LINEAR DYNAMICALLY VARYING H_∞ CONTROL OF CHAOS

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Abstract:
Linear dynamically varying (LDV) systems are introduced as a way to approximate nonlinear dynamical systems running over a compact set. Sub-optimal linear state dependent H_∞ controllers are introduced and shown to stabilize both LDV systems and chaotic systems about any trajectory on the whole attractor. However, since the controllers are based linear approximations, the chaotic system is guaranteed to be stable only if the initial error between the actual and desired orbit is small enough.

Keywords: Chaos, H-infinity Control

1. INTRODUCTION

Recently there has been much research on controlling chaotic systems. The objective of this effort is to take advantage of sensitive dependence on initial conditions and achieve control with very small control force. There has been much success when the desired orbit is a fixed or periodic orbit. In these cases the controller has been designed by applying control techniques of linear time invariant or linear periodically varying systems to the linear approximation of the nonlinear chaotic system. The aim of in this paper is to apply state dependent linear control techniques to the linear approximation of the chaotic system along every orbit. Along a particular orbit, such a linear approximation is a time varying linear system whose parameters vary according to the chaotic system. Such linear systems are called linear dynamically varying (LDV) systems. Much of the control theory for linear time invariant systems can be extended to the LDV case. This results in a time invariant, but spatially varying linear controller which is defined on the whole attractor. This controller is applicable to most chaos control scenarios. For example, controlling to a periodic or fixed point (Hammad *et al.* 1996), targeting (Shinbrot *et al.* 1992), anti-control (Garfinkel *et al.* 1992) are all achievable with a single LDV controller. Synchronization (Pecora and Carroll 1990) can be achieved with an LDV observer. However, since an LDV system is a linear approximation of a nonlinear system, the control system is locally stable in the sense that the initial error between the actual and desired states must be small.

The optimal LDV quadratic controller was developed in (Bohacek and Jonckheere n.d.). The present paper presents the LDV H_∞ controller. The paper proceeds as follows: Section 2 models the control of chaos problem as an LDV control problem. Section 3 formally develops LDV systems. Section 4 presents the LDV H_∞ controller and section 5 briefly describes of an example of 'anti-control.'

2. CONTROLLING DYNAMICAL SYSTEMS WITH LINEAR DYNAMICALLY VARYING CONTROL

Consider the nonlinear system

$$\tilde{x}(k+1) = f(\tilde{x}(k), u(k))$$

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where

$$f(\cdot,0) : S \to S$$

is a dynamical system map with the following properties:

$f(S,0) = S$, i.e. S is f-invariant,
S is a compact subset of $\mathbb{R}^n$,
and $f \in C^1(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$.

The chaotic features of f are not used in this paper, and thus, the results present here apply to any f satisfying the above requirements. However, if f is chaotic, useful techniques to synthesize controllers developed in (Bohacek and Jonckheere n.d.) can be applied. Note that for simplicity $f(x) := f(x,0)$.

The objective is for $\tilde{x}$ to follow some desired trajectory $x_{traject}$ defined by

$$x_{traject}(k+1) = f(x_{traject}(k),0),$$
$$\text{with } x_{traject}(0) = x_{traject_0}.$$

The problem of forcing $\tilde{x}$ to follow a periodic orbit fits into this framework by setting $x_{traject}(0)$ to be a point on the periodic orbit. If the objective is for $\tilde{x}$ to follow an aperiodic orbit, then $x_{traject}(0)$ is set to be a point on this aperiodic orbit. As discussed in (Bohacek and Jonckheere n.d.), with minor modifications, targeting and anti-control also fit into this framework.

The error dynamics can be approximated by a linear dynamically varying system as follows: Define

$$x(k) = \tilde{x}(k) - x_{traject}(k),$$

so that

$$\begin{aligned} & x(k+1) && (1)\\ &= f(\tilde{x}(k),u(k)) - f(x_{traject}(k),0). \end{aligned}$$

The first degree Taylor approximation of $f(\tilde{x}(k),u)$ around $\tilde{x}(k) = x_{traject}(k)$ and $u(k)=0$ yields

$$\begin{aligned} & x(k+1) && (2)\\ &= A_{x_{traject}(k)}x(k) + B_{2_{x_{traject}(k)}}u(k) \\ &\quad + \eta(x(k),u(k),x_{traject}(k)) \end{aligned}$$

where

$$A_{x_{traject}(k)} = \frac{\partial f}{\partial x}(x_{traject},0),$$
$$B_{2_{x_{traject}(k)}} = \frac{\partial f}{\partial u}(x_{traject},0)$$

and η accounts for nonlinear terms. Since $f \in C^1$, the nonlinear term η can be written as a nonlinear gain, i.e.

