Propositional Proof Complexity of Paris-Harrington Tautologies

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A computational question:

Q: how hard is it to prove a **propositional** theorem?

$=$

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Clote [Clo95] proposes:

Maybe combinatorial independence from some strong arithmetic theory leads to hard propositional tautologies.

This approach has some limits...

- Functions which grow too fast may cause **LARGE** formulas with trivial proof complexity.
  E.g., a formula on N variable and $2^{\frac{N}{100}}$ clauses.
- Useless in the context of Bounded Arithmetic.
- Large gaps between **UPPER and LOWER BOUNDS** makes difficult to properly evaluate what is efficient.

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In this talk...

We do not shy away... instead we focus on a **sufficiently weak** version of the Paris-Harrington numbers. The choice allows non-trivial results.

Proof system as a verification process (refutational style)

A deterministic polynomial time machine $V(\cdot, \cdot)$

- $\psi \in$ UNSAT $\quad$ $V(\psi, \pi)$ accept for some proof π
- $\psi \notin$ UNSAT $\quad$ $V(\psi, \pi)$ reject for any string π

If $V(\psi, \pi)$ accepts then π is a “refutation” of ψ.

Central question of Proof Complexity

Is there a proof system such that the shortest refutation for an unsatisfiable ψ has size $\mathrm{poly}(|\psi|)$?

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There is a proof system such that the shortest refutation for an unsatisfiable ψ has size $\mathrm{poly}(|\psi|)$

$$\Longleftrightarrow$$

$$\mathsf{NP} = \mathsf{coNP}$$

Too difficult, the community focuses on simple proof systems.

We study the **length** of refutations $\psi \vdash \square$ in...

- RES: proof lines are clauses

$$\frac{A \vee x \quad \neg x \vee B}{A \vee B}$$ Cut on literals

- RES (2): proof lines are 2-DNFs

$$\frac{A \vee xy \quad \neg x \vee \neg y \vee B}{A \vee B}$$ Cut on 2-terms.

- BD-FREGE: sequent calculus on bounded depth formulas.

Paris-Harrington Numbers

If a graph is big enough, at least one of a specific set of sub-configurations must be present.

—Ramsey type statements —

Paris-Harrington for graphs

There exists a minimum number $R(k;m)$ such that any graph on the integers $[k, R(k;m)]$ contains

- either a clique of size m,
- or a "large" stable set S (i.e., $|S| = \min S$).

Paris-Harrington formulas

- Vertices $V = \{k, \ldots, N\}$,
- Variables E_{ij} for $\{i,j\} \in \binom{V}{2}$

$$\neg\mathsf{Cli}(X) = \bigvee_{\{i,j\}\in\binom{X}{2}} \neg E_{ij} \qquad \neg\mathsf{Ind}(X) = \bigvee_{\{i,j\}\in\binom{X}{2}} E_{ij}$$

$$\mathsf{PH}(N;k,m) := \left(\bigwedge_{X\subseteq V,|X|=m} \neg\mathsf{Cli}(X)\right) \wedge \left(\bigwedge_{X\subseteq V,|X|=\min X} \neg\mathsf{Ind}(X)\right)$$

"$N \geq R(k;m)$" if and only if $\mathsf{PH}(N;k,m)$ is unsatisfiable.

About Paris-Harrington Numbers

Mills [Mil85] gives a β such that for big enough k and $m \geq 3$

$$R(k;m) \leq k^{2^{\beta m}}$$

We are going to study the proof complexity of $\mathrm{PH}(k^{2^{\beta m}};k,m)$

Previous results

(about Ramsey tautologies for graphs)

Ramsey formulas

RAM$(N;a,b)$: (negation of) Ramsey theorem for graph

"There is a graph of N vertices with no a-cliques and no b-stables"

$$\text{RAM}(N;a,b) := \left(\bigwedge_{X\subseteq\{1\ldots N\},|X|=a} \neg\text{Cli}(X)\right) \wedge \left(\bigwedge_{X\subseteq\{1\ldots N\},|X|=b} \neg\text{Ind}(X)\right)$$

[Pud91] RAM$(4^k;k,k)$ is easy in BD-Frege.

[Kra01] PHP$_n^{n^4}$ hard for RES(2) implies RAM$(4^k;k,k)$ hard for RES.

[Kra11] RAM$(r(k,k);k,k)$ his hard for BD-Frege.

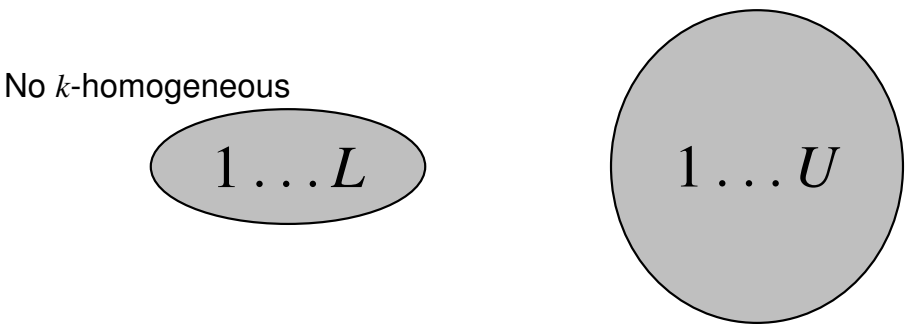
Lower bounds by Krajíček (I): reduction from PHP

PHP_n^m: (negation of) Pigeonhole principle for $m > n$

"There exist an injective mapping from m to n"

Let $r(k,k)$ the critical Ramsey value and $L < r(k,k) \leq U$.

- Fix a satisfying assignment for $\text{RAM}(L; k, k)$.



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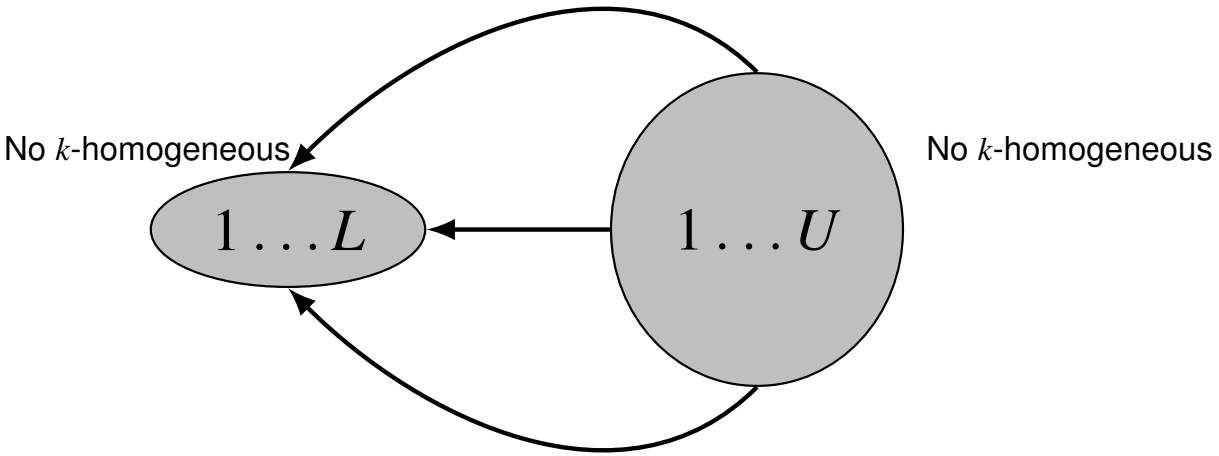

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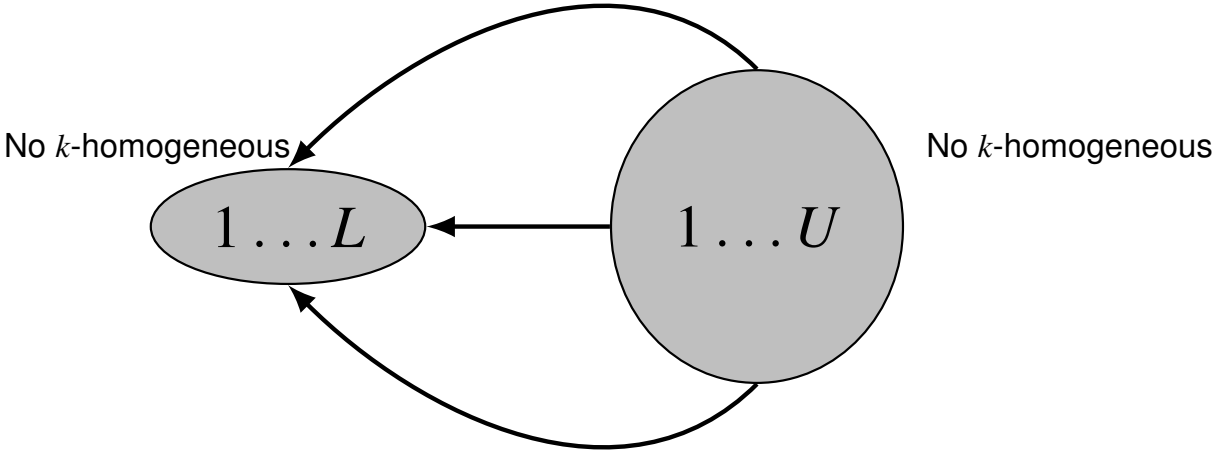
- Fix a satisfying assignment for $\text{RAM}(L;k,k)$.
- An injection from U to L allows to satisfy $\text{RAM}(U;k,k)$.
- Refuting $\text{RAM}(U;k,k)$ allows to exclude such injections.



