Linear Dynamically Varying Linear Quadratic Control of Systems with Complicated Dynamics

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February 26, 2001

Abstract. Linear quadratic controllers for dynamical systems with complicated dynamics are developed. Systems with complicated dynamics are those that run over compact sets and have such features as nontrivial recurrence, periodic and aperiodic orbits. The controllers are based on modeling the nonlinear dynamical systems as linear dynamically varying systems. Necessary and sufficient conditions for the existence of such controllers are linked to the existence of a bounded solution to a functional algebraic Riccati equation. Several methods to solve the functional Riccati equation are presented. Finally, an example of controlling the Hénon map to avoid certain regions is presented.

1. Introduction

This paper develops a control strategy for systems with complicated dynamics. By *complicated dynamics*, we mean a discrete-time, nonlinear dynamical system running over a compact set Θ. By *control*, we mean tracking one of the natural periodic or aperiodic orbits of the dynamics .

A *specific* feature of tracking a natural periodic or aperiodic motion is that the objective can be achieved with low cost control. Therefore, the control is taken to be a very small perturbation of the parameters of the nominal dynamics. More specifically, the nominal and perturbed dynamics are, respectively,

$$\begin{aligned} \theta(k+1) &= f(\theta(k)), & \theta(0) = \theta_0 \in \Theta \\ \varphi(k+1) &= f(\varphi(k), u(k)), & \varphi(0) = \varphi_0 \in \Theta \end{aligned}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is differentiable, $\Theta \subset \mathbb{R}^n$ is compact, and $f(\Theta, 0) \subseteq \Theta$. Thus $\{\theta(k) : k \geq 0\}$ is the desired trajectory and $\varphi(k)$ is the state of the system under control. The objective is to find a control u such that $\lim_{k\to\infty} \|\varphi(k) - \theta(k)\| = 0$.

This paper develops a controller of the form $u(k) = F_{\theta(k)}(\varphi(k) - \theta(k))$, where the feedback F_θ is designed from a linearized approximation of $f(\varphi, u)$ around $(\theta, 0)$, and the gain F_θ is "scheduled" so as to follow the desired trajectory. The controller F_θ exists if a solution to a *functional* algebraic Riccati equation (FARE) exists. The mathematical difficulty with this functional equation is to prove that the relevant solution is continuous, in which case F_θ is continuous.

Typically there is no closed form solution to the FARE. However, dynamical systems on compact sets, subject to some mild additional conditions, are known to have such ergodic properties as recurrence, which can be put into use to construct an approximate solution of arbitrary accuracy.

The proofs of the results in this paper are provided in [1], [3] and [4].

2. Controlling Dynamical Systems with Linear Dynamically Varying Control

Consider a slightly generalized version of the problem posed in Section 1:

$$\theta(k+1) = f(\theta(k), 0) + v(k), \text{ with } \theta(0) = \theta_o \tag{1}$$

$$\varphi(k+1) = f(\varphi(k), u(k)) + w(k), \text{ with } \varphi(0) = \varphi_o. \tag{2}$$

Here we required that $v(k)$ and $w(k)$ are either zero or are small and infrequent enough with $\theta(k+1),\ \varphi(k+1) \in \Theta$. The purpose of v is to allow the desired trajectory to occasionally jump from a point on one orbit to a nearby point on another orbit. The purpose of w on the other hand is to allow for some modeling inaccuracies.

The popular problem of getting φ to follow a periodic orbit fits into this framework by setting $v \equiv w \equiv 0$ and $\theta(0) \in P(f)$, where $P(f)$ is the set of periodic points. However, if the objective is for φ to follow an aperiodic orbit, then $v \equiv w \equiv 0$ and $\theta(0) \in NW(f) \setminus P(f)$, where $NW(f)$ is the set of nonwandering points.

The targeting problem also fits within this framework [12], [13]. Initially, the state is forced to follow a targeting trajectory that originates near the current state, and ends near a point of the desired orbit. If the system is transitive, such a targeting trajectory exists. Once the state has reached the end of the targeting trajectory, a second control is applied to force the system to follow the desired orbit. This switching of trajectories is accomplished by allowing for occasional small jumps in $\theta(k)$. The exogenous input v accounts for such jumps.

