x\rangle$.

*Proof of Claim.* Let $a_0$, $a_1$, $x$, and $j \in \{0,1\}$ be fixed. By (65),

$$i(\langle a_0, a_1, x\rangle) \geq (g^{\mathrm{L}}_{a_j} \circ h)(\langle a_0, a_1, x\rangle). \tag{69}$$

Furthermore, by Claim 1, for all $w < \langle a_0, a_1, x\rangle$,

$$i(w) < (g^{\mathrm{L}}_{a_j} \circ h)(\langle a_0, a_1, x\rangle). \tag{70}$$

Thus,

$$\begin{aligned} &(i^{-1} \circ g^{\mathrm{L}}_{a_j} \circ h)(\langle a_0, a_1, x\rangle) \\ &= \min\{w : i(w) \geq (g^{\mathrm{L}}_{a_j} \circ h)(\langle a_0, a_1, x\rangle)\} && \{\text{by (66)}\} \\ &= \langle a_0, a_1, x\rangle && \{\text{by (69) and (70)}\}. \end{aligned}$$

□ (*Claim 2*)

Recall from Section 2 that, for all $x_1, ..., x_n$, where $n > 2$, $\langle x_1, ..., x_n\rangle \overset{\text{def}}{=} \big\langle x_1, \langle x_2, ..., x_n\rangle\big\rangle$. Thus, Claim 2 also asserts that, for all $a_0$, $a_1$, $q$, $y$, and $j \in \{0,1\}$, $(i^{-1} \circ g^{\mathrm{L}}_{a_j} \circ h)(\langle a_0, a_1, q, y\rangle) = \langle a_0, a_1, q, y\rangle$.
For all $x$ and $z$, let $b$, $c$, and $d$ be as follows.

$$\psi_b(z) = (f^{-\mathrm{L}}_{a_0} \circ \varphi_q)(y), \text{ where } \langle a_0, a_1, q, y\rangle = i^{-1}(z). \tag{71}$$

$$\psi_c(z) = \begin{cases} (f^{-\mathrm{L}}_{a_1} \circ h)(\langle a_0, a_1, q+1, y\rangle), \\ \quad \text{where } \langle a_0, a_1, q, y\rangle = i^{-1}(z). \end{cases} \tag{72}$$

$$\psi_d(x) = h(\langle b, c, 0, x\rangle). \tag{73}$$

For all $q$ and $x$, let $k$ and $k^{-1}$ be as follows.

$$k(\langle q, x\rangle) = \begin{cases} x, & \text{if } q = 0; \\ (g^{\mathrm{R}}_{d} \circ g^{\mathrm{R}}_{c^1 \diamond d} \circ ... \circ g^{\mathrm{R}}_{c^{q-1} \diamond d})(x), & \text{otherwise.} \end{cases} \tag{74}$$

$$k^{-1}(\langle q, x\rangle) = \begin{cases} x, & \text{if } q = 0; \\ (g^{-\mathrm{R}}_{c^{q-1} \diamond d} \circ ... \circ g^{-\mathrm{R}}_{c^1 \diamond d} \circ g^{-\mathrm{R}}_{d})(x), & \text{otherwise.} \end{cases} \tag{75}$$

Clearly, $k$ is computable, and, for all $q$, $k_q$ is 1-1. Clearly, $k^{-1}$ is partial computable. Note that the $-1$ in $k^{-1}$ is to emphasize that, for all $q$, $k_q^{-1}$ is a *left* inverse of $k_q$, i.e., for all $q$,

$$k_q^{-1} \circ k_q = \text{id}. \tag{76}$$

*Claim 3.* For all $q$ and $x$, $\psi_{c^q \diamond d}(x) = h(\langle b, c, q, k_q(x)\rangle)$.

