Stability Issues in Hop-by-Hop Rate Based Congestion Control

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I. Abstract

Continuous in time fluid flow models based on hop-by-hop networks are developed both from a single connection's point of view as well as from the entire networks point of view. Conditions on various control laws that would guarantee stability of such networks are derived.

II. Introduction

The objective of flow control in data networks is to provide a reasonable trade-off between network throughput and quality of service - usually a function of the number of packet losses and end-to-end delay/jitters. The state-of-the-art deployed feedback control mechanisms in the Internet are *end-to-end* flow control methods since the control is enforced only at the source end-host and the intermediate routers on the way to the destination end-host do not take part in flow control (apart from scheduling), e.g. the feedback algorithms used in TCP-suites [1], [2],[3]. There are also *explicit* feedback methods in which a router sets a *congestion bit* in the packet header if its queue length exceeds a certain limit as in DEC-bit [4]. In addition, *hop-by-hop* (HBH) rate based congestion control [5] has also been of interest as a research problem. In HBH, a feedback signal signifying congestion at a node is relayed to the nodes which are sending data to it. In response to the congestion signal received, a node updates its sending accordingly. It has been speculated that HBH is inherently unstable though simulation results for some particular HBH schemes argue against it [5]. The objective of this paper is to study stability of continuous in time fluid flow models based on such networks.

The paper is organized as follows. In Section III, network model from the point of view of a particular connection is described. In Section IV, stability of the model is examined. In Section V, a network model without any concern for a particular connection is obtained and its stability is examined.

III. Problem Formulation

Let us consider a particular unicast connection $C_{s,d}$ which has source s and destination d. Let there be a fixed path from s to d during the lifetime of $C_{s,d}$. Let there be N routers, denoted by r_1 through r_N, on path from s to d. Let the service capacity of r_i at time t be given by $\tilde{\mu}_i(t)$, which w.l.o.g. can be taken to be a time-invariant $\tilde{\mu}_i$. Let the service capacity of r_i at time t for the rest of the connection be $\tilde{\mu}_i^d(t)$ $(\leq \tilde{\mu}_i)$. Then the effective service capacity of r_i for $C_{s,d}$

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at time t is given by

$$\mu_i(t) = \tilde{\mu}_i(t) - \tilde{\mu}_i^d(t). \tag{1}$$

We call r_i *upstream* of r_j if its index in the ordered set $\{s, r_1, \ldots, r_N, d\}$ of the nodes of $C_{s,d}$ is smaller than r_j. Let $\tilde{q}_i(t)$ and $\bar{q}_i$ denote, respectively, the actual and desired queue sizes at r_i at time t. Let the sending rate at r_i for $C_{s,d}$, denoted by $x_i(t)$, be given by $\tilde{x}_i(t)$. Let the desired (pre-set) rate be $\bar{x}_i(t) \doteq \bar{x}(t) = \bar{x} \;\; \forall i, \forall t$. The rationale for pre-set rates is as follows. A connection i is set up only when the service rate for the connection is guaranteed to lie within $[\bar{x} - \delta, \bar{x} + \delta]$ for some $\bar{x}, \delta$. Thus, in the face of new connections coming up or old connections getting terminated, the service rate stays within bounds. Define

$$q_i(t) \doteq \tilde{q}_i(t) - \bar{q}_i(t) \qquad \forall i = 1, 2, \ldots, N \tag{2}$$

$$x_i(t) \doteq \tilde{x}_i(t) - \bar{x}_i(t) \qquad \forall n = 1, 2, \ldots, N. \tag{3}$$

Let us assume dynamics to be linear about $\{x_i, q_i | i = 1, 2, \ldots, N\}$. Let r_i update $x_i(t)$ according to the following law.

$$\dot{x}_i(t) = \lambda x_i(t) + k\; q_{i+1}(t) \qquad \forall n = 1, 2, \ldots, N. \tag{4}$$

The queue dynamics are given by

$$\dot{q}_i(t) = \tilde{x}_{i-1}(t) - \tilde{x}_i(t) + u_i(t) \qquad \forall i = 1, 2, \ldots, N-1 \tag{5}$$

where $u_i(t)$ denotes the difference between incoming traffic and outgoing traffic at r_i due to the rest of the connections. Let $x_0(t)$ define the rate of source at time t. Define $\tilde{x}(t) \doteq [x_0(t)\; x_1(t)\; \ldots\; x_N(t)\; q_1(t)\; \ldots\; q_{N-1}(t)]^T$. For notational simplicity, let us refer to $\tilde{x}$ by x. Define $u(t) \doteq [u_1(t)\; u_2(t)\; \ldots\; u_{N-1}(t)]^T$. Then (4)-(5) can be rewritten as