$$\begin{aligned} & \eta(x,u,x_{traject}) && (3)\\ &= \eta_x(x,u,x_{traject})x + \eta_u(x,u,x_{traject})u \end{aligned}$$

Furthermore, since $f \in C^1$ and S compact, $\|\eta_x\|$ and $\|\eta_u\|$ can be made as small as necessary by limiting the size of u and x. We conclude that if u and x are small, then the error dynamics 1 can be approximated by

$$\begin{aligned} & x(k+1) && (4)\\ &= A_{x_{traject}(k)}x(k) + B_{2_{x_{traject}(k)}}u(k) \end{aligned}$$

System 4 is a linear system with coefficient matrices A and B_2 that vary as $x_{traject}(k)$ varies.

The objective is to design a feedback F such that

$$\begin{aligned} & x(k+1) \\ &= \left(A_{x_{traject}(k)} + B_{2_{x_{traject}(k)}}F_{x_{traject}(k)}\right)x(k) \end{aligned}$$

is uniformly exponentially stable. It is then possible to show that for $\|x(0)\|$ small enough

$$\begin{aligned} & x(k+1) && (5)\\ &= \left(A_{x_{traject}(k)} + B_{2_{x_{traject}(k)}}F_{x_{traject}(k)}\right)x(k) \\ &\quad + \left(\eta_x + \eta_u F_{x_{traject}(k)}\right)x(k) \end{aligned}$$

is a stable system. That is, $x(k) = \tilde{x}(k) - x_{traject}(k) \to 0$. The task of minimizing the effect of the nonlinear term η is well suited for H_∞ controllers, the main topic of this paper.

3. LINEAR DYNAMICALLY VARYING SYSTEMS

Before discussing H_∞ controllers for system 4 it is necessary to formally develop linear dynamically varying systems. A linear dynamically varying (LDV) system is defined as follows:

$$\begin{bmatrix} x(k+1) \\ z(k) \end{bmatrix} = \begin{bmatrix} A_{\theta(k)} & B_{1_{\theta(k)}} & B_{2_{\theta(k)}} \\ C_{\theta(k)} & D_{1_{\theta(k)}} & D_{2_{\theta(k)}} \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \\ u(k) \end{bmatrix} \quad (6)$$
$$\theta(k+1) = f(\theta(k))$$
$$\text{with } \theta(0) = \theta_o \text{ and } x(0) = x_o$$

where

$f : S \to S$, is a continuous map,
$S \subset \mathbb{R}^n$ is compact,
$A : S \to \mathbb{R}^{n\times n}$, $B_1 : S \to \mathbb{R}^{n\times l}$, $B_2 : S \to \mathbb{R}^{n\times m}$, $C : S \to \mathbb{R}^{p\times n}$, $D_1 : S \to \mathbb{R}^{p\times l}$ and $D_2 : S \to \mathbb{R}^{p\times m}$ are maps that need not be continuous,
$\theta \in S$ is the state of the dynamic system, $x(k) \in \mathbb{R}^n$ is the state of the linear system, $u(k) \in \mathbb{R}^m$ is the control input, $w(k) \in \mathbb{R}^l$ is the

disturbance input, and $z(k) \in \mathbb{R}^p$ is the output to be controlled.

Although f need not be chaotic, LDV systems are most relevant when considering chaotic systems and LDV systems are more intuitively motivated when f is chaotic.

Thus a linear dynamically varying system consists of two connected systems. One system is linear with state x. This linear system has parameters that vary according to a second system. The second system is a dynamic system with state θ. It is assumed that both states $x(k)$ and $\theta(k)$ are known at time point k. An LDV system can also be thought of as an uncountable family of time varying linear systems indexed by the initial condition $\theta(0)$.

