Robustness of Interconnected Systems with Controller Saturation and Bounded Delays

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1 Abstract

Stability of a data network topology, in which a source updates its data transmission rate based on delayed congestion information on its own path only, is examined. Delays associated with feedback information are taken to be known constants and bounded uncertainty is associated with feedback information. Use of linear matrix inequality (LMI) and integral quadratic constraint (IQC) techniques to derive sufficiency conditions for stability of such systems is demonstrated.

2 Introduction

This paper deals with stability issues concerning a class of interconnected systems arising in data networks such as Internet. In the state-of-the-art data networks, a user is guaranteed neither a fixed data rate nor a fixed path during the lifetime of the connection. Typically, a user does not have access to the entire network state but has access to only a subset of it through a direct or indirect feedback e.g. if the transport protocol employed is a TCP/IP suite [2], [3], [4], then the feedback information needed to enforce flow control at the user end is derived from advertised window size sent by destination node to source node and computed round trip time (RTT). Because of the possible queuing delays, the feedback is delayed and is subject to uncertainty. In addition, the buffer sizes at the routers are finite and the data transmission rate at a source can be neither negative nor greater than the link speed (or a pre-allocated upper bound if an RSVP mechanism [5] is used). Thus, the stability analysis of such a system is an instance of time-delay systems with saturation nonlinearities.

Noting that flow control in such systems is employed in a decentralized manner relying on local information, the analysis of such systems has been formulated by [6] and [1] as a dynamic game theoretic problem in which the choice of control laws that lead to a stable operating point translates to a Nash equilibrium solution. Therein, it is assumed that relevant network dynamics can be modeled in continuous time by ordinary differential equations, and that delays and saturation nonlinearities can be neglected. In this paper, the assumption that continuous time ordinary differential equations hold is retained but, in addition, the effect of nonlinearities and delays is taken into account. As a rule, inferences drawn from robustness stability analysis become less conservative when more structure is placed on the system under investigation. Therefore, a particular topology is chosen in which one connection interacts with two mutually non-interacting connections. The integral quadratic constraints (IQC) framework with linear matrix inequalities (LMI) is used to examine stability of this system.

This paper is organized as follows. In section 3.1, terminology is introduced and the problem is posed formally. Relevant background material is presented in section 3.2. In section 4, the main results are stated. Discussion and conclusions are presented in section 5.

3 Problem Formulation

3.1 Terminology and Network Topology

Information flow in data network consists of two types of flows, viz. *data* flow and *acknowledgment* flow. These flows are quantized in the form of packets. The network is comprised of three types of nodes, viz. a *source* i.e. the end-host which generates data and receives acknowledgment, *destination* i.e. the end-host which receives data and generates acknowledgment, and *router* which schedules and forwards the data/acknowledgment from an end-host to an end-host, or from an end-host to another router, or from a router to another router, or from a router to an end-host. The link connecting two nodes n_1 and n_2, if it exists, is denoted as $n_1 - n_2$. The link $n_1 - n_2$ is said to exist in the functional sense; i.e., it exists only if there is a connection set up on it, otherwise it is assumed nonexistent even if it is physically present.

Unless otherwise specified, $n_1 - n_2$ is equivalent to $n_2 - n_1$. If a source n_1 is connected to destination n_4 via say $n_1 - n_2, n_2 - n_3$, a *path* $n_1 - n_2 - n_3 - n_4$ is said to to exist. The path taken by data from source to destination is referred to as a *feedforward* path and the path taken by acknowledgment from destination to source is referred to as a *feedback* path. A packet sent from one node to another is subject to *transmission delay* (due to finite link speeds), *processing delay* (due to finite server capacity at a node) and *queuing delay* (due to finite link speeds, finite server capacity and, possibly, interfering interfering traffic competing for the same resource).