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{6}$$

where

$$A = \begin{bmatrix} \Lambda & K \\ W & 0 \end{bmatrix}, \quad B = [\underbrace{0, 0, \ldots, 0}_{N}, 1, 1, \ldots, 1] \tag{7}$$

$$\Lambda = \lambda \cdot I, \quad K = k \cdot I, \quad W = \begin{bmatrix} 1 & -1 & 0 & \ldots & 0 \\ 0 & 1 & -1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & 0 & 0 & 1 \end{bmatrix} \tag{8}$$

and I denotes $N \times N$ identity matrix and 0 denotes $N \times N$ matrix with all entries equal to zero. Let us refer to the system represented by (6) by Σ_1. Σ_1 is a simplified model of the system in which the effect of the dynamics of connections which share links with $C_{s,d}$ is taken to be that of a disturbance. If a state is included to account for each of such flows, model Σ_2 is obtained (see Fig. 1). Let $P_i \doteq \{y_{i,1}, y_{i,2}, \dots, y_{i,n_i}\}$ denote the set of incoming rates at r_{i+1} which share

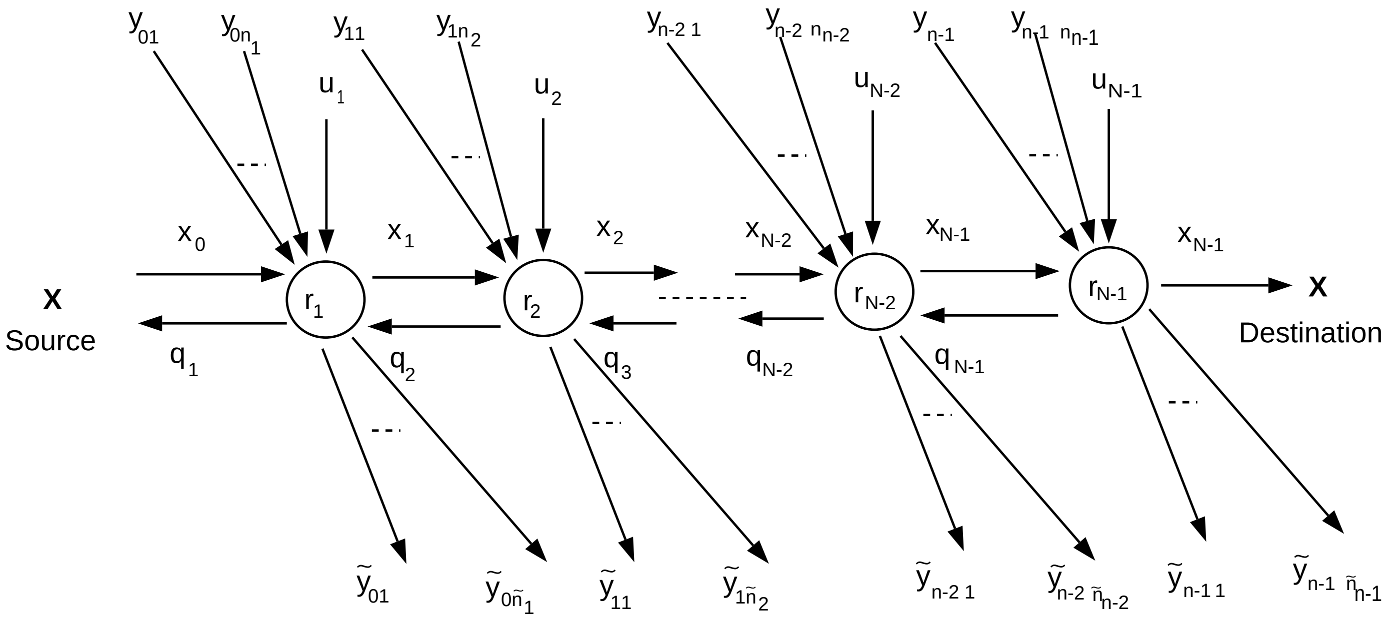

Fig. 1. The figure shows routers r_i exchange feedback information q_i to update data rates x_i for a particular connection $C_{s,d}$. The figure represents Σ_2 with y_{ij}, $\tilde{y}_{ij}$ being the incoming and outgoing flows due to other connections and u_i account for the effect of connections coming up or getting terminated. By setting y_{ij}, $\tilde{y}_{ij}$ to zero, Σ_1 is obtained.