It is often assumed that the system coefficient matrices A, B_1, B_2, C, D_1 and D_2 are continuous. We will refer to such systems as continuous LDV systems. In section 2 it was assumed that $f \in C^1$ and A and B are derivatives of f. Thus the tracking error system 2 can be approximated by a continuous LDV system. However, if a feedback $F : S \to R^{n\times n}$ is used to stabilize a continuous LDV system, then the resulting closed loop system is a continuous LDV systems if and only if F is continuous. Although this paper will focus on stabilizing continuous LDV systems, the continuity of the feedback must be proven. Therefore the definition of an LDV system must allow for possible discontinuous coefficient matrices.

Continuous linear dynamically varying systems are similar to the more general linear parametrically varying (LPV) systems found in (Becker and Packard 1995). In the case of LPV systems, the future values of the parameters are unknown, but confined to a known bounded set, with possibly some bound on the rate of change (Watanabe *et al.* 1996). The better knowledge of the parameters in the LDV case allows for stronger results than in the LPV case. For example (Becker *et al.* 1993), stability of LPV systems can be guaranteed via a single quadratic Lyapunov parameter $X \in \mathbb{R}^{n\times n}$ that satisfies a Lyapunov inequality. In the LDV situation, f is known and a continuous function $X : S \to \mathbb{R}^{n\times n}$ is found that satisfies a Lyapunov equation.

Since a linear dynamically varying system is an uncountable collection of linear time varying systems, the concept of stability is slightly more complex in the dynamically varying case than it is in the time varying case.

Definition 1. The linear dynamically varying system 6 is *uniformly exponentially stable* if for $u \equiv 0$ and $w \equiv 0$, there exist an $0 < \alpha < 1$ and a $\beta < \infty$ such that for all $\theta(0) \in S$

$$\|x(k)\| \leq \beta\alpha^k \|x(0)\|.$$

System 6 is *exponentially stable*, if for $u \equiv 0$, $w \equiv 0$ and for each $\theta(0) \in S$, there exists an $0 < \alpha(\theta(0)) < 1$ and a $\beta(\theta(0)) < \infty$ such that for all $x(j)$ and $j \leq k$

$$\|x(k)\| \leq \beta(\theta(0))\alpha(\theta(0))^{k-j} \|x(j)\|.$$

System 6 is *asymptotically stable* if for $u \equiv 0$ and $w \equiv 0$, any $\|x(0)\| < \infty$ and any $\theta(0) \in S$

$$\|x(k)\| \to 0.$$

Note that an exponentially stable system is stable uniformly in time k, but not uniformly in the initial condition $\theta(0)$. That is, along any given positive trajectory $\left\{f^k(\theta(0)) : k \geq 0\right\}$ an exponentially stable system is (uniformly in time) exponentially stable. The parameters, $\alpha(\theta)$ and $\beta(\theta)$, may vary discontinuously with each trajectory, but remain constant along a positive trajectory; i.e. $\alpha(f(\theta)) = \alpha(\theta)$. It is possible that $\alpha(\theta_i) \to 1$ while $\alpha(\lim_i \theta_i) < 1$ for some sequence $\{\theta_i \in S : i \geq 0\}$, in which case the system is exponentially stable, but not uniformly exponentially stable.

In the case of continuous LDV systems, asymptotic, exponential and uniform exponential stability are equivalent (Proposition 2 in (Bohacek and Jonckheere n.d.)). Since uniformly exponentially stable systems are inherently more robust than exponentially stable systems, it is preferable to remain within the confines of continuous LDV systems. Thus when synthesizing a feedback controlling a continuous LDV system, it is important to ensure that the feedback is not only asymptotically stabilizing, but also continuous. However, to maintain generality, the stabilizability of an LDV system does not require continuity of the feedback.

Definition 2. System 6 is *stabilizable* if there exists a, not necessarily continuous, map $F : \mathbb{N} \times S \to \mathbb{R}^{m\times n}$ such that for all $\theta(0) \in S$ and for all k, $\|F(k, \theta(0))\| \leq \overline{F}(\theta(0)) < \infty$, and

$$x(k+1) = \left(A_{\theta(k)} + B_{2_{\theta(k)}} F(k, \theta(0))\right) x(k)$$
$$\theta(k) = f^k(\theta(0))$$

is exponentially stable.