Lower bounds by Krajíček (II)

Theorem 1 ([Kra11])

$\textsc{Ram}(r(k,k);k,k)$ *is hard for BD-Frege*

Proof.

Fix $n = r(k,k) - 1$. $\textsc{Php}_n^{n+1}$ is hard for BD-Frege. □

Theorem 2 ([Kra01])

If $\textsc{Php}_n^{n^4}$ *is hard for* $\textsc{Res}(2)$ *then* $\textsc{Ram}(4^k;k,k)$ *is hard for* $\textsc{Res}$.

Proof.

It is known that $2^{k/2} < r(k,k) \leq 4^k$. Fix $n = 2^{k/2}$. □

Upper bound by Pudlák (I)

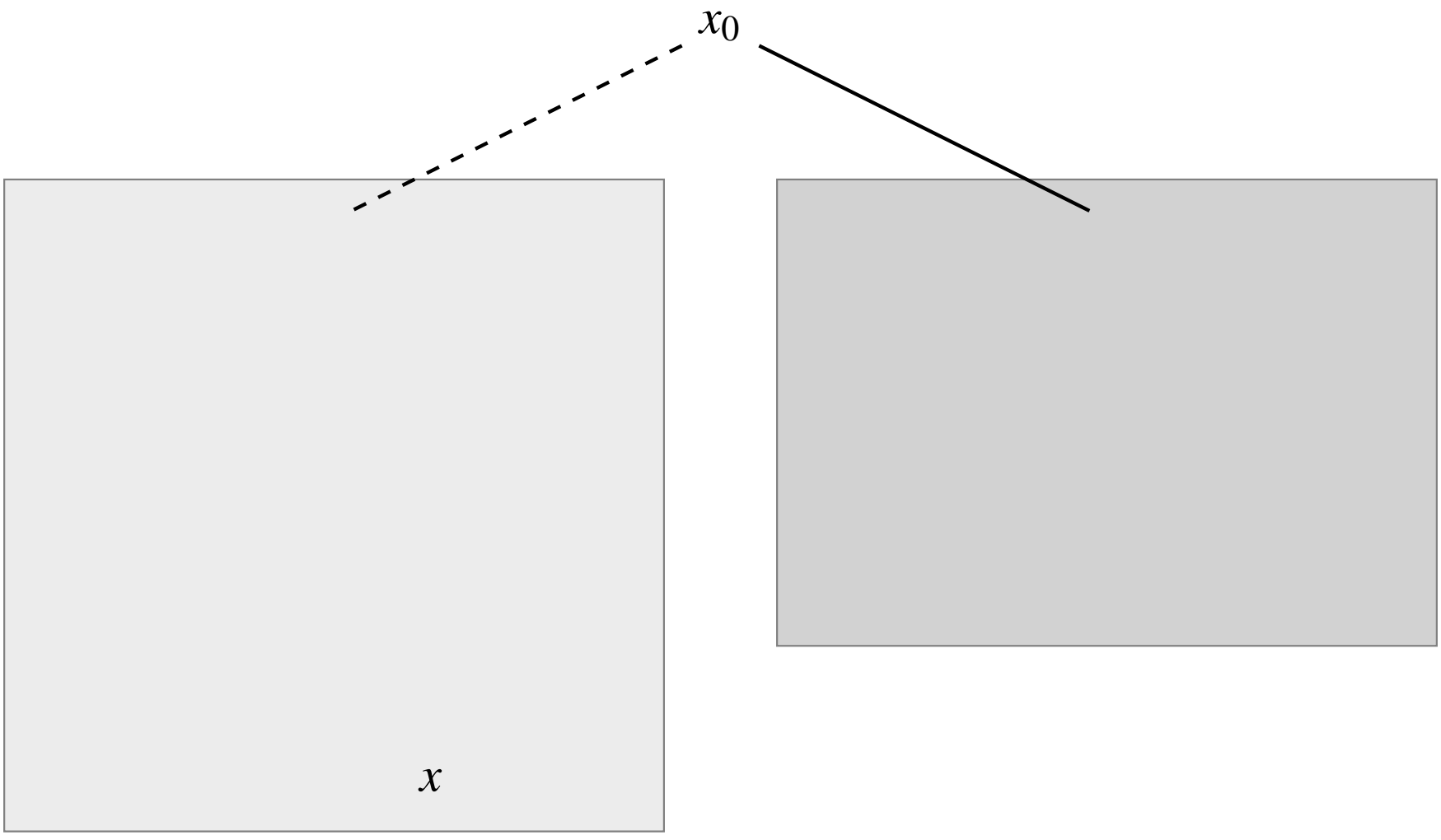
Upper bounds for Ramsey numbers $r(a, b)$ can be proved by defining the following search procedure where each vertex is reachable in a unique way.

Upper bound by Pudlák (II)

x_0

x

Upper bound by Pudlák (II)



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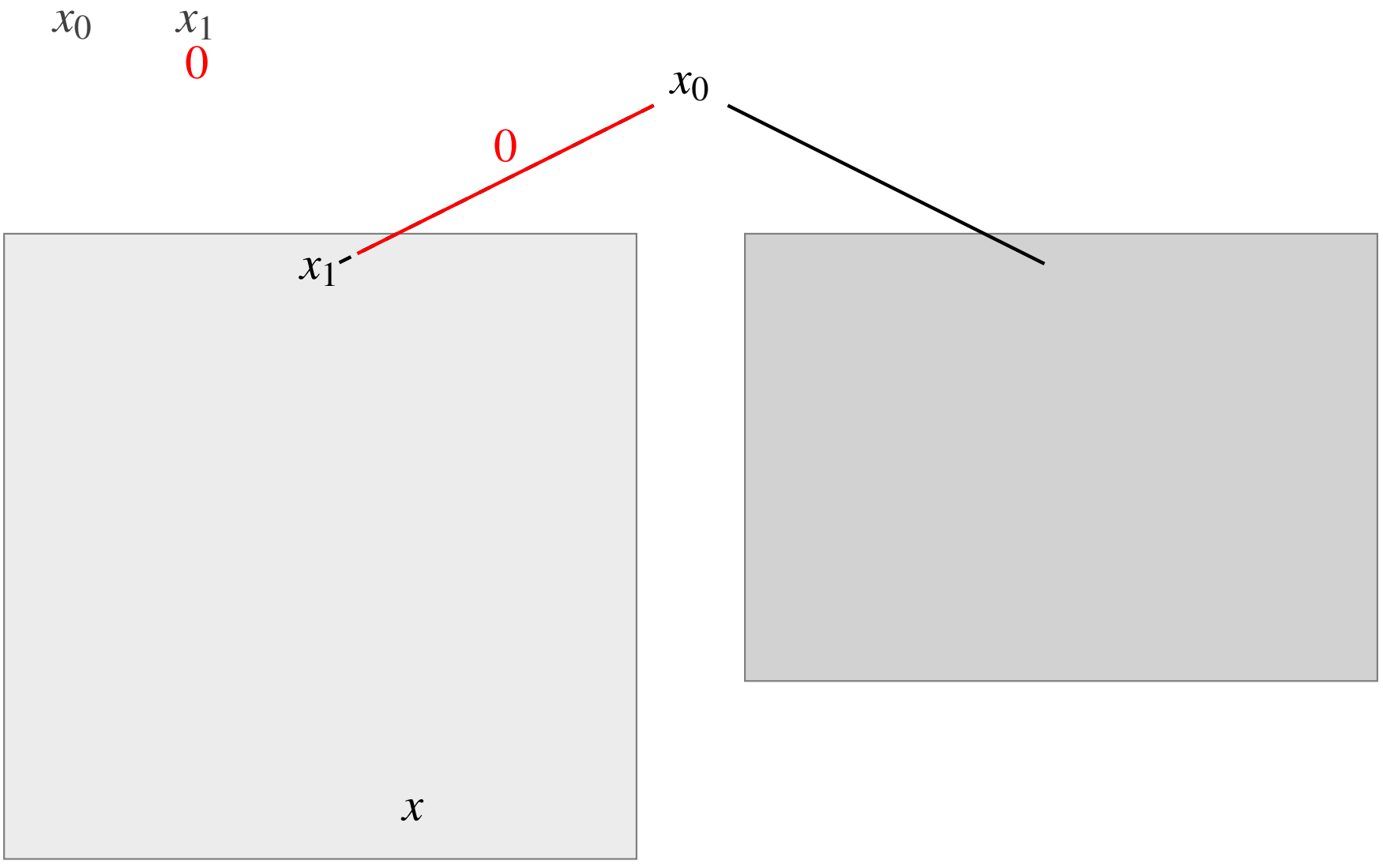