the link r_i-r_{i+1} with $C_{s,d}$. Let $\tilde{y}_{i,1}, \tilde{y}_{i,2}, \dots, \tilde{y}_{i,\tilde{n}_i}$ ($\tilde{n}_i \le n_i$) denote the rates from P_i which do not share link $r_{i+1} - r_{i+2}$ with $C_{s,d}$. Define $Y_i \doteq [y_{i,1}\ y_{i,2}\ \dots\ y_{i,n_i}]^T, Z_i \doteq [\tilde{y}_{i,1}\ \tilde{y}_{i,2}\ \dots\ y_{i,\tilde{n}_i}]^T$. Write $n_x \doteq N + \sum_{i=0}^{N-2} n_i + \tilde{n}_i$. Define

$$x \doteq [x_0\ Y_0\ Z_0\ x_1\ Y_1\ Z_1\ \dots\ x_{N-1}\ q_1\ q_2\ \dots\ q_{N-1}]^T \tag{9}$$

$$\Lambda \doteq diag(\lambda,\ \lambda,\ \dots,\ \lambda) \in R^{n_x \times n_x} \tag{10}$$

$$K_i \doteq [\underbrace{k\ k\ \dots\ k}_{n_i}\ 0\ 0 \dots\ 0]^T \qquad\qquad i = 0, 1, \dots, N-1 \tag{11}$$

$$\phi_i \doteq [0\ 0\ \dots\ 0] \in R^{\sum_{j=0}^{i-1} n_i + \tilde{n}_i} \tag{12}$$

$$W_i \doteq [\phi_i, \underbrace{1,\ 1,\ \dots,\ 1}_{1+n_i},\ \underbrace{-1,\ -1,\ \dots,\ -1}_{1+\tilde{n}_i},\ 0,\ \dots, 0] \tag{13}$$

$$W = [W_1; W_2; \dots; W_N], \quad K \doteq [K_0\ K_1\ \dots\ K_{N-1}]; \tag{14}$$

Then, system dynamics are given by

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{15}$$

where

$$A = \begin{bmatrix} \Lambda & K \\ W & 0 \end{bmatrix} \tag{16}$$

Let us refer to this system by Σ_2. Next consider a modification Σ_3 obtained from Σ_1 by changing rate update law (4) to

$$\dot{x}_i(t) = \lambda x_i(t) + k\ q_{i+1}(t) + \bar{k}\ q_i(t) \qquad\qquad i = 1, 2, \ldots, N. \tag{17}$$

Then, the system dynamics are given by (6),(7) and (8) with

$$K \doteq \begin{bmatrix} k & 0 & 0 & \ldots & 0 \\ \bar{k} & k & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & \bar{k} \end{bmatrix}, \tag{18}$$

IV. Stability Analysis

A. Linearized Models

Let us first consider how the choice of controller affects stability of the linear models Σ_1, Σ_2, Σ_3.

Theorem 1: Let

$$l \leq 0, \quad k \leq 0, \tag{19}$$

$$|l| \geq |\sqrt{l^2 + 4k}| \qquad\qquad \text{if } l^2 + 4k \geq 0 \tag{20}$$

Then, Σ_1 is stable. If either $l > 0$ or $k > 0$, the system is unstable.

Proof. Let ν be an eigenvalue of A and let v be its corresponding right eigenvector. Then

$$A\ v = \nu\ v \Longleftrightarrow \begin{cases} \lambda\ v_i + k\ v_{i+N} = \nu\ v_i & \text{i=1,2,\ldots,N} \\ v_i - v_{i+1} = \nu\ v_{i+N} & \text{i=1,2,\ldots,N-1.} \end{cases} \tag{21}$$

Solving (21), it can be verified that the eigenvalues of A are $\nu = \lambda$ and $\nu = \frac{\lambda \pm \sqrt{\lambda^2+4k}}{2}$. Σ_1 is unstable if an eigenvalue of A lies in open right half s-plane. Σ_1 is stable if all of the eigenvalues of A lie in open left half s-plane. From these definitions, (19) and (20) follow. QED.