Thus a linear dynamically varying system is stabilizable if every linear system in the family of linear systems indexed by $\theta(0)$ is stabilizable. There is no assumption about a global stabilizing feedback F. Thus, the feedback that exists via the definition of stabilizability may not be a bounded or continuous function. However, in the case of continuous LDV systems, it was shown in (Bohacek and Jonckheere n.d.) that a stabilizable system has a well defined, continuous, uniformly exponentially stabilizing feedback $F : S \rightarrow \mathbb{R}^{m\times n}$.

Along with stabilizability, uniform detectability is needed:

Definition 3. System 6 is *uniformly detectable* if there exists a, not necessarily continuous, map $H : S \rightarrow \mathbb{R}^{n\times p}$ such that for $\theta \in S$, $\|H_\theta\| \leq \bar{H} < \infty$ and

$$x(k+1) = \left(A_{\theta(k)} + H_{\theta(k)}C_{\theta(k)}\right)x(k)$$
$$\theta(k) = f^k(\theta(0)),$$

is uniformly exponentially stable. That is, there exists an $\alpha_d < 1$ and a $\beta_d < \infty$ such that for all $\theta(0) \in S$, $\|x(k)\| \leq \beta_d \alpha_d^k \|x(0)\|$.

Definitions 2 and 3 are slightly asymmetric. The definition 3 is a uniform condition, whereas definition 2 is a point wise condition. If f is invertible, then uniform detectability and detectability are equivalent, where detectability is the dual of definition 2. However, to avoid extra assumptions on f, uniform detectability will be assumed.

4. LINEAR DYNAMICALLY VARYING H_∞ CONTROL

In the following, necessary and sufficient conditions for the existence of LDV H_∞ controllers will be presented. The proof rest heavily on the proofs of the linear time invariant case found in (Green and Limebeer 1995), (Stoorvogel 1992) and (Stoorvogel and Weeren 1994). The linear dynamically varying finite horizon case is closely related to the linear time varying case. Proofs of the time varying case can be found in (Halanay and Ionescu 1994) and (Peters and Iglesias 1997).

The objective of the LDV H_∞ problem is to find a uniformly exponentially stabilizing controller F such that if

$$u(k) = F(k, \theta_o)\begin{bmatrix} x(k) \\ w(k) \end{bmatrix},$$

then:

H_∞ **Objective:** For $x(0) = 0$, there exists $\varepsilon > 0$ such that for $w \in l_2$, $\theta_o \in S$,

$$\|z\|^2 - \gamma^2\|w\|^2 \leq -\varepsilon\|w\|^2$$

and if $w = 0$ and $x(0) \neq 0$, then $x(k) \rightarrow 0$.

If this objective is achieved, then $\sup_w \frac{\|z\|}{\|w\|} < \gamma$.

The following assumptions on system 6 are needed:

(1) The triple (A, B_2, f) is stabilizable.
(2) The system parameters A, B_1, B_2, C, D_1 and D_2 are matrix valued continuous functions of θ and S is compact.
(3) $D_{2_\theta}' D_{2_\theta} > 0$ for all $\theta \in S$.
(4) $D_{2_\theta}' \begin{bmatrix} C_\theta & D_{1_\theta} \end{bmatrix} = 0$ and the triple (A, C, f) is uniformly detectable.
(5) $f : S \rightarrow S$ is continuous.

Perhaps these assumptions could be weakened (for example see (Stoorvogel 1992)), but they are common.

Let $\begin{bmatrix} A_{\theta(k)} & \bar{B}_{\theta(k)} \\ \bar{C}_{\theta(k)} & \bar{D}_{\theta(k)} \end{bmatrix} = \begin{bmatrix} A_{\theta(k)} & B_{1_{\theta(k)}} & B_{2_{\theta(k)}} \\ C_{\theta(k)} & D_{1_{\theta(k)}} & D_{2_{\theta(k)}} \\ 0 & I_l & 0 \end{bmatrix}$ and $J = \begin{bmatrix} I_p & 0 \\ 0 & -\gamma^2 I_l \end{bmatrix}$.