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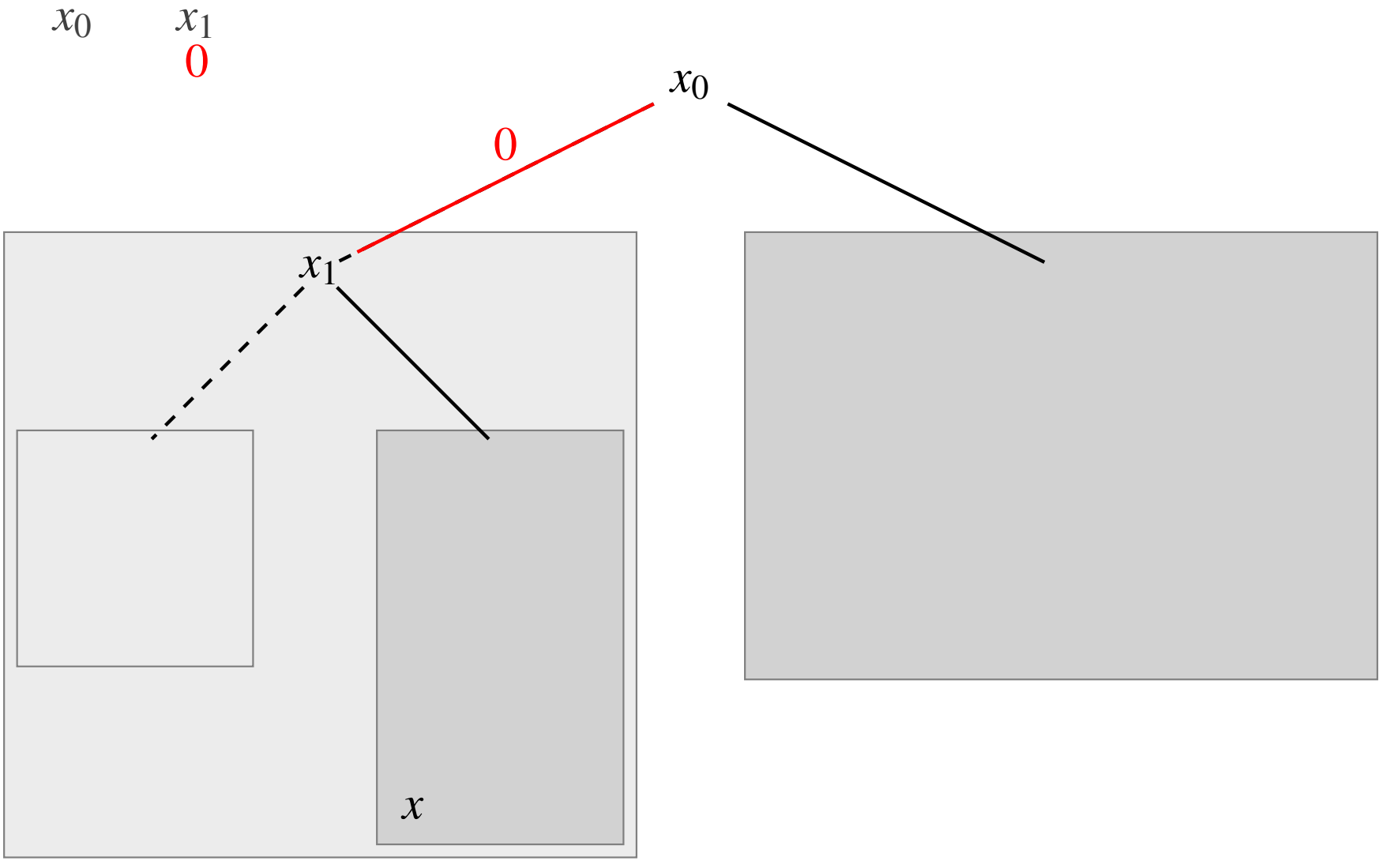

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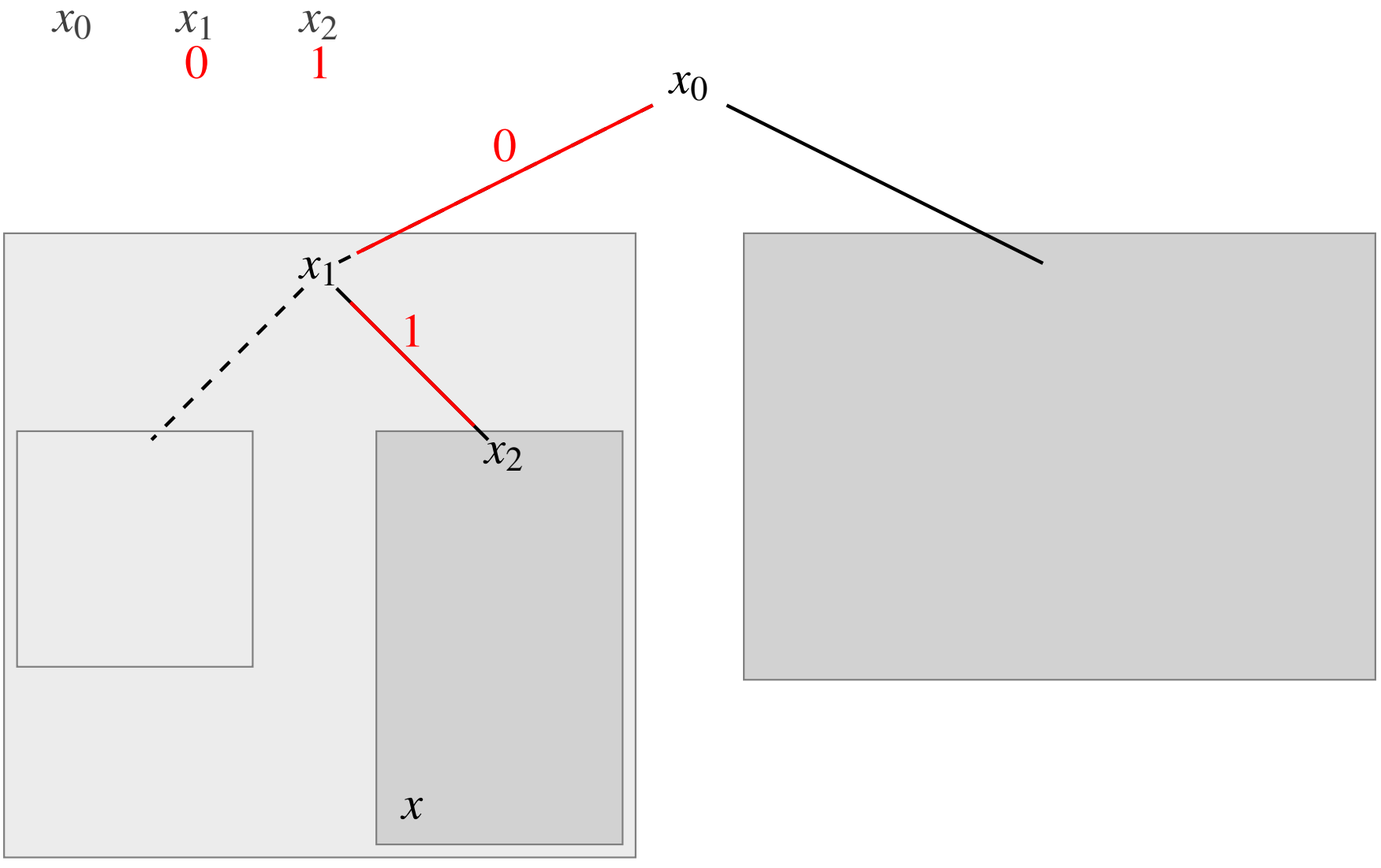