Now we define the tracking error

$$x(k) = \varphi(k) - \theta(k).$$

The linear approximation of $f(\varphi(k), u)$ around $\varphi(k) = \theta(k)$ and $u(k) = 0$ yields

$$x(k+1) \;=\; A_{\theta(k)}x(k) + B_{\theta(k)}u(k) \tag{3}$$

$$+\eta\left(x(k),u(k),\theta(k)\right)-v(k)+w\left(k\right)$$

where

$$A_\theta=\frac{\partial f}{\partial x}\left(\theta,0\right),\quad B_\theta=\frac{\partial f}{\partial u}\left(\theta,0\right)$$

and

$$\begin{aligned}\eta(x(k),u(k),\theta(k)) &= \eta_x\left(x(k),u(k),(\theta)\right)x(k)\\ &\quad +\eta_u\left(x(k),u(k),\theta\left(k\right)\right)u(k)\end{aligned}\tag{4}$$

Furthermore, $\|\eta_x\left(x,u,\theta\right)\|$ and $\|\eta_u\left(x,u,\theta\right)\|$ can be made as small as necessary by limiting the size of u and x.

If u and x are small and $v=w=0$, then we can approximate the error dynamics as

$$\begin{aligned}x(k+1) &= A_{\theta(k)}x(k)+B_{\theta(k)}u(k)\\ \theta\left(k+1\right) &= f\left(\theta\left(k\right)\right)\end{aligned}\tag{5}$$

This is a linear system with coefficient matrices A and B that vary as $\theta(k)$ varies. Since $\theta(k)$ varies according to equation (1), system (5) is a *Linear Dynamically Varying (LDV)* system.

Our objective is to not only force the tracking error to go to zero asymptotically, but go to zero uniformly exponentially fast. We distinguish the different forms of stability as follows.

Definition 1. *The linear dynamically varying system (5) is* uniformly exponentially stable *if for $u(k)=0$, there exist an $0<\alpha<1$ and a $\beta<\infty$ such that for all $\theta(0)\in\Theta$*

$$\|x(k)\|\leq\beta\alpha^k\|x(0)\|.$$

System (5) is exponentially stable, *if for $u(k)=0$ and for each $\theta(0)\in\Theta$, there exists an $0<\alpha(\theta(0))<1$ and a $\beta(\theta(0))<\infty$ such that for all $x(j)$ and $j\leq k$*

$$\|x(k)\|\leq\beta(\theta(0))\alpha(\theta(0))^{k-j}\|x(j)\|.$$

In the case of continuous LDV systems, exponential and uniform exponential stability are equivalent:

Proposition 2. *Assume the function $A:\Theta\to\mathbb{R}^{n\times n}$ is continuous and Θ is compact. Then exponential and uniformly exponential stability are equivalent.*

Since uniformly exponentially stable systems are inherently more robust than exponentially stable systems, it is preferable to remain within the confines of continuous LDV systems. Thus when synthesizing a feedback for controlling a continuous LDV system, it is important to ensure that the feedback is not only stabilizing, but also continuous. However, to maintain generality, an LDV system is considered stabilizable if there exists an exponentially stabilizing feedback, that is:

Definition 3. *System (5) is stabilizable if there exist a, not necessarily continuous, function* $F : \mathbb{N} \times \Theta \to \mathbb{R}^{m \times n}$ *such that for all* $\theta(0) \in \Theta$ *and for all* k, $\|F(k, \theta(0))\| \leq \overline{F}(\theta(0)) < \infty$, *and*

$$\begin{aligned} x(k+1) &= \left(A_{\theta(k)} + B_{\theta(k)} F(k, \theta(0))\right) x(k) \\ \theta(k) &= f^k(\theta(0)) \end{aligned}$$

is exponentially stable.

Our main result can now be formulated:

Theorem 4. *The that system (5) is stabilizable. Then there exists a unique, bounded solution* $X : \Theta \to \mathbb{R}^{n \times n}$ *such that* $X_\theta = X'_\theta \geq 0$,

$$\begin{aligned} &X_\theta = A'_\theta X_{f(\theta)} A_\theta + I \\ &-A'_\theta X_{f(\theta)} B_\theta \left(I + B'_\theta X_{f(\theta)} B_\theta\right)^{-1} B'_\theta X_{f(\theta)} A_\theta + I \end{aligned} \tag{6}$$

and the feedback

$$\begin{aligned} u(k) &= -\left(I + B'_{\theta(k)} X_{f(\theta(k))} B_{\theta(k)}\right)^{-1} \\ &\quad \times B'_{\theta(k)} X_{f(\theta(k))} A_{\theta(k)} x(k) \end{aligned} \tag{7}$$

uniformly exponentially stabilizes system (5). For $\|x(0)\| < \infty$, *this feedback minimizes* $\sum_{k=0}^{\infty} \|z(k)\|^2$ *and* X *is a uniformly continuous function.*