Theorem 2: Let (19) and (20) hold. Then, Σ_2 is stable. If either $l > 0$ or $k > 0$, the system is unstable.

Proof. It can be shown on the lines of the proof for Theorem 1 that the eigenvalues of matrix A of Σ_2 are $\nu = \lambda, \frac{\lambda \pm \sqrt{\lambda^2+4n_i k}}{2}$ where n_i is as defined before. Note that since $k \leq 0$

$$\sqrt{\lambda^2 + 4n_i k} < \sqrt{\lambda^2 + 4k} \tag{22}$$

$$\Longleftrightarrow \frac{\lambda \pm \sqrt{\lambda^2 + 4k}}{2} < 0 \Longrightarrow \frac{\lambda \pm \sqrt{\lambda^2 + 4n_i k}}{2} < 0. \tag{23}$$

Hence Σ_2 is stable if (19)-(20) are satisfied and is unstable otherwise. QED.

Theorem1 and Theorem2 together show that from stability point of view, it is sufficient to check the stability of Σ_1 rather than consider the more involved model Σ_2. On similar lines the following theorem can be proved.

Theorem 3: Let $\lambda, k, \bar{k}$ be such that the roots of

$$P(x) = x^4 - 2\lambda\, x^3 + (-\lambda^2 - 2(k - \bar{k}))\, x^2 + 2(k - \bar{k})\lambda\, x + (k - \bar{k})^2 + k\bar{k} = 0 \tag{24}$$

are nonpositive. Then, Σ_3 is stable.

Next, the effect of finite (and positive) queuesizes is considered.

B. Effect of Nonlinearity

First some basic definitions are stated.

Definition 1 [6]: Let $f : R \mapsto R$ with $f(0) = 0$. Then, $f \in \text{sector}[k_1, k_2]$ iff $k_1 e^2 \leq e f(e) \leq k_2 e^2,\ \forall e \in R\ (e \neq 0)$.

Definition 2 [7]: Let $u = \Delta v$. Then, Δ is said to satisfy integral quadratic constraint (IQC) given by Π if

$$\int_{-\infty}^{\infty} \begin{bmatrix} v(j\omega) \\ u(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} v(j\omega) \\ u(j\omega) \end{bmatrix} \mathrm{d}\omega \geq 0 \quad \forall v \in L_2 \tag{25}$$

where L_2 denotes the space of square integrable functions and $*$ denotes Hermitian operator.

In reality, $0 \leq \tilde{q}(t) \leq q^*\ \forall t$. Because of this saturation nonlinearity, the pure integral action in (5) is replaced by $\dot{q}(t) = f(\zeta)$ where $\zeta(t) = \tilde{x}_{i-1}(t) - \tilde{x}_i(t) + u_i(t)$ and

$$f(\zeta) = \begin{cases} 1 & \text{if } |\tilde{q}(t)| \leq q^* \\ & \text{or if } \tilde{q}(t) = q^* \text{ and } sgn(u) = -1 \\ & \text{or if } \tilde{q}(t) = -q^* \text{ and } sgn(u) = 1 \\ 0 & \text{else.} \end{cases} \tag{26}$$

Note that $f \in \text{sector}[0, 1]$. Let hop-to-hop delay be $\tau\ \ (0 \leq \tau \leq \tau^*)$. This modeled as $(\Delta u)(t) = u(t - \tau^*)$. For simplicity let us consider Σ_4 with source and destination separated by 3 routers. Tapping signals u_i and v_i as shown in fig 2, the plant P and uncertainty Δ decomposition is given as $v = P\ u,\ \ u = \Delta\ v$ where

$$P = \begin{bmatrix} 0 & 0 & -K(s) & 0 & K(s)/s & 0 \\ 0 & 0 & 0 & -K(s) & \bar{K}(s)/s & K(s)/s \\ 0 & 0 & 0 & 0 & 1/s & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/s \\ -1 & 0 & -K(s) & K(s) & (K(s) - \bar{K}(s))/s & -K(s)/s \\ 0 & -1 & 0 & -K(s) & \bar{K}(s)/s & (K(s) - \bar{K}(s))/s \end{bmatrix} \tag{27}$$

$$\Delta = diag(1 - e^{-\tau s},\ 1 - e^{-\tau s},\ 1 - e^{-\tau s},\ 1 - e^{-\tau s},\ f(\cdot),\ f(\cdot)). \tag{28}$$