Theorem 1. Suppose assumptions 1-5 hold. There exists a exponentially stabilizing controller $u(k) = F_u(k, \theta(0))\begin{bmatrix} x(k) \\ w(k) \end{bmatrix}$ satisfying the H_∞ objective if and only if there exists a uniformly bounded map $X_\infty : S \rightarrow \mathbb{R}^{n\times n}$ such that

$$X_\infty(\theta) = C_\theta' C_\theta + A_\theta' X_\infty(f(\theta)) A_\theta - L_\theta' R_\theta^{-1} L_\theta \tag{7}$$

where

$$R(\theta) = \overline{D}_\theta' J \overline{D}_\theta + \bar{B}_\theta' X_\infty(f(\theta))\bar{B}_\theta \tag{8}$$
$$L(\theta) = \overline{D}_\theta' J \bar{C}_\theta + \bar{B}_\theta' X_\infty(f(\theta)) A_\theta,$$

and

$$x(k+1) = \left(A_{\theta(k)} - \bar{B}_{\theta(k)} R(\theta(k))^{-1} L(\theta(k))\right) x(k) \tag{9}$$

is uniformly exponentially stable and for some $\varepsilon > 0$, and all $\theta \in S$,

$$X_\infty(\theta) \geq 0 \tag{10}$$
$$\nabla(\theta) = R_1(\theta) - R_2'(\theta) R_3^{-1}(\theta) R_2(\theta) \leq -\varepsilon I.$$

In this case the control

$$u_\infty(k) = -R_3^{-1}(\theta(k)) \begin{bmatrix} L_2(\theta(k)) & R_2(\theta(k)) \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} \tag{11}$$

satisfies the H_∞ objective, X_∞ is continuous and the closed loop system with control 11, that is

$$x(k+1) = Acl\left(f^k(\theta_o)\right) x(k) + N\left(f^k(\theta_o)\right) w(k),$$

is a uniformly exponentially stable system where $Acl(\theta) = \left(A_\theta - B_{2_\theta} R_3(\theta)^{-1} L_2(\theta)\right)$ and $N(\theta) = \left(B_{1_\theta} - B_{2_\theta} R_3(\theta)^{-1} L_2(\theta)\right)$.

Since (A, C) is uniformly detectable, $D_2' D_2 > 0$ and (A, B_2) is stabilizable, the optimal stabilizing LDV quadratic control exists (Bohacek and Jonckheere n.d.). That is, there exists a unique, continuous and bounded function $X_2 : S \to \mathbb{R}^{n \times n}$ such that $X_2'(\theta) = X_2(\theta) \geq 0$ and

$$X_2(\theta) = C_\theta' C_\theta + A_\theta' X_2(f(\theta)) A_\theta \tag{12}$$
$$- \left(B_{2_\theta}' X_2(f(\theta)) A_\theta\right)' T(\theta)^{-1} \left(B_{2_\theta}' X_2(f(\theta)) A_\theta\right).$$

with $T(\theta) = \left(D_{2_\theta}' D_{2_\theta} + B_{2_\theta}' X_2(f(\theta)) B_{2_\theta}\right)$. Furthermore, for $w \equiv 0$,

$$\inf_{u \in l_2} \|z\|^2 = x_o' X_2(\theta) x_o$$

and this infimum is attained with

$$u(k) = u_{LQ}(k)$$
$$= -T(\theta(k))^{-1} B_{2_{\theta(k)}}' X_2(f\theta(k)) A_{\theta(k)} x(k)$$

and

$$x(k+1) = A_{\theta(k)} x(k)$$
$$-B_{2_{\theta(k)}} T(\theta(k))^{-1} B_{2_{\theta(k)}}' X_2(f(\theta(k)) A_{\theta(k)} x(k)$$

is uniformly exponentially stable.

Assume the conditions of theorem 1 are true. Let $X(k, N+1, \theta) \geq 0$ denote the solution to the finite horizon Riccati equation with terminal condition $X_2\left(f^{N+1}(\theta)\right)$. That is,

$$X(k, N+1, \theta) =$$
$$A_{f^k(\theta)}' X(k+1, N+1, \theta) A_{f^k(\theta)} + \overline{C}_{f^k(\theta)}' J \overline{C}_{f^k(\theta)}$$
$$-L'(k, N+1, \theta) R^{-1}(k, N+1, \theta) L(k, N+1, \theta)$$

where

$$L(k, N+1, \theta) =$$
$$\bar{D}_{f^k(\theta)}' J \bar{C}_{f^k(\theta)} + \bar{B}_{f^k(\theta)}' X(k+1, N+1, \theta) A_{f^k(\theta)},$$
$$R(k, N+1, \theta) =$$
$$\bar{D}_{f^k(\theta)}' J \bar{D}_{f^k(\theta)} + \bar{B}_{f^k(\theta)}' X(k+1, N+1, \theta) \bar{B}_{f^k(\theta)}$$

and

$$X(N+1, N+1, \theta) = X_2(f^{N+1}(\theta))$$

with X_2 the solution to the functional Riccati equation 12. It is possible to show that

$$\lim_{N \to \infty} X(0, N+1, \theta) = X_\infty(\theta)$$

solves equations 7 and 8, satisfies equations 10, and that system 9 is uniformly exponentially stable. If the map f is chaotic, the methods of approximating X_2 developed in (Bohacek and Jonckheere n.d.) and (Jonckheere and Bohacek 1998) can be used to approximate X_∞.