Conversely, if there exists a bounded solution, X, *to equation (6) such that* $X_\theta = X'_\theta \geq 0$, *then system (5) is stabilizable and* X *is continuous. In this case, if* $X(k, N, \theta)$ *solves the finite horizon Riccati equation, i.e.*

$$\begin{aligned} &X(k, N, \theta) = A'_{f^k(\theta)} X(k+1, N, \theta) A_{f^k(\theta)} \\ &+I - A'_{f^k(\theta)} X(k+1, N, \theta) B_{f^k(\theta)} \\ &\times \left(I + B'_{f^k(\theta)} X(k+1, N, \theta) B_{f^k(\theta)}\right)^{-1} \\ &\times B'_{f^k(\theta)} X(k+1, N, \theta) A_{f^k(\theta)} \end{aligned} \tag{8}$$

with

$$X(N, N, \theta) = I,$$

then $X(0, N, \theta) \to X_\theta$ *uniformly in* θ.

Remark 1. *Stabilizability is a rather weak assumption. Indeed, stabilizability merely assumes that every trajectory is stabilizable. Of course, one could not hope to stabilize every trajectory if some trajectories are not stabilizable. Given this obviously necessary condition, it is interesting that continuity and compactness are all that is needed to prove the existence of a continuous and uniformly stabilizing controller.*

Remark 2. *The continuity of the cost* X *is counter-intuitive in the case where* f *is sensitively dependent on initial conditions. Intuitively, this continuity means that, although small changes in the initial conditions may lead to drastic changes in the trajectories, the cost to stabilize these trajectories does not change very much.*

Remark 3. *Since the LDV controller yields a uniformly exponentially stable LDV system, the LDV controller will locally stabilize the original nonlinear system (3). Hence, if* $\|\varphi(0) - \theta(0)\|$ *is small enough and the control given by (7) is applied, then* $\varphi(k) \to \theta(k)$ *as* $k \to \infty$ *where* $\varphi(k)$ *and* $\theta(k)$ *is given by (1) and (2) respectively. Furthermore, if the perturbations* w *and* v *are small enough and/or infrequent enough, then the system remains stable. See [1] for details.*

3. Exploiting Ergodicity to Solve the Functional Algebraic Riccati Equation (FARE)

Solving any function equation is a difficult task. Here we take advantage of different properties of complicated dynamics to solve the FARE.

3.1. Solving the FARE on a Transitive Orbit. A rather brute force method is based on the finite horizon approximation of the infinite horizon linear quadratic control problem. To this end, one simply defines $X(0, N, \theta_o)$ as the solution to the finite horizon Riccati equation as in (8). It can be shown that $X(0, N, \theta_o) \to X_{\theta_o}$ as $N \to \infty$, uniformly in θ_o. Hence, for N large enough, $X(0, N, \theta_o)$ is a good approximation of X_{θ_o}. Furthermore, if $X(0, N, \theta_o)$ is known, then $X(0, N-1, f(\theta_o))$ is also known, and if $N-1$ is large enough, then $X(0, N-1, f(\theta_o)) \approx X_{f(\theta_o)}$. Similarly, we compute $X\left(0, N-k, f^k(\theta)\right) \approx X_{f^k(\theta_o)}$ for $k \leq K$ where $N-K$ is large. Now, if θ_o is a transitive point, then $E = \left\{f^k(\theta_o) : 0 \leq k \leq K\right\}$ is a ε-net, and solving X_θ on E provides a good approximation of X everywhere. It is possible to show that this approach will only fail if the LDV system is unstabilizable [1].

Remark 4. *This method is appealing because all that is needed is a transitive trajectory and* A_θ *and* B_θ. *The function* f *does not need to be known.*

3.2. Solving the FARE on Periodic Orbits. It is a generic property of diffeomorphisms on a compact set to have a dense set of periodic points [10]. In this case, since the solution to the FARE is continuous, if we solve the FARE on the set of periodic points, then we know the solution everywhere. Similarly, if we solve the FARE on the set of periodic points with period less than N, where N is large enough, then we approximately know the solution everywhere. It is possible to solve the FARE on periodic points very efficiently.