For N hop case, the form of Δ is obvious and P can be seen to be a $3N \times 3N$ matrix with $N-1 \times N-1$ component matrices (see Fig 2) bearing relationship with 2×2 counterparts in (2) as

$$\begin{bmatrix} K(s)/s & 0 \\ \bar{K}(s)/s & K(s)/s \end{bmatrix} \mapsto \begin{bmatrix} K(s)/s & 0 & 0 & 0 & \dots & 0 \\ \bar{K}(s)/s & K(s)/s & 0 & 0 & \dots & 0 \\ 0 & \bar{K}(s)/s & K(s)/s & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \bar{K}(s)/s & K(s)/s \end{bmatrix} \quad \text{etc.}$$

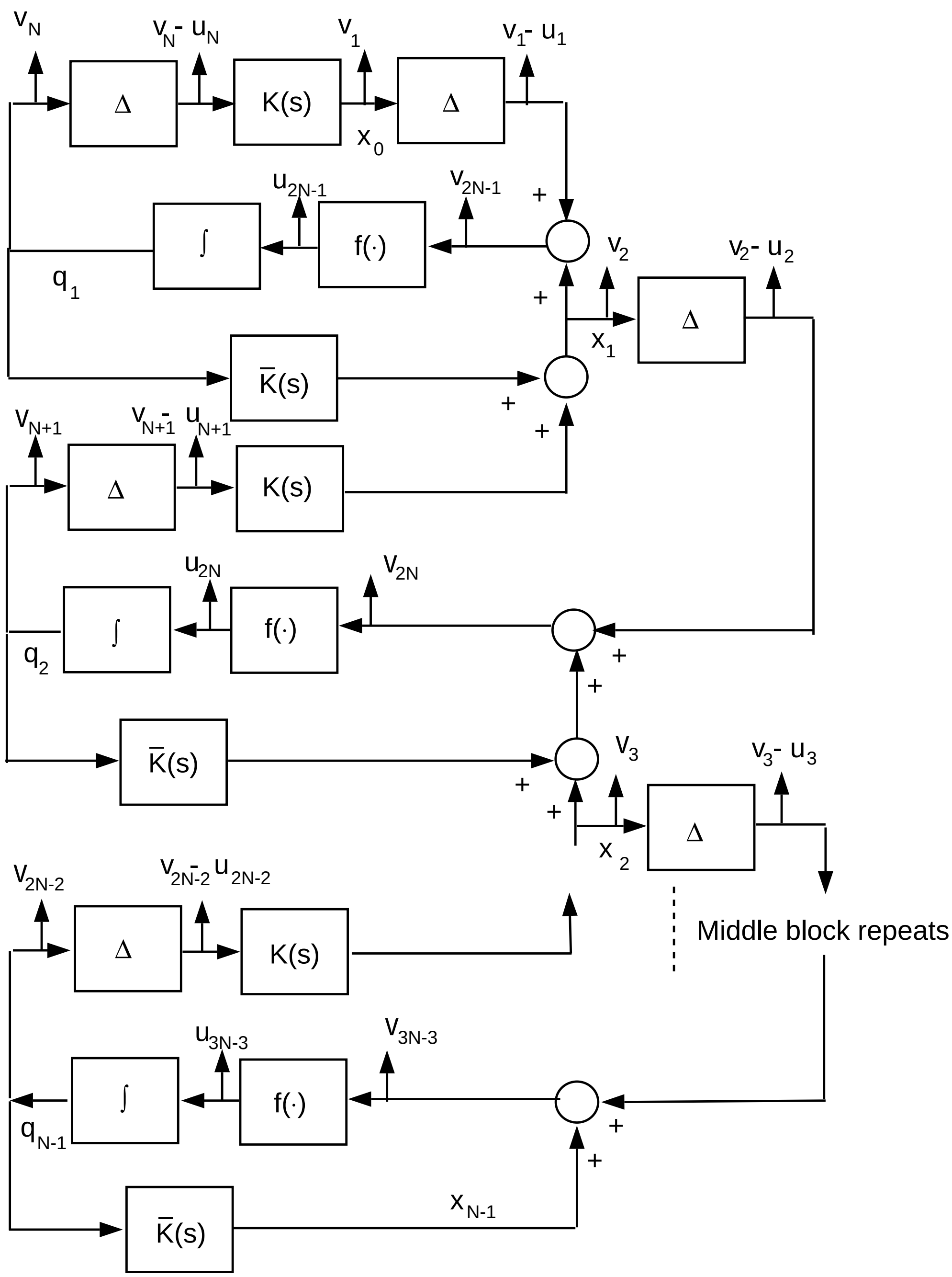

Fig. 2. The figure shows tapping of signals used to decompose the system into uncertainty Δ and plant P seen by Δ.

