Linear Dynamically Varying LQ Control of Nonlinear Systems over Compact Sets

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ABSTRACT

Linear quadratic controllers for tracking natural and composite trajectories of nonlinear dynamical systems evoluting over compact sets are developed. Typically, such systems exhibit "complicated dynamics," i.e., have nontrivial recurrence. The controllers, which use small perturbations of the nominal dynamics as input actuators, are based on modeling the tracking error as a linear dynamically varying (LDV) system. Necessary and sufficient conditions for the existence of such a controller are linked to the existence of a bounded solution to a functional algebraic Riccati equation (FARE). It is shown that, despite the lack of continuity of the asymptotic trajectory relative to initial conditions, the cost to stabilize about the trajectory, as given by the solution to the FARE, is continuous. An ergodic theory method for solving the FARE is presented. Furthermore, it is shown that wrapping the LDV controller around the nonlinear system secures a stable tracking dynamics. Finally, an example of controlling the Hénon map to follow an aperiodic orbit is presented.

I. INTRODUCTION

A typical feature of nonlinear dynamical systems running over compact sets is that their phase portraits exhibit a variety of trajectories ranging from the trivial periodic orbits to the nonperiodic transitive orbits [16]. Sensitive dependence on initial conditions and other parameters [21] allows a preselected trajectory to be tracked — despite offset in initial conditions, extraneous disturbances and uncertainty on the dynamics — via a cheap control that acts as a small perturbation of the nominal dynamics. To formalize the above ideas, define the nominal and perturbed dynamics, respectively, as

$$\theta(k+1) = f(\theta(k)) \ , \qquad \theta(0) = \theta_0 \in \Theta, \tag{1}$$

$$\varphi(k+1) = f(\varphi(k), u(k)), \qquad \varphi(0) = \varphi_0 \in \Theta. \tag{2}$$

In the above, $\{\theta(k) \in \mathbb{R}^n : k \geq 0\}$ is the desired trajectory and $\varphi(k)$ is the state of the system under control $u(k) \in \mathbb{R}^m$, which is taken to be a small perturbation of the nominal dynamics, viz., $f(\varphi(k), 0) = f(\varphi(k))$. The dynamics is differentialable, viz, $f \in C^1(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$ and the motion is restricted to a compact f-invariant ($f(\Theta) \subseteq \Theta$) subset of $\mathbb{R}^n$. The broad objective is to find a control u such that $\lim_{k\to\infty} \|\varphi(k) - \theta(k)\| = 0$.

This paper develops a controller of the form $u(k) = F_{\theta(k)}(\varphi(k) - \theta(k))$, where the feedback F_θ is designed, every for $\theta \in \Theta$, from a linear approximation of $f(\varphi, u)$ around $(\theta, 0)$, and the gain $F_{\theta(k)}$ is "scheduled" so as to force $\varphi(k)$ to follow the desired trajectory $\theta(k)$. This is a specialized version of the LPV scheme [18] [2], where the parameter vector θ, instead of being uncertain, is dynamically modeled. This justifies the terminology of *Linear Dynamically Varying (LDV) control.*

An unusual feature of the LDV controller viewed as a tracking controller is that the gain is spatially varying and defined all over Θ. As the first and most generic application, given an *arbitrary* desired trajectory $\{\theta(k) : k = 0, \cdots\}$, evaluating F_θ along the trajectory $\{\theta(k) : k = 0, \cdots\}$ yields the time-varying controller $F_{\theta(k)}$ that makes the nonlinear system $\varphi(k+1) = f(\varphi(k), F_{\theta(k)}(\varphi(k) - \theta(k)))$ asymptotically track $\theta(k+1) = f(\theta(k))$. More importantly, the globally defined controller F_θ becomes fully motivated in those specialized applications where there is a need to quickly adapt

the tracking controller to a new reference trajectory without recomputing a new time-varying controller along the new trajectory. Specifically, having reached $\varphi(k_0) \approx \theta(k_0)$, one could track the trajectory $\left\{\tilde{\theta}(k) : k \geq k_0\right\}$ starting at $\tilde{\theta}(k_0) \approx \theta(k_0)$ by switching from $\left\{F_{\theta(k)} : k = 0, \cdots, k_0 - 1\right\}$ to $\left\{F_{\tilde{\theta}(k)} : k \geq k_0\right\}$. As shown in [5], switching among natural trajectories allows for such broader control objectives as targeting and periodic orbit avoidance. In orbital mechanics, switching among free orbits proved instrumental in NASA's experiment of steering a decommissioned satellite to a rendez-vous with the Giaccobini-Zinner comet [9]. Along the related line of application of the X-33 program, there is a need to adapt the spacecraft controller to a change of launch to landing trajectory in case of failure and/or a change of landing site [13], [15] .