Let θ be a fixed point, i.e., $f(\theta) = \theta$. Then, by equation (6), we have

$$X_\theta = A'_\theta X_\theta A_\theta - A'_\theta X_\theta B_\theta \left(I + B'_\theta X_\theta B_\theta\right)^{-1} B'_\theta X_\theta A_\theta + I$$

This can be written more succinctly as $X_\theta = \rho_\theta(X_\theta)$, where $\rho_\theta : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ and depends on θ.

Similarly, if θ is a periodic point with period N, we can write

$$\rho_\theta \circ \rho_{f^1(\theta)} \circ \cdots \rho_{f^{N-1}(\theta)}(X_\theta) = X_\theta.$$

Hence, X_θ is a fixed point of some map. It turns out that solving the above equation is equivalent to solving the following invariant subspace problem [6],

$$\left(\prod_{k=0}^{N-1} \begin{bmatrix} \alpha_{f^k(\theta)} & \beta_{f^k(\theta)} \\ \gamma_{f^k(\theta)} & \delta_{f^k(\theta)} \end{bmatrix}\right) \begin{bmatrix} I \\ X_\theta \end{bmatrix} \subset \left\{ \begin{bmatrix} I \\ X_\theta \end{bmatrix} \right\}$$

where $\left\{ \begin{bmatrix} I \\ X_\theta \end{bmatrix} \right\}$ is the span of $\begin{bmatrix} I \\ X_\theta \end{bmatrix}$ and where

$$\begin{aligned} \alpha_\theta &= A_\theta^{-1},\ \beta_\theta = A_\theta^{-1} B_\theta B'_\theta, \\ \gamma_\theta &= A_\theta^{-1},\ \delta_\theta = A'_\theta + A_\theta^{-1} B_\theta B'_\theta, \end{aligned}$$

This invariant subspace problem can be solved very efficiently [14].

As in the case of solving the FARE along a transitive orbit, one can solve the FARE along every periodic orbit if and only if the LDV system is stabilizable [1].

3.3. Solving the FARE on Recurrent Set. The previous two methods of solving the FARE provide some insight into the stabilizability and the relationship between the stabilizability of different orbits. However, these methods do not necessarily lead to efficient computationally methods to solve the FARE. The method based on transitive points has the difficulty that the convergence can be very slow. The method based on periodic points has the drawback that it is not always possible to exactly determine the periodic points. Now a method is presented that solves the FARE

along a segment of a recurrent orbit and does not suffer from these drawbacks. It should be noted that in many cases (e.g. hyperbolic systems [10]) almost all orbits are recurrent. Hence, there is no difficulty in determining recurrent orbits.

The approach here is to approximate a segment of a recurrent orbit as a periodic orbit. Once this approximately periodic orbit in hand, one can solve the FARE along this orbit using the efficient methods based on periodic orbits discussed above. To be more specific, define $\rho_\theta : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ as above and let $\theta \in R(f)$ where $R(f)$ is the set of recurrent points. Since θ is recurrent, for $\delta > 0$ there exists an $N < \infty$ such that $\left\|\theta - f^N(\theta)\right\| < \delta$. Since X is continuous, for δ small enough, $\left\|X_\theta - X_{f^N(\theta)}\right\| < \varepsilon$ and

$$\left\|\rho_\theta \circ \rho_{f(\theta)} \circ \cdots \circ \rho_{f^{N-1}(\theta)}\left(X_{f^N(\theta)}\right) - X_{f^N(\theta)}\right\| < \varepsilon.$$

Therefore, $X_{f^N(\theta)}$ is "nearly" the fixed point of

$$\rho_\theta \circ \rho_{f^1(\theta)} \circ \cdots \rho_{f^{N-1}(\theta)}(Y) = Y. \tag{9}$$

In fact,

Theorem 5. *Let θ be a recurrent point and let $\varepsilon > 0$. Then there exists a $\delta > 0$ and $N < \infty$ such that $\|f^N(\theta) - \theta\| < \delta$ and $\|X_\theta - Y\| < \varepsilon$ where X solves the FARE and Y solves (9).*

Remark 5. *The proof relies on the closing lemma and on the structural stability of the FARE [3].*

Remark 6. *This method is appealing because all that is needed is a recurrent trajectory and A_θ and B_θ. The function f does not need to be known.*