Let $\tilde{x}_{ij}$ denote the rate of data flow from n_i to n_j. We consider deviations of sending rate about an operating point as given by $x_{ij} \doteq \tilde{x}_{ij} - x^*_{ij}$ for some x^*_{ij}. Due to rate limitations, which may exist purely because of the finite bandwidth of the outgoing link at a node or because of a quality of service type limitations imposed by an RSVP type scheme, the actual transmission rates are given by

$$\tilde{x}_{ij} = \begin{cases} u_{ij} & \text{if } |x_{ij}| > u_{ij} \\ x_{ij} & \text{if } |x_{ij}| \leq u_{ij}. \end{cases} \quad (1)$$

Consider the network topology shown in Fig 1. Let $q_3(t)$ and $q_5(t)$ denote instantaneous queue sizes at n_3 and n_5 respectively. We assume that an estimate of queue sizes (instantaneous or averaged over short interval) at only those the routers over which its data packets are traversing is available at a source. If the total queuelength experienced over feedforward loop increases (i.e., if delay in the feedforward loop increases), then the source decreases its rate of transmission. The control law considered is as follows.

$$\dot{x}_{13}(t) = \lambda x_{13}(t) + K(q_3(t - \tau_1(t)) + q_5(t - \tau_2(t))) \quad (2)$$

$$\dot{x}_{23}(t) = \lambda x_{23}(t) + K q_3(t - \tau_3(t)) \quad (3)$$

$$\dot{x}_{65}(t) = \lambda x_{56}(t) + K q_5(t - \tau_4(t)) \quad (4)$$

Delays $\tau_i(\cdot)$ ($\tau_i \in [0, \tau_i^*]$) associated with feedback information are state dependent. For ease of analysis (primarily, notational convinience), they are taken to be a known constant τ and $q_i(t - \tau_i(t))$ replaced by $\alpha_i(t) q_i(t - \tau)$ where $\alpha_i(\cdot)$ is a time-varying scalar ($i = 3, 5$), thereby associating an uncertainty with $q_i(t - \tau)$. Note that $\alpha_i(t)$ ($\in [-\zeta, \zeta]\ \forall t$) is inversely proportional to outgoing link bandwidth available at n_i and is directly proportional to queue size at n_i. Also, note that the smaller the value of $\tau_i(t)$, the smaller the value of ζ_i which implies greater confidence in the $q_i(t - \tau)$ and hence its uncertainty will be smaller. The queue dynamics are given by

$$\dot{q}_3(t) = \begin{cases} \max\{\tilde{x}_{13}(t) + \tilde{x}_{23}(t) - \mu_3, 0\} & \text{if } q_3(t) = -q^* \\ \tilde{x}_{13}(t) + \tilde{x}_{23}(t) - \mu_3 & \text{if } |q_5(t)| < q^* \\ \min\{\tilde{x}_{13}(t) + \tilde{x}_{23}(t) - \mu_3, 0\} & \text{if } q_3(t) = q^* \end{cases} \quad (5)$$

$$\dot{q}_5(t) = \begin{cases} \max\{\tilde{x}_{35}(t) - \mu_5, 0\} & \text{if } q_5(t) = -q^* \\ \tilde{x}_{35}(t) - \mu_5 & \text{if } |q_5(t)| < q^* \\ \min\{\tilde{x}_{35}(t) - \mu_5, 0\} & \text{if } q_5(t) = q^* \end{cases} \quad (6)$$

where μ_3, μ_5 are the respective service capacities at n_3 and n_5 for the deviations, dominated mostly by the outgoing link capacity if first-in-first-out queuing policy is used.

Our goal here is to examine stability of the network. Several related practical questions can now be posed. For example, having decided on uncertainty bounds, it would be of interest to find the minimum destabilizing delay τ^* associated with them. Alternatively, having fixed the nominal delay τ^* associated with feedback information, the associated minimum destabilizing uncertainty bound could be found out. The problem to be solved depends on the degree of confidence placed in the these variables. In order to pose the problem formally, we present the necessary background material in the next subsection.

3.2 Mathematical Preliminaries

Let $L_2^n[0,\infty)$ denote the space of signals $x(\cdot) \in R^n$ square integrable on $[0,\infty)$ with inner product defined as $< x, y >_{L_2^n} \doteq \int_{-\infty}^{\infty} y^*(t)x(t)dt$ and induced norm given by $||x||_{L_2^n} \doteq \sqrt{< x, x >_{L_2^n}}$. Unless otherwise specified, L_2^n refers to $L_2^n[0,\infty)$ and $||x||$ denotes $||x||_{L_2^n}$. Given a signal $x(\cdot) \in R^n$, $\hat{x}(j\omega)$ denotes its Fourier transform. An *operator* denotes a function $F : L_2^n \longrightarrow L_2^m$ and its *gain* is given by

$$||F|| \doteq \sup_{f \in L_2^n, f \neq 0} \{||F(f)||/||f||\}. \quad (7)$$

If $||F||$ is finite, F is said to be *bounded* . F is said to be stable if there exist $\gamma \geq 0$ and β such that

$$||Fu||_2 \leq \gamma ||u||_2 + \beta \qquad \forall u \in L_2. \quad (8)$$

Definition 1 Let $\phi(\cdot, t) : R \longrightarrow R$ denote a function such that

$$\alpha u^2 \leq \phi(u,t) \leq \beta u^2 \qquad , \forall u \in R, \forall t \geq 0. \quad (9)$$

Then, ϕ is said to belong to sector $\{\alpha, \beta\}$.