The controller F_θ exists if and only if a solution X_θ to a *functional* algebraic Riccati equation (FARE) linking X_θ and $X_{f(\theta)}$ exists. The mathematical difficulty with this functional equation is to prove that the relevant solution is continuous, in which case F_θ is continuous.

Typically there is no closed form solution to the FARE. However, dynamical systems on compact sets, subject to some mild additional conditions, are known to have such ergodic properties as recurrence and transitivity [16], which can be put into use to construct an approximate solution of arbitrary accuracy.

The paper proceeds as follows: Section II. formalizes the tracking control problem of interest and shows how the tracking error can be approximated as a linear dynamically varying (LDV) system. Section III. formally develops LDV systems and the optimal linear-quadratic controllers for this class of systems. Section IV. shows that these linear controllers are suitable for robustly stabilizing the nonlinear systems. Section V. develops numerical techniques to compute the solution of the FARE. Section VI. gives an example. Finally, the Appendix provides the proofs of the technical results.

II. LINEAR DYNAMICALLY VARYING TRACKING ERROR DYNAMICS

Define the tracking error

$$x(k) = \varphi(k) - \theta(k).$$

Then we obtain

$$x(k+1) = f\left(\varphi(k), u(k)\right) - f\left(\theta(k), 0\right)$$

The first degree Taylor approximation of $f\left(\varphi(k), u\left(k\right)\right)$ around $\varphi(k) = \theta(k)$ and $u(k) = 0$ yields

$$x(k+1) = A_{\theta(k)}x(k) + B_{\theta(k)}u(k) + \eta\left(x(k), u(k), \theta(k)\right) \tag{3}$$

where

$$A_\theta = \frac{\partial f}{\partial x}\left(\theta, 0\right), \quad B_\theta = \frac{\partial f}{\partial u}\left(\theta, 0\right)$$

and $\eta\left(x(k), u(k), \theta(k)\right)$ accounts for nonlinear terms. to be specific, since $f \in C^1$,

$$\eta(x(k), u(k), \theta(k)) = \eta_x(x(k), u(k), \theta(k))x(k) + \eta_u(x(k), u(k), \theta(k))u(k) \tag{4}$$

where

$$\eta_{x_{i,j}}\left(x, u, \theta\right) = \int_0^1 \left(\frac{\partial f_i}{\partial x_j}\left(tx + \theta, tu\right) - \frac{\partial f_i}{\partial x_j}(\theta, 0)\right) dt \tag{5}$$

and

$$\eta_{u_{i,j}}\left(x, u, \theta\right) = \int_0^1 \left(\frac{\partial f_i}{\partial u_j}\left(tx + \theta, tu\right) - \frac{\partial f_i}{\partial u_j}(\theta, 0)\right) dt \tag{6}$$

Since $f \in C^1$ and Θ is compact, if x and u are bounded, then $\frac{\partial f_i}{\partial x_j}\left(tx + \theta, tu\right) - \frac{\partial f_i}{\partial x_j}(\theta, 0)$ is uniformly continuous in x and u for $t \in [0, 1]$. In particular, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $\left\|x\right\|, \left\|u\right\| < \delta$, then $\left|\frac{\partial f_i}{\partial x_j}\left(tx + \theta, tu\right) - \frac{\partial f_i}{\partial x_j}\left(\theta, 0\right)\right| < \varepsilon$ for $t \in [0, 1]$. Therefore, $\left\|\eta_x\left(x, u, \theta\right)\right\|$ and $\left\|\eta_u\left(x, u, \theta\right)\right\|$ can be made as small as necessary by limiting the size of u and x.

If u and x are small, the error dynamics can be approximated as

$$x(k+1) = A_{\theta(k)}x(k) + B_{\theta(k)}u(k) \tag{7}$$

This is a linear system with coefficient matrices A and B that vary as $\theta(k)$ varies. Since $\theta(k)$ varies according the dynamical to equation (1), such an interconnection as (1) and (3) is called a *Linear Dynamically Varying (LDV)* system.

LDV and LPV systems can be unified under the so-called Linear Set Valued Dynamically Varying (LSVDV) systems characterized by a *set valued* map f [15]. LDV and LPV [4], [3] systems are the two extremes, the former characterized by the fact that $f\left(\theta\right)$ is reduced to a

point, the latter characterized by $f(\theta) = \Theta$. Somewhere between the two extremes lies the case of slow systems characterized by $\left\|\theta' - \theta\right\| < \Delta$, $\forall \theta' \in f(\theta)$ [23] [2]. Under this condition, it is customary to postulate existence of an analytic map $X : \Theta \to \mathbb{R}^{n \times n}$ such that $A'_\theta X_{\theta+\delta} A_\theta - X_\theta < 0$ for $\|\delta\| < \Delta$ from which stability follows. Here, instead of *postulating* analyticity or any other convenient property of some solution to an *inequality*, we *prove* continuity of the solution to a relevant *equation* (see (46)).