*Proof of Claim.* By induction on $q$. In the case when $q = 0$,

$$\psi_{c^q \diamond d}(x) = \psi_d(x) = h(\langle b, c, 0, x\rangle) = h(\langle b, c, q, k_q(x)\rangle).$$

So suppose, inductively, that the claim holds for $q$. Then, for all $x$,

$$\begin{aligned}
&\psi_{c^{q+1} \diamond d}(x) \\
&= \psi_{c \diamond c^q \diamond d}(x) && \{\text{by (62)}\} \\
&= (f_c^{\mathrm{L}} \circ \psi_c \circ g_c^{\mathrm{L}} \circ \psi_{c^q \diamond d} \circ g_{c^q \diamond d}^{\mathrm{R}})(x) && \{\text{by the choice of } \diamond\} \\
&= (f_c^{\mathrm{L}} \circ \psi_c \circ g_c^{\mathrm{L}} \circ h)(\langle b, c, q, (k_q \circ g_{c^q \diamond d}^{\mathrm{R}})(x)\rangle) && \{\text{by the i.h.}\} \\
&= (f_c^{\mathrm{L}} \circ \psi_c \circ g_c^{\mathrm{L}} \circ h)(\langle b, c, q, k_{q+1}(x)\rangle) && \{\text{by (74)}\} \\
&= (f_c^{\mathrm{L}} \circ f_c^{-\mathrm{L}} \circ h)(\langle b, c, q+1, k_{q+1}(x)\rangle) && \{\text{by (72) and Claim 2}\} \\
&= h(\langle b, c, q+1, k_{q+1}(x)\rangle) && \{\text{by (63)}\}.
\end{aligned}$$

□ (*Claim 3*)

*Claim 4.* For all $q$ and $x$, $\psi_{b \diamond c^q \diamond d}(x) = (\varphi_q \circ k_{q+1})(x)$.

*Proof of Claim.* For all $q$ and $x$,

$$\begin{aligned}
&\psi_{b \diamond c^q \diamond d}(x) \\
&= (f_b^{\mathrm{L}} \circ \psi_b \circ g_b^{\mathrm{L}} \circ \psi_{c^q \diamond d} \circ g_{c^q \diamond d}^{\mathrm{R}})(x) && \{\text{by the choice of } \diamond\} \\
&= (f_b^{\mathrm{L}} \circ \psi_b \circ g_b^{\mathrm{L}} \circ h)(\langle b, c, q, (k_q \circ g_{c^q \diamond d}^{\mathrm{R}})(x)\rangle) && \{\text{by Claim 3}\} \\
&= (f_b^{\mathrm{L}} \circ \psi_b \circ g_b^{\mathrm{L}} \circ h)(\langle b, c, q, k_{q+1}(x)\rangle) && \{\text{by (74)}\} \\
&= (f_b^{\mathrm{L}} \circ f_b^{-\mathrm{L}} \circ \varphi_q \circ k_{q+1})(x) && \{\text{by (71) and Claim 2}\} \\
&= (\varphi_q \circ k_{q+1})(x) && \{\text{by (63)}\}.
\end{aligned}$$

□ (*Claim 4*)

By **PKRT** in $\varphi$ [20], there exists $e_0, e_1, ...$ such that $\lambda p \centerdot e_p$ is computable, and, for all $p$,

$$\varphi_{e_p} = \varphi_p \circ k_{e_p+1}^{-1}. \tag{77}$$

*Claim 5.* For all $p$, $\psi_{b \diamond c^{e_p} \diamond d} = \varphi_p$.

*Proof of Claim.* For all $p$,

$$\begin{aligned}
\psi_{b \diamond c^{e_p} \diamond d} &= \varphi_{e_p} \circ k_{e_p+1} && \{\text{by Claim 4}\} \\
&= \varphi_p \circ k_{e_p+1}^{-1} \circ k_{e_p+1} && \{\text{by (77)}\} \\
&= \varphi_p && \{\text{by (76)}\}.
\end{aligned}$$

□ (*Claim 5*)

Thus, $\lambda p \centerdot b \diamond c^{e_p} \diamond d$ witnesses that $\psi$ is acceptable. □ (*Theorem 5*)