3.4. Jump Linear Approximation. There is a strong connection between LDV systems and *jump linear systems* [7]. Jump linear systems are described by

$$x^{JL}(k+1) = A^{JL}_{\sigma(k)} x(k) + B^{JL}_{\sigma(k)} u(k)$$

where $\sigma(k)$ is a Markov chain. It is possible to approximate the LDV system as a jump linear system as follows: Partition Θ into a finite set of cells R_i, i.e. $\Theta = \bigcup_{i=1}^N R_i$. For each cell R_i choose a point $r_i \in R_i$. Define $A^{JL}_i = A_{r_i}$ and $B^{JL}_i = B_{r_i}$. Define the transition probabilities $p_{i,j} = \frac{\lambda(f(R_i)\cap R_j)}{\lambda(R_i)}$ where λ is Lebesgue measure.

The optimal jump linear controller for a jump linear system is given by $u(k) = F_{\sigma(k)} x(k)$ where

$$F_\sigma = -\left(I + B'_\sigma \hat{Y}_\sigma B_\sigma\right)^{-1} B'_\sigma \hat{Y}_\sigma A_\sigma$$

and

$$\begin{aligned} Y_\sigma &= A'_\sigma \hat{Y}_\sigma A_\sigma + I \\ &\quad - A'_\sigma \hat{Y}_\sigma B_\sigma \left(I + B'_\sigma \hat{Y}_\sigma B_\sigma \right)^{-1} B'_\sigma \hat{Y}_\sigma A_\sigma \end{aligned} \tag{10}$$

for all $\sigma = 1, \cdots, N$ and $\hat{Y}_\sigma = \sum_{j=1}^{N} p_{\sigma,j} Y_j$. The equations given by (10) form a set of coupled Riccati equations whose solution has been extensively investigated in [5]. In [4] it was shown that as $\max_i diam\,(R_i) \to 0$, the solution to the coupled Riccati equations (10) approaches the solution to FARE.

4. Example

In the following an LDV controller is devised for the Hénon system. The Hénon system is defined as

$$\begin{aligned} \begin{bmatrix} \theta_1(k+1) \\ \theta_2(k+1) \end{bmatrix} &= \begin{bmatrix} f_1(\theta(k), u(k)) \\ f_2(\theta(k), u(k)) \end{bmatrix} \\ = \begin{bmatrix} 1 - (a + u(k))\,\theta_1(k)^2 + \theta_2(k) \\ b\theta_1(k) \end{bmatrix} \end{aligned}$$

with control input u. In this example $a = 1.4$ and $b = 0.3$. For these parameters values, it is known that the Hénon function has an attractor [8]. It is not known whether the Hénon function is chaotic, but numerical iteration indicates that the attractor is transitive [11] and hence nonwandering. That is, it has been tested that for many initial conditions the trajectory enters every ε-neighborhood of every point in the attractor for very small ε. Thus we define Θ to be the attractor.

Define

$$\begin{aligned} A_{\theta(k)} &= \left.\frac{df}{d\theta}\right|_{\theta(k),0} = \begin{bmatrix} -2\,a\,\theta_1(k) & 1 \\ b & 0 \end{bmatrix} \\ \text{and } B_{\theta(k)} &= \left.\frac{df}{du}\right|_{\theta(k),0} = \begin{bmatrix} -\theta_1^2(k) \\ 0 \end{bmatrix}. \end{aligned} \tag{11}$$

Since a trajectory that enters every ε-neighborhood for small ε is easily found, the FARE may be solved using the methods from Section 3.1 or 3.3. In this way the function $F : \Theta \to \mathbb{R}^{1\times 2}$ is obtained and the closed-loop tracking error dynamics (as discussed in Section 2) is

$$\begin{aligned} &\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \\ &A_{\theta(k)} + B_{\theta(k)} F_{\theta(k)} \begin{bmatrix} x_1(k) \\ x_2(k)) \end{bmatrix} + \begin{bmatrix} \eta_1(k) \\ \eta_2(k) \end{bmatrix} \end{aligned} \tag{12}$$

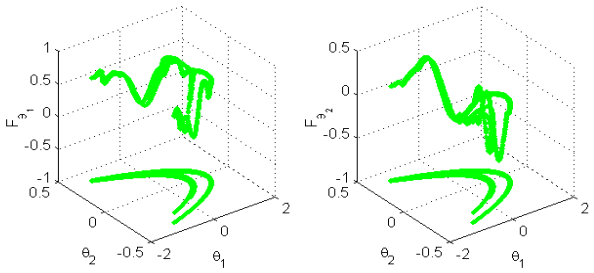

Figure 1: Feedback $F_\theta = \begin{bmatrix} F_{\theta_1} & F_{\theta_2} \end{bmatrix}$ for the Hénon map. This shows that the feedback is continuous in θ. A plot of the attractor of the Hénon map in the $(\theta_1, \theta_2, -6)$ plane is included for reference.

where η account for the error in linearization.