Definition 2 Two signals $w \in L_2^m[0,\infty)$ and $v \in L_2^n[0,\infty)$ are said to satisfy the IQC defined by $\Pi(j\omega)$ if

$$\int_{-\infty}^{\infty} \begin{bmatrix} \hat{v}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \hat{v}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix} d\omega \geq 0. \quad (10)$$

Definition 3 A bounded operator $\Delta : R^n \longrightarrow R^m$ is said to satisfy IQC defined by $\Pi(j\omega)$ if (10) holds for all $w = \Delta v$ where $v \in L_2^n$.

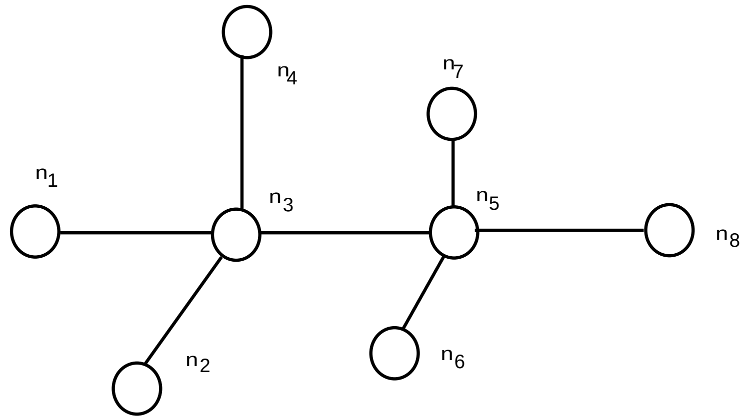

Figure 1: The figure shows network topology under examination. Nodes n_1, n_2, n_6 are source nodes with n_8, n_4, n_7 as the corresponding destinations. Nodes n_3, n_5 are routers.

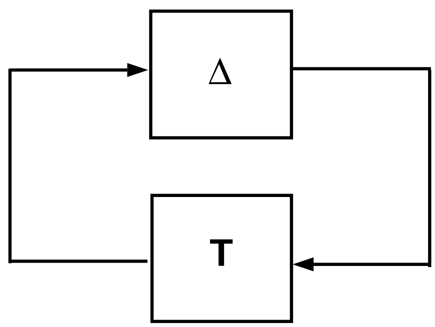

Figure 2: The interconnection $\mathcal{I}_1$. T is the linear time-invariant plant *seen* by the uncertainty $\mathbf{\Delta}$.

Modulo change of variables, most robust control analysis problems can be studied as an instance of the canonical uncertain system $\mathcal{I}_1$, shown in Fig 2 where $\Delta \in \mathbf{\Delta}$ (Δ diagonal) represents the uncertainties and T denotes the plant (augmented by controller, if any) seen by the uncertainty [7], [11]. The robustness analysis problem can be interpreted as a 'topological separation' of the graph of $T(j\omega)$ and the inverse graph of $\Delta(j\omega)$ as follows (rephrased from [9]).

Theorem 1 Suppose T contains all its poles in the open left half s-plane. Let Δ be a bounded causal operator and assume that $\forall \rho \in [0,1]$, $\mathcal{I}_1$ is well-posed (in the sense of [9]). Consider a measurable Hermitian function $\Pi : jR \longrightarrow C^{(l+m)\times(l+m)}$ where l is the size of T and m is the size of Δ. Let IQC defined by Π be satisfied by $\tau\Delta$ $\forall \tau \in [0,1]$. Then, $\mathcal{I}_1$ is stable if

$$\begin{bmatrix} T(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} T(j\omega) \\ I \end{bmatrix} < 0 \tag{11}$$

holds $\forall \omega \in R$.

We redraw system represented by (1)-(6) as shown in Fig 3.