Using IQCs for the uncertainties (see [8] for a list of IQCs), sufficiency conditions for the stability of the system can be derived. If τ is assumed to be negligible, as would be the case with high-speed data links, $\Delta = diag(f(\cdot),\ f(\cdot))$ and satisfies IQC given by

$$\Pi(j\omega) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2 \end{bmatrix}. \tag{29}$$

Theorem 4: Σ_4 is stable if $K(s), \bar{K}(s)$ are chosen such that

$$((\frac{K(s)-\bar{K}(s)}{s})^* + (\frac{K(s)-\bar{K}(s)}{s} - 2))^2 - |(\frac{\bar{K}(s)-K(s)}{s})|^2 < 0. \tag{30}$$

Proof: It can be seen that the plant P in this case is

$$P = \begin{bmatrix} (K(s)-\bar{K}(s))/s & -K(s)/s \\ \bar{K}(s)/s & (K(s)-\bar{K}(s))/s \end{bmatrix}. \tag{31}$$

For stability, it is sufficient if [7]

$$\begin{bmatrix} P \\ I \end{bmatrix}^* \Pi \begin{bmatrix} P \\ I \end{bmatrix} < 0 \quad \text{on } s = jR. \tag{32}$$

Expanding and using Schur's compliment [9], the result follows. Q.E.D.

V. Model of the Entire Network

Now we extend the continuous time, linear system model and controller to the entire network.

A. System model

Define the matrix $[\alpha_{i,j}]$ by

$$\alpha_{i,j} = \begin{cases} 1 & \text{if router } i \text{ sends data to router } j \\ 0 & \text{otherwise} \end{cases}$$

Let q_i denote the queue size at r_i, $\bar{q}_i$ denote the short term average queue size r_i, $x_{i,j}$ denote the rate at which data is being sent from r_i to r_j, $\bar{x}_{i,j}$ denote the short term average rate at which data is sent from r_i to r_j, $v_{i,j}$ denote the deviation from current distribution, d_in_i denote the rate at which data enters the network (from the un-modeled network) at r_i, d_out_i denote the rate at which data leaves the network (to the un-modeled network) at r_i and let $u_i \doteq d\_in_i - d\_out_i$. The queue dynamics are

$$\dot{q}_i = -\sum_j \alpha_{i,j} x_{i,j} + \sum_j \alpha_{j,i} x_{j,i} + u_i. \tag{33}$$

Since $x_{i,j}$ is not defined for each i,j we defined $\alpha_{i,j} x_{i,j}$ to be zero if $\alpha_{i,j} = 0$. We impose the following rate update law

$$\dot{x}_{i,j} = -\lambda\left(x_{i,j} - \bar{x}_{i,j}\right) - k_1\left(q_j - \bar{q}_j\right) + k_2\left(q_i - \bar{q}_i\right) + v_{i,j} \tag{34}$$

where, for simplicity, $k_1, k_2, \lambda > 0$ are constant. For the moment, assume that $v_{i,j} = 0$. As will be shown, the $-\lambda\left(x_{i,j} - \bar{x}_{i,j}\right)$ term forces the system to be stable. From (34), it is clear that a steady state is achieved when $x_{i,j} = \bar{x}_{i,j}$ and $q_j = \bar{q}_j$ for all i,j. That is, the rate $x_{i,j}$ is the same as $\bar{x}_{i,j}$. Since, it is not known what the average rate at any router should be, we cannot predefine $\bar{x}_{i,j}$. Thus, we simply let $\bar{x}_{i,j}$ to be a low pass filtered version of $x_{i,j}$. Likewise, $\bar{q}_j$ is a low pass version of q_j. In particular,

$$\begin{aligned} \bar{x}_{i,j} &= -\delta \bar{x}_{i,j} + \delta x_{i,j} \\ \bar{q}_{i,j} &= -\gamma \bar{q}_{i,j} + \gamma q_{i,j} \end{aligned} \tag{35}$$