Figure 1 shows the feedback gain, F, for the LDV system (12). Note that the feedback is continuous on Θ, the attractor of the Hénon map.

The objective in this example is to avoid a region around a fixed point of the Hénon map. The fixed point has an unstable and a stable eigenvalue. Furthermore, computer simulations show that any open set containing the fixed point is visited infinitely often by a generic orbit [11], hence $\inf_k \left\| f^k\left(\varphi\left(0\right)\right) - \theta_{fixed} \right\| = 0$ for most $\varphi\left(0\right)$.

Since the Hénon map is ergodic and has sensitive dependence on initial conditions, a small control force will have a noticeable effect on the state only after a number of iterations. Therefore, to avoid the fixed point, the control must be applied well in advance to the state entering the forbidden region around the fixed point. However, accurately predicting the future state of a system with complicated dynamics is difficult. Thus, a compromise is made, and control is applied only if the state is predicted to enter the forbidden region around the fixed point in L iterations or less.

To be more specific, the objective is

$$\varphi\left(k\right) \notin FB := B\left(\theta_{fixed}, R_{Too\_Close}\right) \text{ for all } k,$$

where $B\left(\theta_{fixed}, R_{Too\_Close}\right)$ is the open ball around θ_{fixed}. Define $l^* = \min\left\{l : \varphi\left(k_o\right) \in f^{-l}\left(FB\right)\right\}$. If $l^* \leq L$, then the control is initiated. The control is determined by finding a $\theta_o \in \Theta$ such that $\left\|\theta_o - \varphi\left(k_o\right)\right\| < R_{Capture}$ and when $\varphi\left(k_o + l\right)$ is controlled to track $f^l\left(\theta_o\right)$, $\left\|\varphi\left(k_o + l\right) - \theta_{fixed}\right\| > R_{TooClose}$ for $l \leq L$. By careful choice of $R_{TooClose}$ and L, the existence of such a θ_o can be guaranteed.

In this example $R_{Too\_Close} = 0.01$, $R_{Capture} = 0.0005$ and $L = 20$. Note that

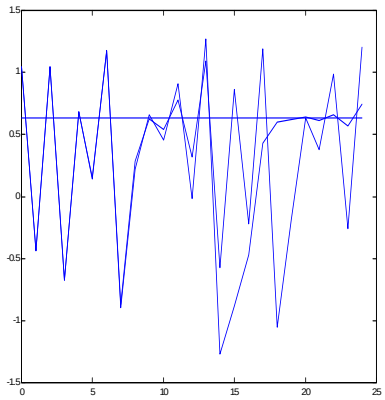

Figure 2: Avoidance: The first element of the state, φ_1, of the controlled (solid line), the uncontrolled (dotted line) and fixed point (dashed line) of the Henon map is shown. At time $k = 0$, it is determined that unless control is initiated, $\varphi(20)$ will enter the forbidden region around the fixed point. Therefore, at $k = 0$ control is applied to keep φ from entering the forbidden region. At $k = 20$, φ is far from the fixed point and control is terminated. Note that at $k = 20$ the uncontrolled trajectory is very close to the fixed point.

$R_{Capture}$ is very small; this leads to a very small control force. The ability to control with very small control force is specific of control of systems with sensitive dependence on initial conditions. Figure 2 shows the trajectory of φ_1 when control is applied, when control is not applied, and the fixed point.

5. Conclusion

Linear Dynamically Varying controllers for nonlinear dynamical systems running over compact sets have been developed. These controllers are useful for tracking periodic and aperiodic orbits. The necessary and sufficient conditions for the existence of such controllers are rather weak and are equivalent to the existence of a positive semi-definite solution to a functional algebraic Riccati equation (FARE). If the dynamical system has adequate ergodic properties, there are many techniques for computing the solution to the FARE.

The continuous-time counterpart of the present approach, with special emphasis on the differential-geometric features inherent to a dynamical system running over a manifold, is available in [9]. The H^∞ counterpart is available in [2].

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