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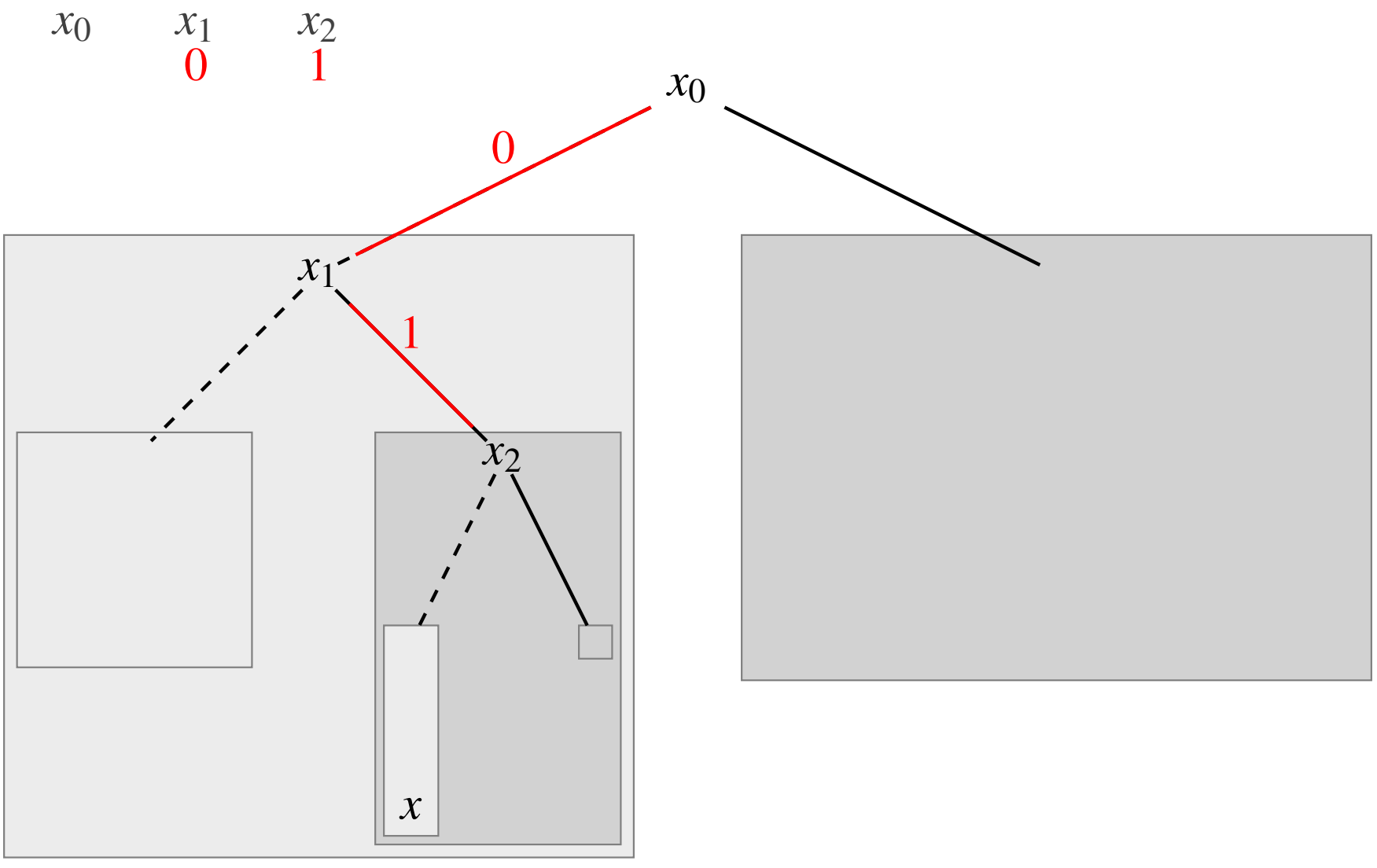

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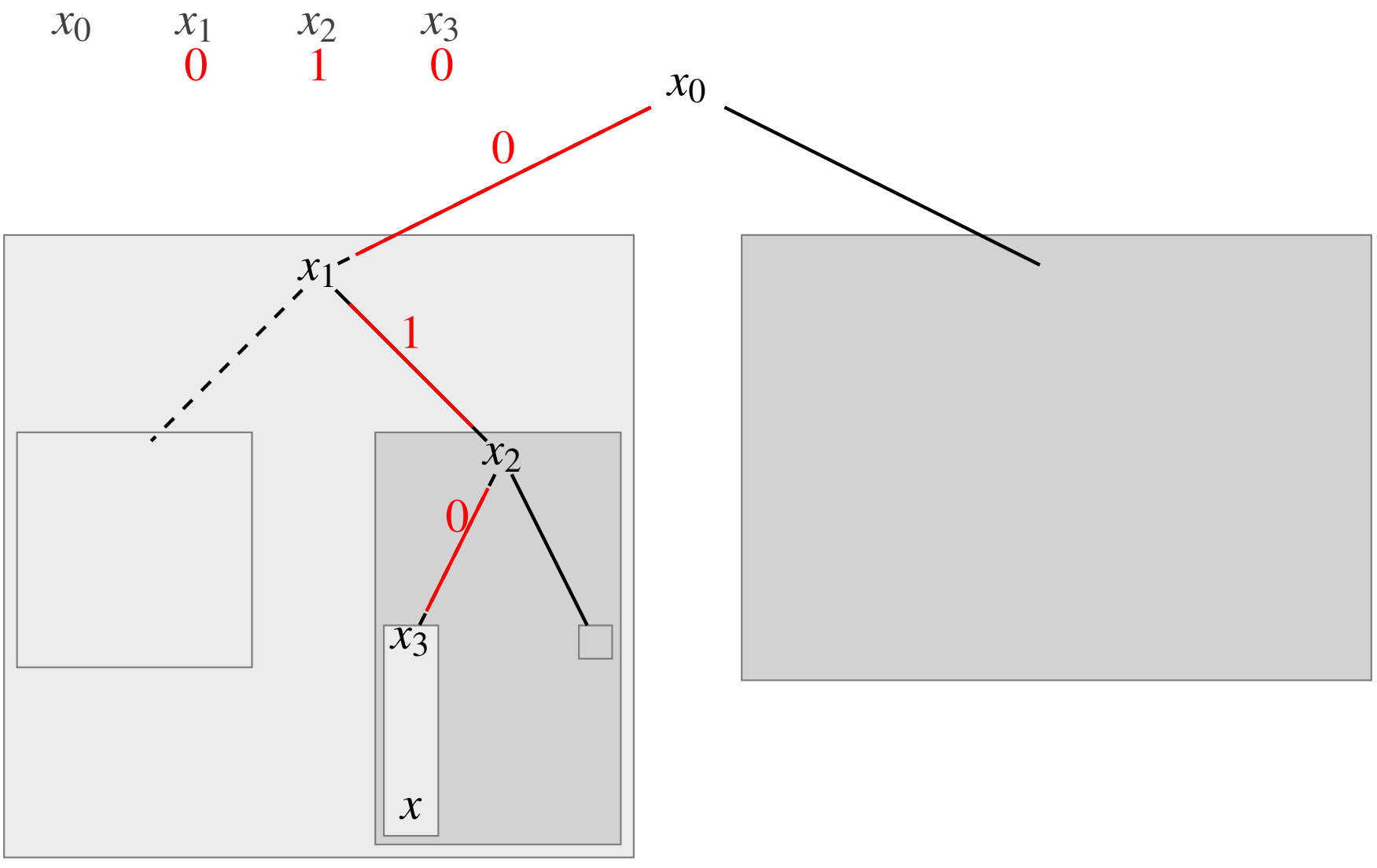