where γ and δ are small. The exogenous inputs u_i are used to account for the interaction of the modeled network with un-modeled network - either routers or end-hosts. By definition, if $u_i > 0$, then more data is entering the network at this router than is leaving. At the steady state, the same amount of data enters the network as it leaves. Thus $\sum_i u_i = 0$. Now suppose that a new connection begins with data entering the network at r_j and leaving the network at r_k. Furthermore, before this new connection, suppose that no data was entering or leaving the network through these routers, i.e. $u_j = u_k = 0$. As the connection begins, u_j becomes positive. But, since the data has not yet reached r_k, $u_k = 0$, thus $\sum_i u_i > 0$. In time, the data reaches router k and $u_k < 0$ since by assumption the rest of the connections are operating at steady state, so that $\sum_i u_i = 0$. After the connection terminates, $u_j = 0$. However, until the last packet of data has reached r_k, we have $u_k < 0$ and $\sum_i u_i < 0$. Finally, when the last packet reaches r_k and $u_k = 0$ and $\sum_i u_i = 0$. Thus

$$\int_{-\infty}^{\infty} \sum_i u_i = 0 \tag{36}$$

but $\int_{t_1}^{t_2} \sum_i u_i$ might not be zero for a choice of t_1, t_2 $(t_1 < t_2)$. We assume that the connections are set so that the data enters and leaves at the same rate. The exact value of this rate is not be discussed.

The input u does not account for all the variability that can occur. In particular, u can remain constant and the connections may change. The inputs $v_{i,j}$ account for these other changes. Define $xq_{i,j}$ to be the rate at which data is entering r_i with destination r_j. Define some nonlinear system f such that when

$$v_{i,j} = f\left(x_{i,j}, \sum_k \alpha_{i,k} x_{i,k}, xq_{i,j}, \sum_k \alpha_{i,k} xq_{i,k}, q_i\right)$$

the short term average of the fraction of data that leaves r_i for r_j is the same as the short term average of the fraction of data entering the r_i that is to be sent to r_j. One possibility is

$$v_{i,j} = \frac{dxq_{i,j}}{dt} - \left(-\lambda \tilde{x}_{i,j} - k_1 (q_j - \bar{q}_j) + k_2 (q_i - \bar{q}_i)\right).$$

For our purposes, the exact form of $v_{i,j}$ is not important. However, it is important to note that

$$\sum_j v_{i,j} = 0 \qquad \text{for all } i. \tag{37}$$

That is, if the fraction of data destined for one router increases, the fraction must decrease for some other routers. Furthermore, $v_{i,j}$ are not used to model an increase in the flow of data leaving the router. The total flow of data out of a router can only change by some queues filling up or emptying.

B. Stability of Network

It will be proven that with the rate update law defined by equation (34), the entire network is stable. That is, the linear system mapping inputs u and v to the queuelengths $q_i - \bar{q}_i$ and data rates $x_{i,j} - \bar{x}_{i,j}$ are asymptotically stable. After some manipulation, the transfer function from u_i and $v_{i,j}$ to $\tilde{q}_i$ is found to be

$$\frac{s+\delta}{s(s+\lambda+\delta)}\left(\frac{s(s+\gamma)(s+\lambda+\delta)}{s+\delta} I - (k_1 P_1 + k_2 P_2)\right)\tilde{q} = u + \left(\frac{\delta + s}{s(s+\lambda+\delta)}\right) v$$

where P_1 and P_2 are some matrices, $\tilde{q}_i \doteq q_i - \bar{q}_i$, $\tilde{q} \doteq [\tilde{q}_1 \cdots \tilde{q}_n]^T$, $u \doteq [u_1 \cdots u_n]^T$, $v \doteq [\sum_j \alpha_{j,1} v_{j,1} \cdots \sum_j \alpha_{j,n} v_{j,n}]^T$. The poles of the linear system which has input u and ouput $\tilde{q}$ correspond to the values of s such that

$$\left(\frac{s(s+\gamma)(s+\lambda+\delta)}{s+\delta} I - (k_1 P_1 + k_2 P_2)\right) \tag{38}$$

is not invertible. Clearly this matrix is not invertible if and only if $\frac{s(s+\gamma)(s+\lambda+\delta)}{(s+\delta)}$ is an eigenvalue of $(k_1 P_1 + k_2 P_2)$. The next lemma describes the eigenvalues of $(k_1 P_1 + k_2 P_2)$.