III. LINEAR DYNAMICALLY VARYING SYSTEMS AND CONTROLS

Before controllers for LDV systems can be developed, such systems must be formalized and the relevant stability and detectability concepts must be defined. For the purpose of control, a linear dynamically varying (LDV) system is defined as follows:

$$\begin{aligned} x(k+1) &= A_{\theta(k)} x(k) + B_{\theta(k)} u(k) \\ z(k) &= \begin{bmatrix} C_{\theta(k)} x(k) \\ D_{\theta(k)} u(k) \end{bmatrix} \\ \theta(k+1) &= f(\theta(k)) \\ \text{with } \theta(0) &= \theta_o \text{ and } x(0) = x_o \end{aligned} \tag{8}$$

where $f : \Theta \to \Theta$ is a continuous map, $\Theta \subset \mathbb{R}^n$ is compact and f-invariant, $A : \Theta \to \mathbb{R}^{n \times n}$, $B : \Theta \to \mathbb{R}^{n \times m}$, $C : \Theta \to \mathbb{R}^{p_1 \times n}$, and $D : \Theta \to \mathbb{R}^{p_2 \times m}$ are functions that need not be continuous, $\theta(k) \in \Theta$ is the state of the dynamical system, $x(k) \in \mathbb{R}^n$ is the state of the linear system, $u(k) \in \mathbb{R}^m$ is the control input, and $z(k) \in \mathbb{R}^{p_1 + p_2}$ is the output to be controlled. It is assumed that both states $x(k)$ and $\theta(k)$ are known at time point k.

It is often assumed that the system coefficient matrices A, B, C and D are continuous. We refer to such systems as *continuous* LDV systems. In Section II., it was assumed that $f \in C^1$ and A and B were defined to be the matrices of partial derivatives of f, so that A and B were continuous. Thus, the tracking error dynamics associated with system (1) and (2) can be approximated by a *continuous* LDV system. However, if a feedback $F : \Theta \to R^{m \times n}$ is used to stabilize a continuous

LDV system, then the resulting closed-loop system is a continuous LDV systems only if F is continuous. Although this paper will focus on continuous LDV systems, we cannot a priori assume that the stabilizing feedback is continuous. Therefore, the definition of an LDV system should allow for discontinuous coefficient matrices. Another motivation for allowing discontinuous LDV systems is to define a class large enough to encompass jump linear systems as discussed in [8].

From a mathematical perspective, a linear dynamically varying system is a family of linear time-varying systems indexed by the initial condition $\theta(0)$. If $\theta(0)$ is a fixed point, then the linear system with index $\theta(0)$ is time-invariant. If $\theta(0)$ is a periodic point, the linear system with index $\theta(0)$ is a periodically varying linear system. If $\theta(0)$ is an aperiodic point, the linear system is a linear time-varying system.

Since a linear dynamically varying system is an uncountable collection of linear time varying systems, the concept of stability is slightly more complex in the dynamically varying case than it is in the time varying case.

Definition 1 *The linear dynamically varying system (8) is* uniformly exponentially stable *if for* $u(k) = 0$, *there exist an* $0 < \alpha < 1$ *and a* $\beta < \infty$ *such that for all* $\theta(0) \in \Theta$

$$\|x(k)\| \leq \beta\alpha^k \|x(0)\|.$$

System (8) is exponentially stable *if for* $u(k) = 0$ *and for each* $\theta(0) \in \Theta$, *there exist an* $0 < \alpha(\theta(0)) < 1$ *and a* $\beta(\theta(0)) < \infty$ *such that for all* $x(j)$ *and* $j \leq k$

$$\|x(k+j)\| \leq \beta(\theta(0))\alpha(\theta(0))^j \|x(j)\|.$$

System (8) is asymptotically stable *if for* $u(k) = 0$, *any* $\|x(0)\| < \infty$ *and any* $\theta(0) \in \Theta$

$$\|x(k)\| \to 0.$$

Thus a linear dynamically varying system is exponentially stable if every linear system in the family of linear systems indexed by $\theta(0)$ is exponentially stable. The parameters, $\alpha(\theta)$ and $\beta(\theta)$, remain constant along a positive trajectory; i.e. $\alpha(f(\theta)) = \alpha(\theta)$, but may vary discontinuously across different trajectories. Another difficulty with this stability concept is that it is possible that

$\lim_i \alpha(\theta_i) = 1$ while $\alpha(\lim_i \theta_i) < 1$ for some sequence $\{\theta_i \in \Theta : i \geq 0\}$, in which case the system is exponentially stable, but not uniformly exponentially stable.