Remark 1. Theorem 1 can be extended to the strictly casual feedback case where $u(k)$ does not depend on $w(k)$.

Remark 2. Observe that the if θ is a fixed point, the Riccati equations 7 and 12 reduce to the linear time invariant Riccati equations; if θ is a periodic point, equations 7 and 12 reduce to the periodically varying Riccati equations. In case of slow variation of θ, vis $f(\theta) \approx \theta$, equations 7 and 12 reduce to the state dependent Riccati equation.

Remark 3. If f is chaotic, the continuity of X_∞ and X_2 is counterintuitive. Due to extreme sensitivity to initial conditions, a small change in initial conditions $\theta(0)$ will lead to a drastic change in the linear system $\left(A_{\theta(k)}, B_{2_{\theta(k)}}\right)$. For example, since f is chaotic, fixed points are arbitrarily close to transitive points. Thus the continuity of X_∞ and X_2 imply that the cost to stabilize a time invariant system is about the same as the cost to stabilize a time varying system with system parameters that are, eventually, very different from the time invariant system parameters.

Remark 4. The controller F is globally defined, time invariant, but spatially varying. Thus, to track $x_{traject}(k) = f^k(x_{traject}(0))$, it suffices to implement $u(k) = F_{x_{traject}(k)}(\tilde{x}(k) - x_{traject}(k))$.

5. ANTI-CONTROL

Once the controller is designed many control objectives can be implemented. One objective, 'anti-control', is to prevent the state from entering a certain forbidden region of the attractor. This objective is of particular interest in physiology. For example, it is currently believed that a healthy heart is chaotic (Skinner *et al.* 1990) and fibrillation is associated with a periodic orbit with frequency of 8-10Hz (Gray *et al.* 1998), (**?**). Therefore, it is postulated that avoiding fibrillation may be accomplished by controlling the heart to avoid the region around a periodic orbit.

Assume that f is chaotic, let FB denote the forbidden region and assume that FB has positive measure. The objective of anti-control is to apply small control force to keep the state from entering FB. Due to sensitivity to initial conditions, a small control force will have a noticeable effect on the state only after a number of iterations. Therefore, the control must be applied well in advance

Fig. 1. Anti-Control: Control is applied to prevent the state (solid line) from getting too close to the fixed point (dotted line at 0.63). If no control had been applied, the state would have neared the fixed point at time index 20 (dotted line).

to the state entering FB. The further in advance the control is applied, the smaller the necessary control force. However, accurately predicting the future state of a chaotic system is difficult. Thus a compromise is made and control is applied only if the state is predicted to enter FB in L iterations or less. Anti-control is implemented as follows: Since FB has positive measure, $\bigcup_{l=0}^{\infty} f^{-l}(FB) = S$ modulo a set of zero measure. Let $\tilde{x}(k)$ be the current state and $\tilde{x}(k) \in \bigcup_{l=0}^{\infty} f^{-l}(FB)$. Define $l^* = \inf\left\{l : \tilde{x}(k) \in f^{-l}(FB)\right\}$. If $l^* < L$, then control is applied. The control is defined by finding a $y \in S$ such that $\|y - \tilde{x}(k)\| < \varepsilon$ and $f^j(y) \notin FB$ for $0 \leq j \leq l^*$. The control is applied to force $\tilde{x}(k+j)$ to track $f^j(y)$ for $j \leq l^*$, i.e. $u(k) = F_{f^j(y)}\left(\tilde{x}(k+j) - f^j(y)\right)$. When $j > l^*$ control is terminated until the state is once again predicted to enter into FB. Figure 1 shows a time series plot of anti-control applied to the Hénon map. In this example the forbidden region is a region around the fixed point.

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