$\mathcal{N}_1 \doteq \mathrm{diag}(\mathrm{sat}(\cdot), \mathrm{sat}(\cdot), \mathrm{sat}(\cdot))$ denotes the rate saturation nonlinearity, $\mathcal{N}_2 \doteq \mathrm{diag}(\Delta_1, \Delta_2)$ captures the queue saturation nonlinearity, $\mathcal{N}_3 \doteq \mathrm{diag}(\Delta_3, \Delta_4)$ captures the uncertainty associated with queue information and $\bar{\mathcal{N}}_4 \doteq e^{-\tau s} I$ is the known delay (where I is the identity matrix). Write $\mathcal{N}_4 \doteq (\frac{2-\tau s}{2+\tau s} - e^{-\tau s})I$ The system in Fig 3 can be seen to be an instance of $\mathcal{I}_1$ with

$$\Delta \doteq \mathrm{diag}(\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \mathcal{N}_4) \tag{12}$$

$$T \doteq \begin{bmatrix} 0 & 0 & q(s) & P_1(s) \\ I & 0 & 0 & 0 \\ 0 & P_2(s) & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix} \tag{13}$$

where $q(s) = \frac{2-\tau s}{2+\tau s}$, $P_1(s)$ is a 3×3 matrix of controller dynamics and $P_2(s) = \mathrm{diag}(1/s, 1/s)$. Then, the problem of determining stability of the system can be stated as that of determining an IQC $\Pi(j\omega)$ which gives the desired topological separation.

4 Main Result

First, IQCs for component nonlinearities are stated.

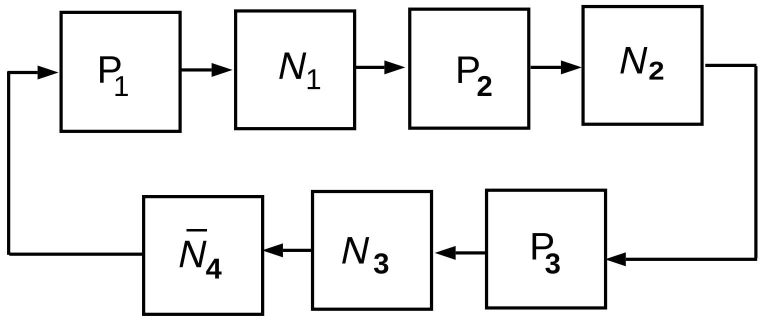

Figure 3: A block digram decomposition of the investigated network topology.

Lemma 1 Saturation nonlinearity in $\mathcal{N}_1$ satisfies IQC given by

$$\Pi_1(j\omega) \doteq \begin{bmatrix} 0 & 1+H(j\omega) \\ 1+H(-j\omega) & -2(1+\mathrm{Re}(H(j\omega)) \end{bmatrix} \quad (14)$$

where $H \in RL_\infty$ such that l_1 norm of H is less than unity.

Proof. $\mathcal{N}_1$ is a slope bounded monotonic odd nonlinearity with slope equal to 1. The result follows from [10] (see also [9]).

Lemma 2 Nonlinearity in $\mathcal{N}_2$ satisfies IQC given by

$$\Pi_2(j\omega) \doteq \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}. \quad (15)$$

Proof. $\mathcal{N}_2$ is a sector bounded nonlinearity with $\alpha = 0, \beta = 1$. The IQC follows from [10], [9].

Lemma 3 Time varying uncertainty in $\mathcal{N}_3$ satisfies IQC given by

$$\Pi_3(j\omega) \doteq \begin{bmatrix} (1+\rho)(H^*H + \frac{\psi(H,d)^2}{\rho} I) & 0 \\ 0 & -H^*H \end{bmatrix} \quad (16)$$

where $\psi(H,d) \doteq \int_{-\infty}^{\infty} ||h(t)|| \min\{2, D|t|\} dt$ and D is the bound on rate of change of variation and H is a stable, causal transfer function and $\rho > 0$ is the parameter to be adjusted.

Proof. See [9].

Lemma 4 Delay term in $\mathcal{N}_4$ satisfies IQC given by

$$\Pi_4(j\omega) \doteq \begin{bmatrix} p(j\omega) & 0 \\ 0 & 1 \end{bmatrix} \quad (17)$$

where $p(j\omega) \doteq 2(1 - \frac{4-\omega^2\tau^2}{4+\omega^2\tau^2}\cos(\omega\tau) - \frac{4\omega\tau}{4+\omega^2\tau^2}\sin(\omega\tau))$.

Proof. Follows from algebraic manipulations and is omitted.

