1 Intro

For the crop circle problem, we need to compute the total area encompassed by circles that may intersect. A restriction given in the problem was that no point ever belongs to 3 or more circles. This limits us to having to compute, at most, the area of two intersecting circles. So here’s some quick and dirty notes on how to compute the area. A diagram is shown in figure 1.

Without loss of generality, we assume the center of the first circle is at $C_1 = (0,0)$ and the center of the second circle is at $C_2 = (d,0)$. We can always put any pair of circles into this coordinate system.

We want to compute the area encompassed by both circles. Adding the areas of both circles will overestimate the total area because the area of the intersection will be counted twice. So we have to subtract that part out. The intersection of two circles is referred to as a lens. We therefore have:

$$A = \pi R_1^2 + \pi R_2^2 - \text{Area(lens)}. \quad (1)$$

The area of the lens is not necessarily trivial to compute since the lens is not a familiar polygon shape. The lens is shown in a closeup in figure 2.

The area of the lens is given by the sum of the areas of the two circular segments, which are separated by the chord $a$. (By abuse of notation, we will refer to the chord as $a$ and its length also as $a$.) So we now need to compute the area of the circular segments.

2 Area of a Circular Segment

A circular segment is formed by drawing a chord in a circle. The “shaded” area in figure 3 shows the circular segment we are interested in.

To form the circular segment, we draw a chord, $a$, in the circle. We run two radial spokes to the endpoints of the chord. The angle between the spokes is $\theta$. The area of the circular segment, $A$, is given by

$$A = A_c(\theta) - A_T \quad (2)$$

where $A_c(\theta)$ is the partial area of the circle swept out by the angle $\theta$, and $A_T$ is the area of the triangle with sides $R, R, a$. 

Figure 1: Example of overlapping circles with radii $R_1$ and $R_2$. The circle centers are a distance $d$ apart.
Figure 2: A closeup of the intersection region of the two circles. This is a lens. The lotus points are the two intersection points of the circles given by \((x_c, y_c)\) and \((x_c, -y_c)\). \(a\) is the chord length.

Figure 3: The shaded area is the circular segment of interest.

The area of the portion of the circle swept out by an angle of \(\theta\) is given by integrating the area differential in polar coordinates, \(dA = dr \, r \, d\theta\). So we have

\[
A_c(\theta) = \int_{r=0}^{R} \int_{\theta'=0}^{\theta} r \, dr \, d\theta'
= \frac{1}{2} R^2 \theta
\]

(3)

Note that if \(\theta = 2\pi\), we get the area of a full circle, \(\pi R^2\). If, for some reason, you don’t like calculus, you can compute the area as a ratio to the area of a full circle, i.e.,

\[
\frac{2\pi}{\pi R^2} = \frac{\theta}{A_c(\theta)}
A_c(\theta) = \frac{1}{2} R^2 \theta
\]

(4)
The area of the triangle is given by Heron’s rule,

$$ A_T = \sqrt{s(s-a)(s-R)(s-R)} \quad s \equiv \frac{a+2R}{2} \quad (5) $$

We now have

$$ A = \frac{1}{2} R^2 \theta - \sqrt{s(s-a)(s-R)(s-R)} \quad s \equiv \frac{a+2R}{2} \quad (6) $$

The only thing left is figure out $\theta$. Draw a perpendicular bisector in the triangle and look at either half. The result is shown in figure 4.

![Figure 4: Half of the triangle formed by the chord, $a$.](image)

From the figure, it is clear that we must have:

$$ \sin \left( \frac{1}{2} \theta \right) = \frac{a}{2R} \quad \theta = 2 \sin^{-1} \left( \frac{a}{2R} \right) \quad (7) $$

So, putting it all together, we finally have the area of a circular segment formed by a chord of length $a$:

$$ A = R^2 \sin^{-1} \left( \frac{a}{2R} \right) - \sqrt{s(s-a)(s-R)^2} \quad s \equiv \frac{a+2R}{2} \quad (8) $$

We now need the length of $a$.