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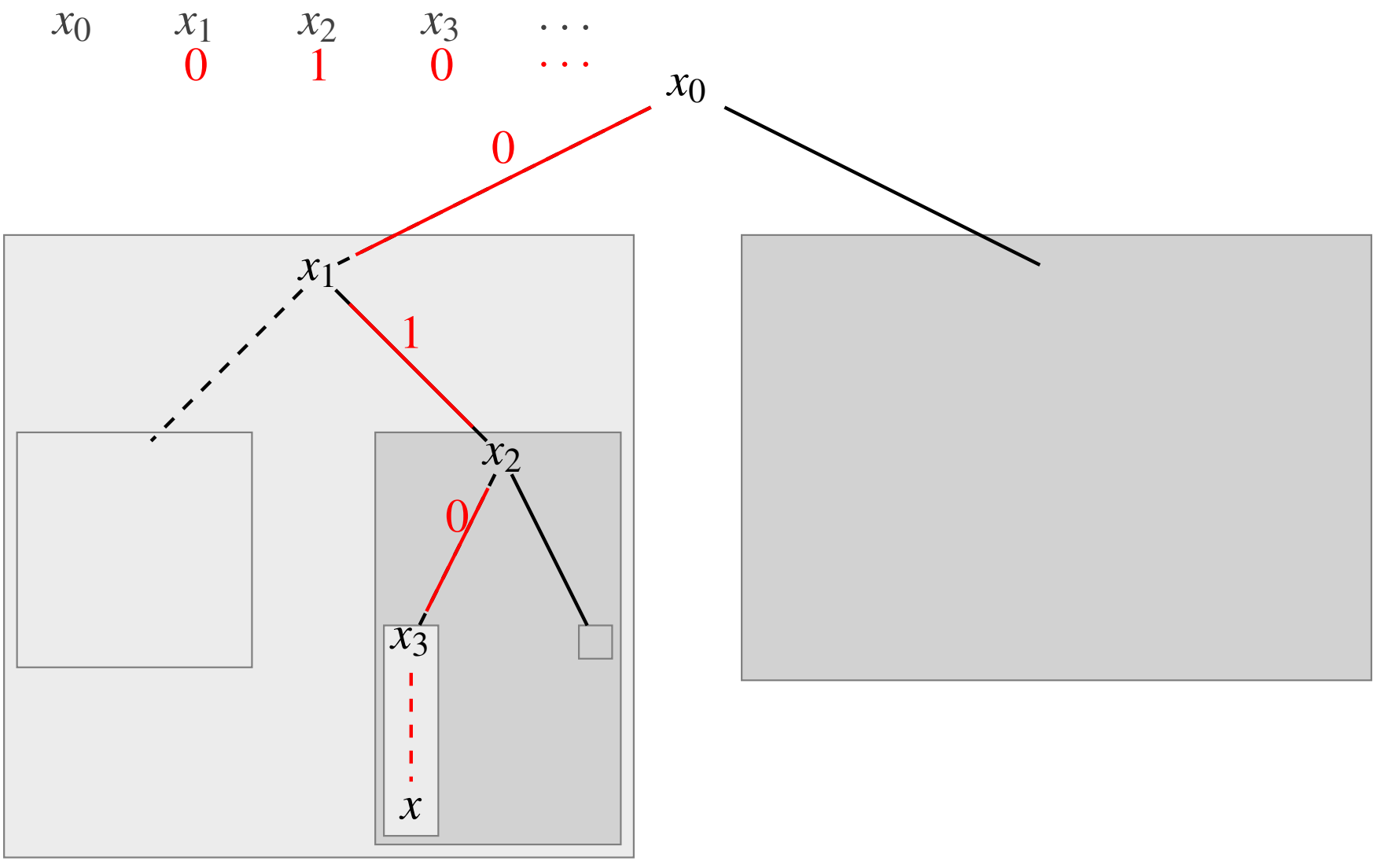

Upper bound for BD-Frege (III): reduction to PHP

There is a known injective mapping that associate vertices of a graph to 0–1 strings in such a way that:

- $k-1$ zeros imply a stable sets of size k;
- $k-1$ ones imply a clique of size k.
- No k-homogeneous sets $\longrightarrow$ **FEW** possible strings.

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- $k-1$ ones imply a clique of size k.
- No k-homogeneous sets $\longrightarrow$ **FEW** possible strings.

If vertices there are **twice** the number of possible strings then BD-Frege can prove a contradiction efficiently.

Proof.

PHP_n^{2n} has short proof in BD-Frege. □

New results

(on PH tautologies)

New result

Carlucci, Galesi, L. (2011)

There are α and β as in [EM81, Mil85] such that, for

$$L = k^{2^{\alpha m}} < R(k,m) \leq N = k^{2^{\beta m}}$$

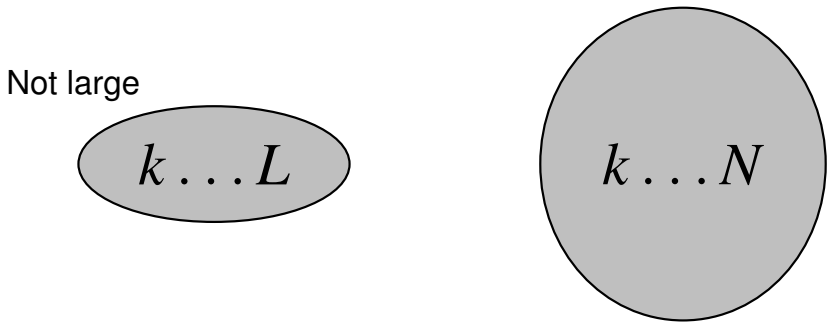
- $\textsc{Ph}(N;k,m)$ has a BD-Frege refutation of size $2^{\Theta(N \log\log N)}$
- If $\textsc{Res}(2)$ complexity of $\textsc{Php}_L^{N-o(N)}$ is $2^{\Omega(L^{1/2+\epsilon})}$ then $\textsc{Ph}(N;k,m)$ requires $\textsc{Res}$ refutation of size $2^{\Omega(L^{1/2+\epsilon})}$

Lower bound

...conditional

Lower bound: first attempt

Let us reduce from pigeonhole principle, as in Ramsey formulas lower bounds. For $L < R(k, m) \leq N$



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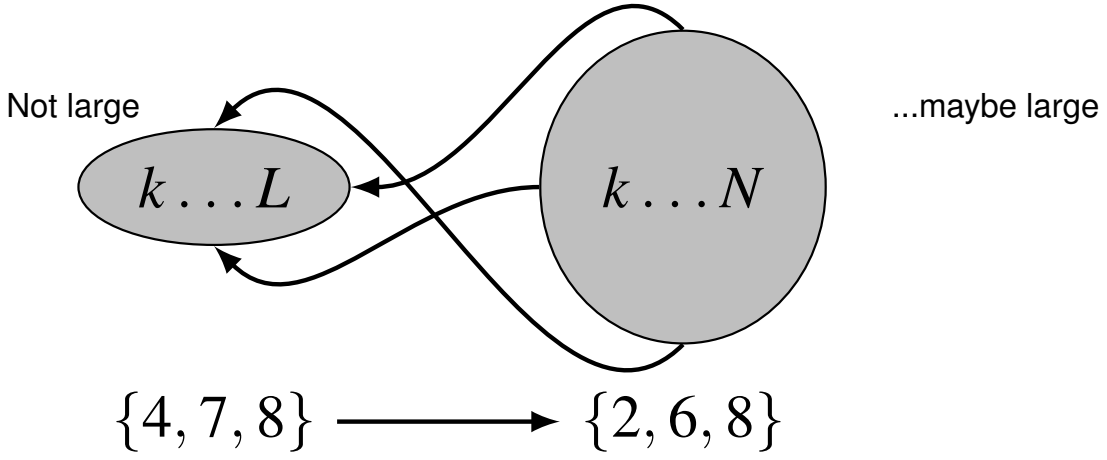

We need to force the mapping to be non-decreasing, but such a "non-decreasing pigeonhole principle" has short RES refutations.

Lower bound: reduction from off-diagonal Ramsey [EM81]

$$[k, N] = \overbrace{[n_0, n_1 - 1] \cup [n_1, n_2 - 1] \ldots [n_{\frac{m}{2}-3}, n_{\frac{m}{2}-2} - 1] \cup [M, N]}^{\frac{m}{2}-1 \text{ intervals}}$$

Where

- $n_0 := k$ and
- $n_{i+1} := r(3, n_i) + n_i - 1$.
- $M := n_{m/2-2}$.

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- Edges between intervals are all present.
- Fix subgraph $[n_i, n_{i+1} - 1]$ such that it has no triangles and no stable sets of size n_i.

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- Edges between intervals are all present.
- Fix subgraph $[n_i, n_{i+1} - 1]$ such that it has no triangles and no stable sets of size n_i.
- Any large stable implies an M-stable in $[M, N]$.
- Any m-clique implies a triangle in $[M, N]$.

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- Any large stable implies an M-stable in $[M, N]$.
- Any m-clique implies a triangle in $[M, N]$.
- We use Krajíček l.b. technique for $\textsc{Ram}(|[M, N]|; 3, M)$.