Lemma 1: The eigenvalues of $(k_1P_1 + k_2P_2)$ are real non-positive. There is one eigenvalue at zero. The eigenvector associated with the zero eigenvalue is the constant vector, i.e. $\begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}^T$.

Since $(k_1P_1 + k_2P_2)$ has an eigenvalue at zero, the matrix (38) is not invertible for $s = 0$. Thus the linear system has a pole at zero. However, since u is multiplied by s, this pole is not excited by u. The other poles are the values of $s \neq 0$ where s is such that $\frac{s(s+\gamma)(s+\lambda+\delta)}{s+\delta}$ is an eigenvalue of $(k_1P_1 + k_2P_2)$. By Lemma 1, s must be such that $\frac{s(s+\gamma)(s+\lambda+\delta)}{s+\delta} = -r$, for some $r \geq 0$. For $r = 0$, $s = 0$ corresponds to the pole at zero. For $r > 0$,

$$s(s+\gamma)(s+\lambda+\delta) + r(s+\delta) = 0 \tag{39}$$

Applying the Routh stability criterion, we find that since all the constants are positive, the solutions to (39) has negative real parts if $(\lambda+\gamma+\delta)(\gamma(\lambda+\delta)+r) - r\delta > 0$, which is true for all $r > 0$. Therefore, we conclude that all poles occur for values of s such that $Re(s) < 0$. Hence, the linear system from inputs u to $\tilde{q}$ is asymptotically stable.

Now we examine the mapping from v to q, that is

$$\left(\frac{(s+\gamma)(s+\lambda+\delta)s}{s+\delta} I - (k_1P_1 + k_2P_2)\right)\tilde{q} = \begin{bmatrix} \sum_j \alpha_{j,1}v_{j,1} \\ \vdots \\ \sum_j \alpha_{j,1}v_{j,n} \end{bmatrix}.$$

Again, there is a pole at zero that is not excited by the input v. This can be explained as follows: Setting $s = 0$, we examine the mapping

$$(-(k_1P_1 + k_2P_2))\tilde{Q}(0) = V(0) \tag{40}$$

where $\tilde{Q}(0)$ and $V(0)$ are the Laplace transform of $\tilde{q}$ and v evaluated at 0. That is,

$$V(0) = \begin{bmatrix} \int_{-\infty}^{\infty} \sum_j \alpha_{j,1}v_{j,1} & \cdots & \int_{-\infty}^{\infty} \sum_j \alpha_{j,n}v_{j,n} \end{bmatrix}$$

and $\tilde{Q}(0) = \begin{bmatrix} \int_{-\infty}^{\infty} \tilde{q}_1 & \cdots & \int_{-\infty}^{\infty} \tilde{q}_n \end{bmatrix}$. From (40)

$$\tilde{Q}(0) = (k_1P_1 + k_2P_2)^{-1} V(0) \tag{41}$$

where $\tilde{Q}(0)$ is defined only if $V(0)$ is orthogonal to the direction that $(k_1P_1 + k_2P_2)^{-1}$ is infinite, that is, if $V(0)$ is orthogonal to the eigenspace of zero eigenvalue of $(k_1P_1 + k_2P_2)$. By Lemma 1, this eigenspace is spanned by the vector $\begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}^T$. And by equation (37), $\sum_i \sum_j \alpha_{j,i}v_{j,i} = 0$. Thus $V(0)$ is orthogonal to $\begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}^T$; the mapping (41) is well defined and $\tilde{Q}(0)$ is finite whenever is. Hence, the pole at zero is not excited by the input v.

In the same way the transfer function from u to $\tilde{q}$ is stable, the other poles of the transfer function from v to $\tilde{q}$ are stable. Therefore, the linear system from u and v to $\tilde{q}$ is asymptotically stable. Since $\lambda > 0$, equation (34) implies, the mappings from u and v to $\tilde{x}$ are asymptotically stable. Of course, the mapping to q, $\bar{q}$, x and $\bar{x}$ are only marginally stable. For if u or v are nonzero, these values 'charge up' to some non zero value. This is to be expected, if $x_{i,j}$ is zero then no data flows. Furthermore, without direct control over u, we cannot make the queues go to zero.

VI. Conclusion

Several models of continuous in time fluid flow networks addressing the issues of hop-by-hop networks have been studied under different control strategies. The results characterize control laws purely from stability stand point.

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