### 3 Chord Length

We now need to figure out the chord length for the chord $a$ shown in figure 2. The endpoints of the chord are given by $(x_c, y_c)$ and $(x_c, -y_c)$ due to how we laid out our coordinate system. So, clearly $a = 2y_c$. The chord endpoints are located at the intersection points of our circle. The points must therefore obey the equations:

$$ x_c^2 + y_c^2 = R_1^2 \quad (9) $$

$$ (x_c - d)^2 + y_c^2 = R_2^2 \quad (10) $$

We now have to figure out $y_c$. Start by subtracting (10) from (9). We get

$$ x_c^2 - (x_c - d)^2 = R_1^2 - R_2^2 $$

which simplifies to

$$ x_c = \frac{d^2 + R_1^2 - R_2^2}{2d}. \quad (11) $$
Putting this into (9), we get

\[ y_c = \left[ R_1^2 - \frac{1}{4d^2} (d^2 + R_1^2 - R_2^2)^2 \right]^{1/2} \]  
\[ = \frac{1}{2d} \sqrt{4d^2 R_1^2 - (d^2 + R_1^2 - R_2^2)^2} \]  
\[ = \frac{1}{2d} \sqrt{(2dR_1 - (d^2 + R_1^2 - R_2^2)) (2dR_1 + (d^2 + R_1^2 - R_2^2))} \]  
\[ = \frac{1}{2d} \sqrt{(R_2^2 - (d - R_1)^2) (d + R_1)^2 - R_2^2} \]  
\[ y_c = \frac{1}{2d} \sqrt{(-d + R_1 + R_2) (d - R_1 + R_2) (d + R_1 - R_2) (d + R_1 + R_2)} \] (12)

Some references include the extra factorization to get \( y_c \) down to (13).
We therefore have
\[ a = \frac{1}{d} \sqrt{(-d + R_1 + R_2) (d - R_1 + R_2) (d + R_1 - R_2) (d + R_1 + R_2)} \] (13)

4 Area of the Lens

We can now get back to computing the area of the lens. As seen in figure 2, the lens consists of two circular segments, one from each circle and the chord \( a \). The result is:

\[ \text{Area(lens)} = R_1^2 \sin^{-1} \left( \frac{a}{2R_1} \right) - \sqrt{s_1 (s_1 - a)(s_1 - R_1)^2} \]  
\[ + R_2^2 \sin^{-1} \left( \frac{a}{2R_2} \right) - \sqrt{s_2 (s_2 - a)(s_2 - R_2)^2} \] (15)

with
\[ s_1 = \frac{a + 2R_1}{2} \]  
\[ s_2 = \frac{a + 2R_2}{2} \] (16)

As a quick check, there is no lens if the two circles are tangential to each other, i.e., \( d = R_1 + R_2 \). We can see from (14) that \( a = 0 \) for this case. So the sin\(^{-1} \) terms will be 0. Further, for this case, \( s_1 = R_1 \) and \( s_2 = R_2 \), which eliminates the square root terms in (15), showing the area of the lens would be 0 if the circles are tangential.

5 Area Encompassed by the Circles

Finally, we can return to the area encompassed by the circles. From (1) and (15), we have:

\[ A = \pi \left( R_1^2 + R_2^2 \right) - R_1^2 \sin^{-1} \left( \frac{a}{2R_1} \right) - R_2^2 \sin^{-1} \left( \frac{a}{2R_2} \right) \]  
\[ + \sqrt{s_1 (s_1 - a)(s_1 - R_1)^2} + \sqrt{s_2 (s_2 - a)(s_2 - R_2)^2} \] (17)

with
\[ s_1 = \frac{a + 2R_1}{2} \]  
\[ s_2 = \frac{a + 2R_2}{2} \]
\[ a = \frac{1}{d} \sqrt{(-d + R_1 + R_2) (d - R_1 + R_2) (d + R_1 - R_2) (d + R_1 + R_2)} \]