Lower bound: proof scheme

Given a RES refutation of size S for $\mathrm{PH}(N;k,m)$ there are

$$2^{2^{k/2}} < M < \sqrt{N} \qquad L := r(3,M) - 1 \qquad \text{(see Mills [Mil85])},$$

such that we get

1. a RES refutation of size S of $\mathrm{RAM}(N - o(N); 3, M)$;
2. a RES(2) refutation of size $S \cdot 2^{O(M \log M)}$ of $\mathrm{PHP}_L^{N-o(N)}$, see [Kra01].
3. Knowing that $L \approx \frac{M^2}{\log M}$ we get the statement

If RES(2) complexity of $\mathrm{PHP}_L^{N-o(N)}$ is $2^{\Omega(L^{1/2+\epsilon})}$ then $\mathrm{PH}(N;k,k)$ require RES refutation of size $2^{\Omega(L^{1/2+\epsilon})}$

Comments on the lower bound

If $\textsc{Res}(2)$ complexity of $\textsc{Php}_L^{N-o(N)}$ is $2^{\Omega(L^{1/2+\epsilon})}$ then $\textsc{Ph}(N;k,k)$ require $\textsc{Res}$ refutation of size $2^{\Omega(L^{1/2+\epsilon})}$

- Is the assumption believable? (I think it is)
- Formula $\textsc{Php}_L^{N-o(N)}$ ranges between

$$\textsc{Php}^n_{n/\log n} \quad \ldots \quad \textsc{Php}^{2^{\log^c n}}_n .$$

  depending on L position relative to known bounds.
- If $2^{h(k)}$ is a lower bound then $\sqrt{h(k)} \leq R(k,m)$.

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Better combinatorial bounds are needed for better results. (Vacuously true, but we provide actual numbers!)

Upper bound

Upper bound: proof scheme

1. A recursive scheme reduces $\text{PH}(N; k, m)$ to $\text{PH}(N; \sqrt{N}, 3)$;
2. $\text{PH}(N; \sqrt{N}, 3)$ is reduced to $\text{RAM}(N - c\sqrt{N}; \sqrt{N}, 3)$;
3. We use Pudlák proof of Ramsey tautologies in BD-Frege.

The recursive scheme at step 1 costs $2^{O(Nm)}$.

BD-Frege refutation of $\text{PH}(N; \sqrt{N}, 3)$ costs $2^{O(N)}$ (step 2 and 3).

Upper bound: the idea of the recursive scheme

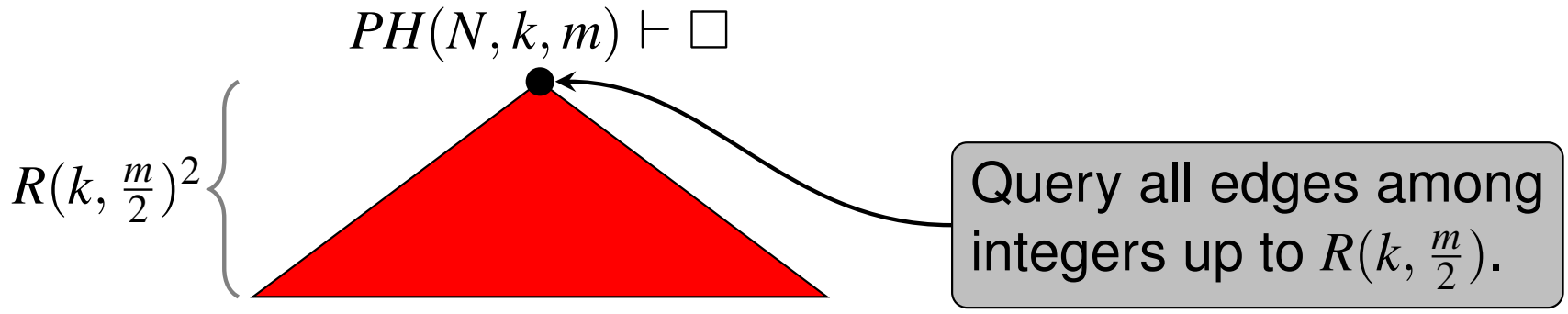
Assume the clique parameter is $m = 3 \cdot 2^l$.

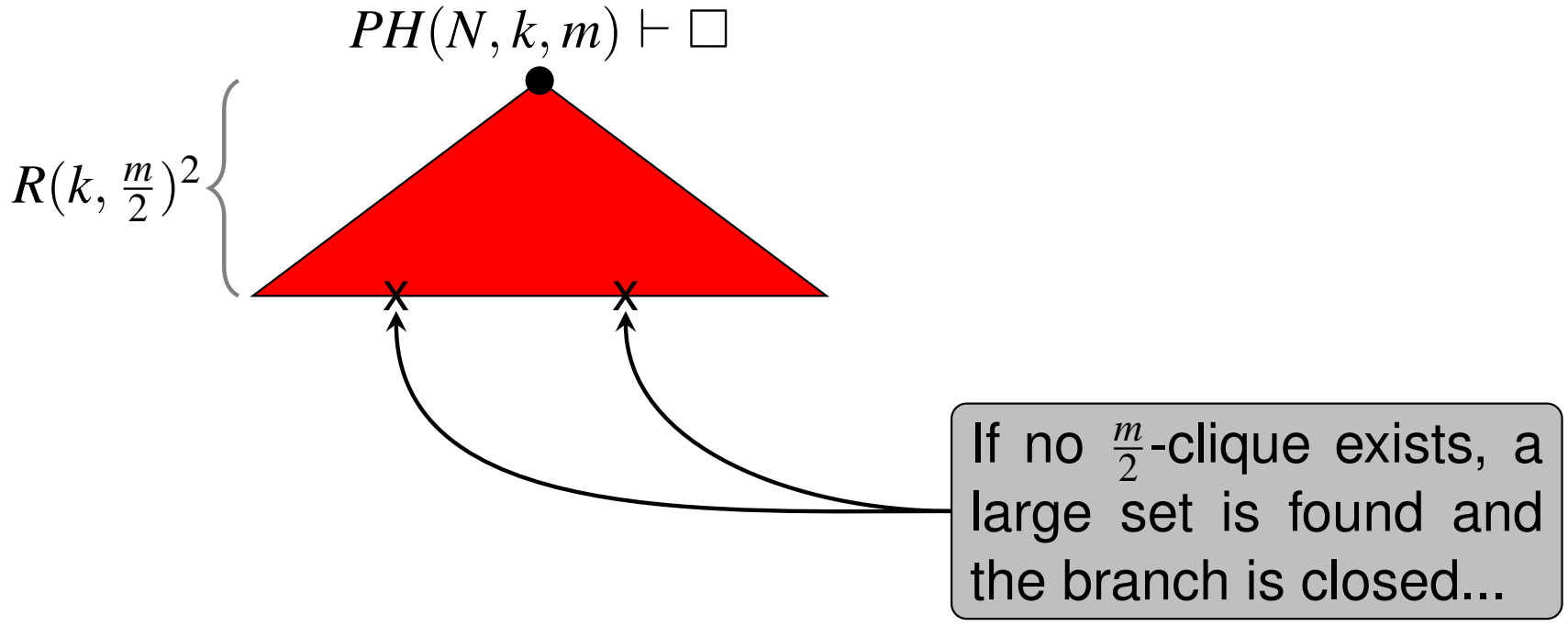
Mills' idea for the upper bound is to self-reduce the problem to the case of $m/2$-clique.