Combining the above IQCs, sufficiency conditions for the stability of this system can be checked by using different IQC parameters. Use of linear matrix inequality (LMI) techniques to analyze stability of dynamical systems subject to fixed unknown delays (see [8], [12]) or bounded norm disturbance ([8]) has been vigorously pursued over the last few years. If the system defined by (1)-(6) is put in state-space form, it is possible to use standard LMIs for its stability analysis. Let $x(t) \doteq [x_{13}(t)\ x_{23}(t)\ x_{65}(t)\ q_3(t)\ q_6(t)], w(t) \doteq [\tilde{x}_{13}(t)\ \tilde{x}_{23}(t)\ \tilde{x}_{65}(t)\ \tilde{q}_3(t)\ \tilde{q}_6(t)]$. Let $\delta \doteq \tau_i(t) - \tau$. For small δ, Taylor series expansion of $q(t-\tau+\delta)$ gives

$$q(t-\tau+\delta) = q(t-\tau) + \delta\dot{q}(t-\tau) \quad (18)$$

as the first order approximation so that, for example,

$$\dot{x}_{23}(t) = \lambda x_{23}(t) + Kq_3(t - \tau_2(t))$$

can be written as

$$\dot{x}_{23}(t) = \lambda x_{23}(t) + Kq_3(t-\tau_2) + \Phi_2(t)(x_{13}(t-\tau_2) + x_{23}(t-\tau_2))$$

which is of the form

$$\dot{x}_{23}(t) = \lambda x_2 3(t) + Kq_3(t-\tau_2) + w_2(t-\tau_2)$$

where $w_2(t) \doteq \Phi_2(t)C_2x(t)$ and $C_2 \doteq [W_2\ W_2\ 0\ 0\ 0]$. Likewise, $w_i(t) \doteq \Phi_i(t)C_ix(t)$ for appropriate C_i ($i = 1,3$). Note that $\frac{||w_i||}{||x||} \le 1$ ($i = 1,2,3$). With this substitution, (1)-(6) can be rewritten as

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{3} A_i x(t-\tau_i) + \sum_{i=1}^{3} B_i w_i(t-\tau_i) \quad (19)$$

where matrices A, A_i and B_i are defined appropriately from (1)-(6), Then, stability can be examined by combining LMIs for delays with LMIs for norm bounded disturbances (see [8]) . A sufficiency condition for stability is

$$\begin{bmatrix} X & Y \\ Y' & Z \end{bmatrix} < 0$$
$$\lambda \ge 0 \text{ and } P, P_i > 0 \qquad i = 1,2,3 \quad (20)$$

where $X \doteq A'P + PA + \sum_{i=1}^{3} P_i$, $Y \doteq [PA_1 \ \ldots \ PA_3 \ PB_1 \ldots PB_3]$ and

$$Z \doteq \begin{bmatrix} -P_1 + \lambda C_1' C_1 & 0 & 0 & 0 \\ 0 & -P_2 + \lambda C_2' C_2 & 0 & 0 \\ 0 & 0 & -P_3 + \lambda C_3' C_3 & 0 \\ 0 & 0 & 0 & -I \end{bmatrix}$$

5 Discussion and Conclusion

Stability analysis of the state-of-the-art data networks is a formidable problem because they exhibit a high degree of interaction and have nonlinear uncertain dynamics. In the present note, we have formulated the problem such that realistic constraints are accounted for. For example, it is assumed that a source exerts flow control law based on the knowledge of the queue sizes at the routers on its own path to destination only. The case of rate updates based on a function of queue sizes (instantaneous or average) can easily be handled. Assuming next generation routers can handle this functionality, tolerance of a connection to (un-modeled) interfering traffic can be characterized using appropriate IQCs. This assumption is not unrealistic since currently, it is possible to set a bit in a packet header to 0 if the queue size is below a certain threshold or to 1 if it is above a certain threshold (see DEC bit scheme [13]). The adherence to ordinary differential equations is not too restrictive since data transmission is observed to be a reasonably continuous process, on a coarse time scale, for bulk data transfer connections, even for acknowledgment clocked transport protocols such as TCP/IP suites though it is unsuitable to describe dynamics of short lived acknowledgment clocked connections. In the prevalent literature, it is customary to analyze data networks by considering the case of a single source and its bottleneck router, i.e. the most congested router on the feedforward path to destination, subject to random, usually i.i.d. packet losses ([14]). The present work explicitly takes into account the effect of interfering traffic and of other routers on the feedforward path to destination in addition to bottleneck router.

6 Acknowledgment

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