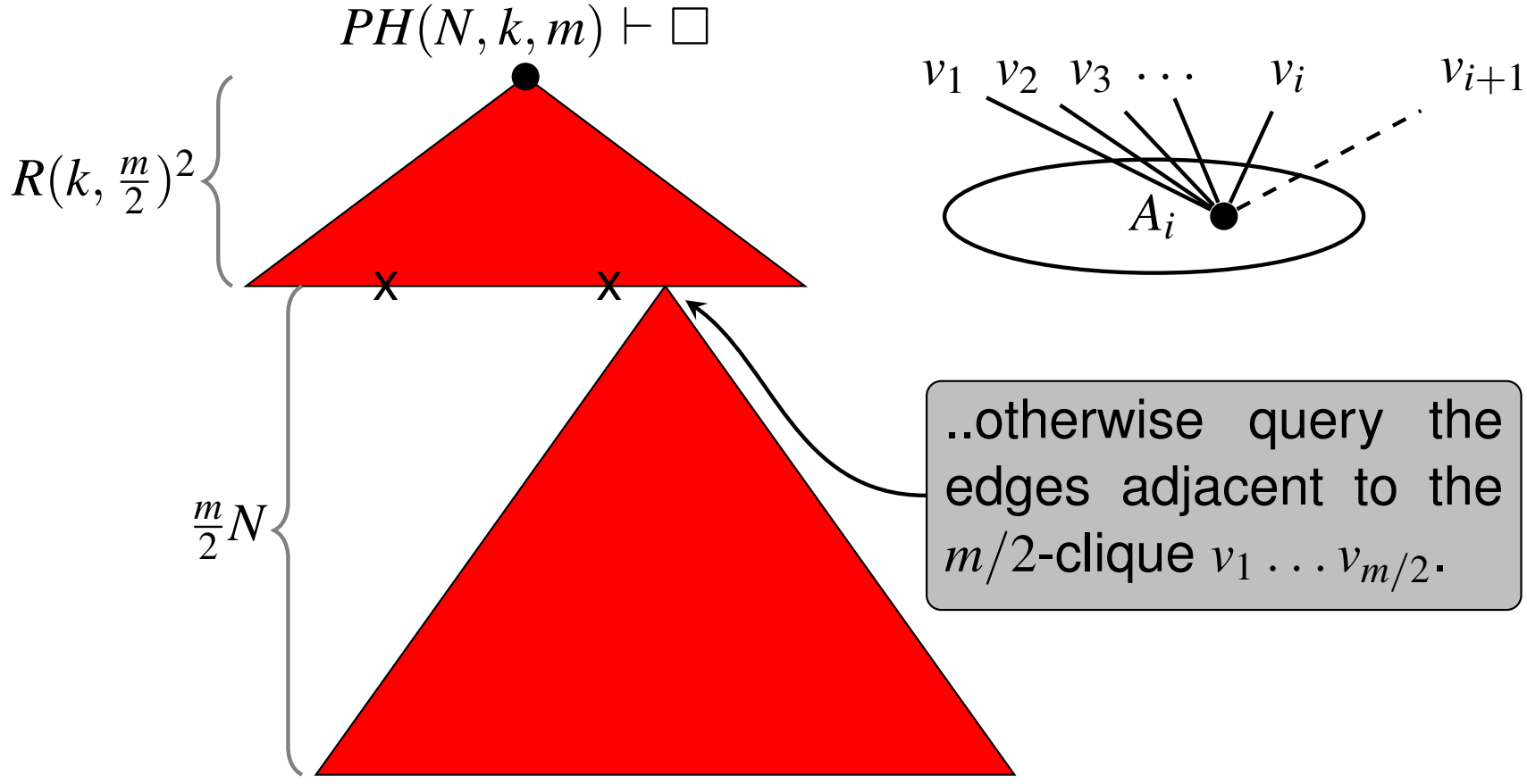
We pay for the halving of m with a shrinking of the interval.

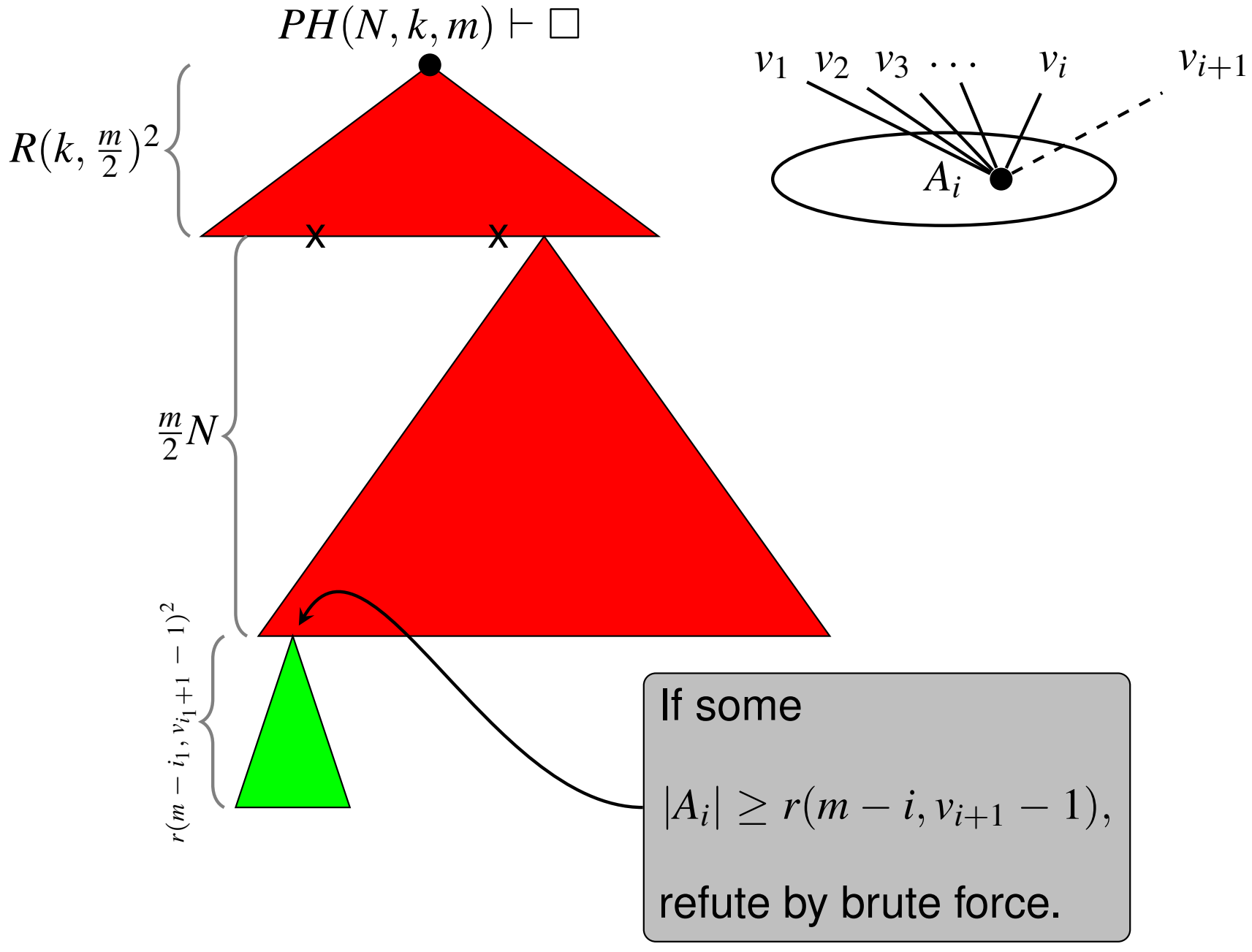
For $N = k^{2^{\beta m}}$ the interval remains big enough for the reduction to be possible until the clique parameter reaches 3.

$$PH(N, k, m) \vdash \square$$


$PH(N,k,m) \vdash \square$
$R(k,\frac{m}{2})^2$
Query all edges among integers up to $R(k,\frac{m}{2})$.


$PH(N,k,m) \vdash \square$
$R(k, \frac{m}{2})^2$
If no $\frac{m}{2}$-clique exists, a large set is found and the branch is closed...


$PH(N,k,m) \vdash \square$
$R(k, \frac{m}{2})^2$
$\frac{m}{2}N$
v_1 v_2 v_3 $\cdots$ v_i v_{i+1}
A_i
..otherwise query the edges adjacent to the $m/2$-clique $v_1 \ldots v_{m/2}$.


$PH(N,k,m) \vdash \square$
$R(k,\frac{m}{2})^2$
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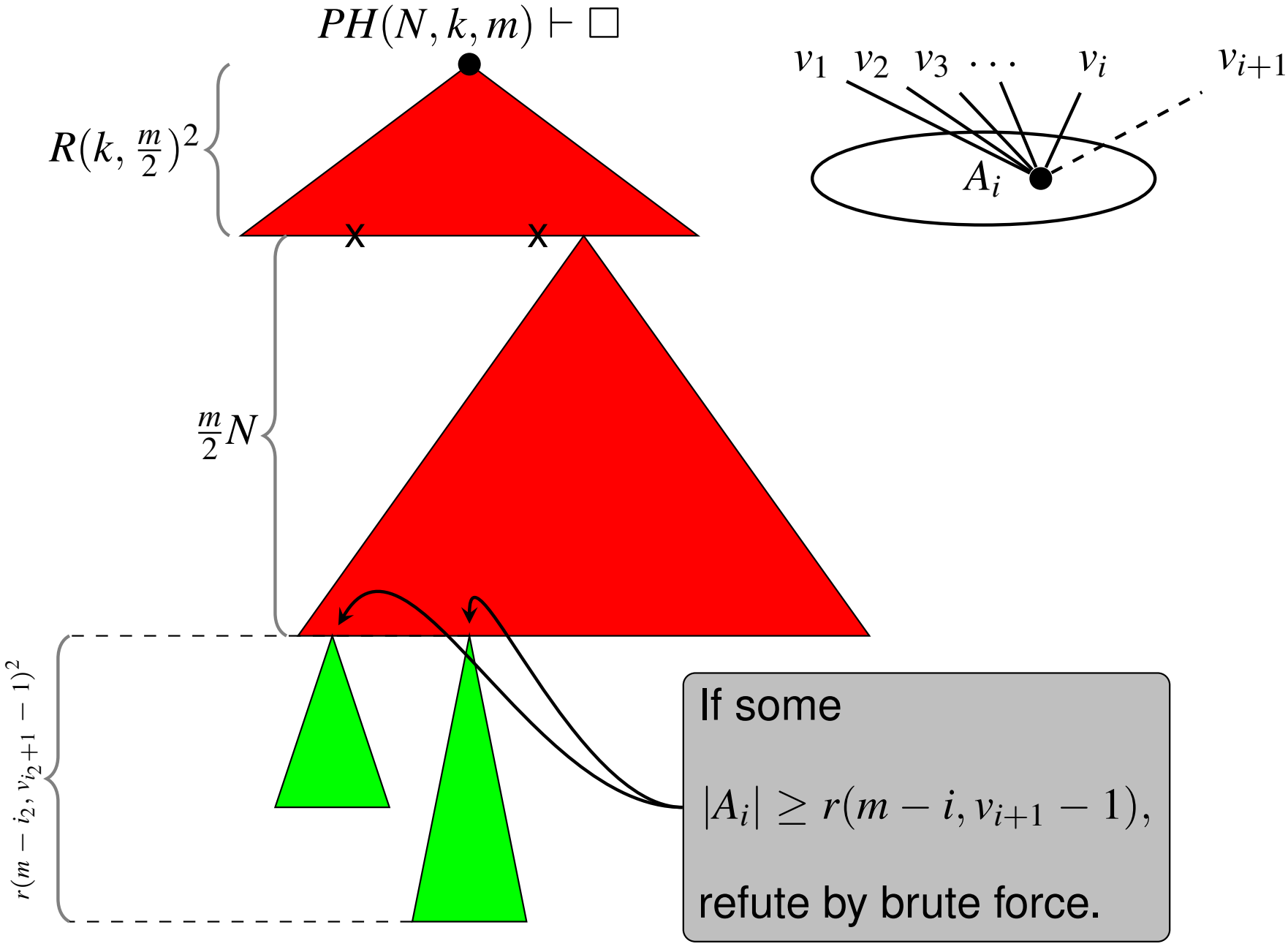
$r(m-i_1, v_{i_1+1}-1)^2$
v_1 v_2 v_3 $\cdots$ v_i v_{i+1}
A_i
If some
$|A_i| \geq r(m-i, v_{i+1}-1)$,
refute by brute force.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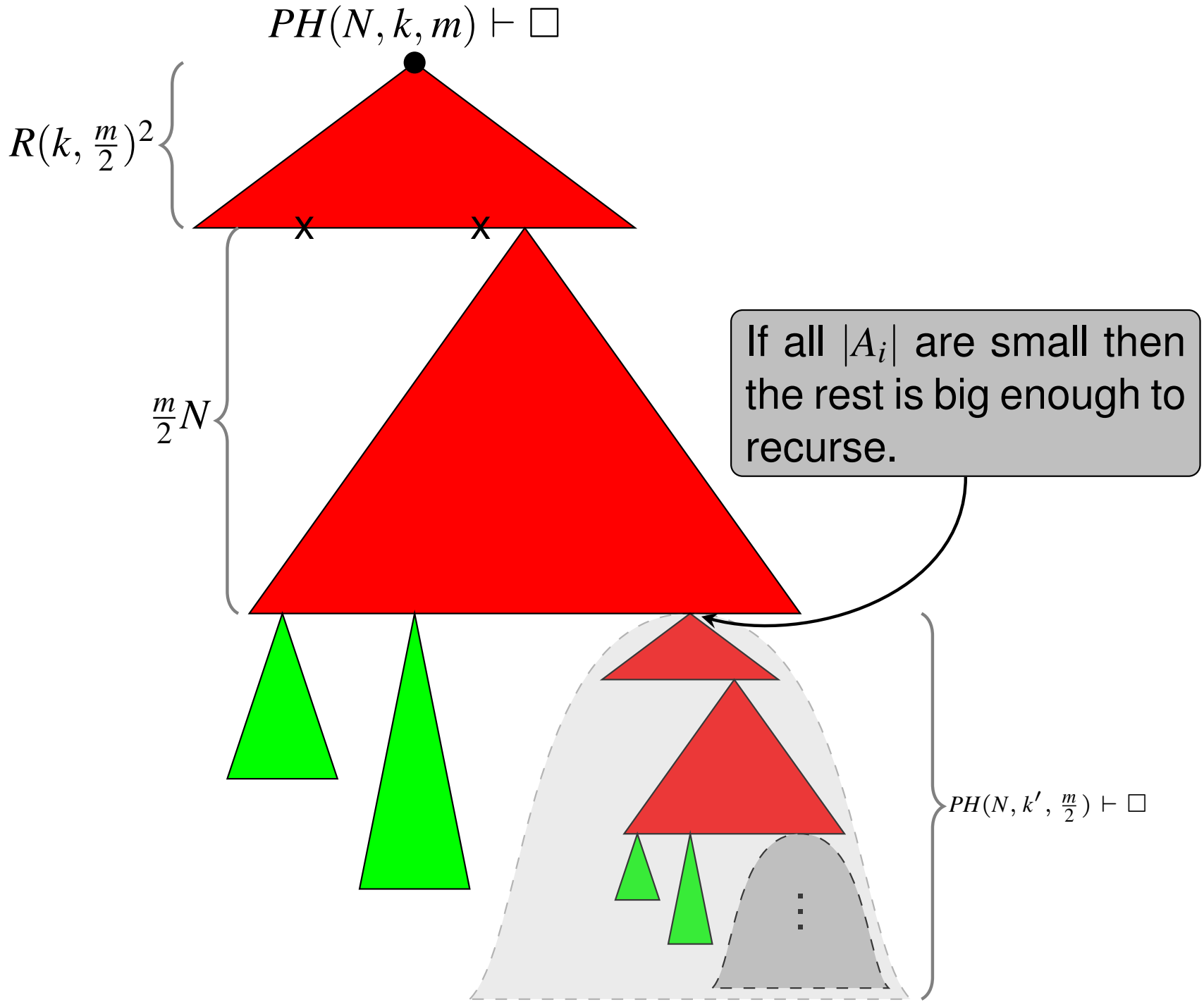
$PH(N,k,m) \vdash \square$
$R(k, \frac{m}{2})^2$
$\frac{m}{2}N$
If all $|A_i|$ are small then the rest is big enough to recurse.
$PH(N, k', \frac{m}{2}) \vdash \square$

Comments on the upper bound

PH$(N;k,k)$ has a BD-Frege refutation of size $2^{O(N \log\log N)}$

- PH$(N;k,k)$ has $2^{\Theta(N)}$ clauses (we do almost polynomial).
- A trivial refutation has size 2^{N^2} (we do much better).
- A (less) trivial refutation is $2^{R(k,k)^2}$ (we don't know).

If $R(k,k) \ll \sqrt{N}$ refutation is much smaller than the formula.

Open Problems

- Improve the upper bound from $2^{O(N \log \log N)}$ to $2^{O(N)}$.
- Prove the lower bound with weaker assumptions.
- Solve the $\text{RES}(2)$ proof complexity of weak pigeonhole principle.

- Improve the upper bound from $2^{O(N \log \log N)}$ to $2^{O(N)}$.

Hint: reduce the use of brute force in our proof.

- Prove the lower bound with weaker assumptions.

Hint: reduce from more "balanced" Ramsey formulas.

- Solve the $\text{RES}(2)$ proof complexity of weak pigeonhole principle.

Hint: if you have any, please tell me!

Further details

Conference version:

L. Carlucci, N. Galesi, M. Lauria.
Paris-Harrington Tautologies
26th Conference on Computational Complexity, 2011.

Full version:

L. Carlucci, N. Galesi, M. Lauria.
Paris-Harrington Tautologies
`http://eccc.hpi-web.de/report/2010/153/`

Thank You


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