In the case of *continuous* LDV systems, asymptotic, exponential and uniform exponential stability are equivalent:

Proposition 2 *Assume that the function $A : \Theta \to \mathbb{R}^{n \times n}$ is continuous and Θ is compact. Then asymptotic, exponential and uniformly exponential stability are equivalent.*

Proof. The proof is withheld until section VIII.. ■

Note that for general time varying systems, exponential stability and asymptotic stability are not equivalent. However, in the case of continuous LDV's, continuity and compactness lead to the equivalence of these two forms of stability.

Since uniformly exponentially stable systems are inherently more robust than exponentially stable systems, it is preferable to remain within the confines of continuous LDV systems. Thus, when synthesizing a feedback for controlling a continuous LDV system, it is important to ensure that the feedback is not only asymptotically stabilizing, but also continuous. However, to maintain generality, an LDV system is considered stabilizable if there exists an exponentially stabilizing feedback, that is:

Definition 3 *System (8) is stabilizable if there exists a, not necessarily continuous, function $F : \mathbb{N} \times \Theta \to \mathbb{R}^{m \times n}$ such that for all $\theta(0) \in \Theta$ and for all k, $\|F(k, \theta(0))\| \leq \overline{F}(\theta(0)) < \infty$ and the system*

$$\begin{aligned} x(k+1) &= \left(A_{\theta(k)} + B_{\theta(k)} F(k, \theta(0))\right) x(k) \\ \theta(k) &= f^k(\theta(0)) \end{aligned}$$

is exponentially stable, that is, there exists $\alpha(\theta(0))$ and $\beta(\theta(0))$ such that

$$\left\| \prod_{i=j}^{k-1} \left(A_{f^i(\theta(0))} + B_{f^i(\theta(0))} F(i, \theta(0))\right) \right\| \leq \beta(\theta(0)) \alpha^{k-j}(\theta(0)),$$

where the factors of the matrix product are taken in the proper order.

Thus, the feedback that exists via the definition of stabilizability may not be uniformly bounded nor even continuous in $\theta(0)$. A feedback that is uniformly bounded and making the closed loop system uniformly exponentially stable will be said to be uniformly stabilizing.

Along with stabilizability, a detectability concept is needed.

Definition 4 *System (8) is uniformly detectable if there exists a, not necessarily continuous, function* $L : \Theta \to \mathbb{R}^{n\times p}$ *such that for* $\theta \in \Theta$, $\|L_\theta\| \leq \overline{L} < \infty$ *and the system*

$$\begin{aligned} x(k+1) &= \left(A_{\theta(k)} + L_{\theta(k)} C_{\theta(k)}\right) x(k) \\ \theta(k) &= f^k\left(\theta(0)\right), \end{aligned}$$

is uniformly exponentially stable. That is, there exist an $\alpha_d < 1$ *and a* $\beta_d < \infty$ *such that for all* $\theta(0) \in \Theta$, $\left\| \prod_{i=j}^{k-1} \left(A_{f^i(\theta(0))} + L_{f^i(\theta(0))} B_{f^i(\theta(0))}\right) \right\| \leq \beta_d \alpha_d^k \|x(0)\|$.

The conditions of stabilizability and uniform detectability, required to secure existence of an LDV controller, are slightly asymmetric. However, as can easily be shown by a duality argument, if the function f is invertible, then uniform detectability can be weakened to detectability, which is *exactly* the dual of stabilizability.

Since stabilizability depends on A, B *and* f, we will say that the triple (A, B, f) is stabilizable to mean that system (8) is stabilizable. Similarly, we say that the triple (A, C, f) is uniformly detectable to mean that system (8) is uniformly detectable.

Under the following assumptions, the existence of a uniformly stabilizing continuous linear dynamically varying quadratic controller is proved:

Assumption 1 The functions $A : \Theta \to \mathbb{R}^{n\times n}$, $C : \Theta \to \mathbb{R}^{p\times n}$and $f : \Theta \to \Theta$ are such that (A, C, f) is uniformly detectable.

Assumption 2 The functions $A : \Theta \to \mathbb{R}^{n\times n}$, $B : \Theta \to \mathbb{R}^{n\times m}$ and $f : \Theta \to \Theta$ are such that (A, B, f) is stabilizable.

Assumption 3 The functions A, B, C, D and f are continuous, Θ is compact and $D_\theta' D_\theta > 0$ for all $\theta \in \Theta$.

Our main result can now be formulated:

Theorem 5 *Suppose assumptions 1, 2 and 3 hold. Then there exists a unique, bounded solution* $X : \Theta \to \mathbb{R}^{n \times n}$ *to the functional algebraic Riccati equation (FARE)*

$$X_\theta = A'_\theta X_{f(\theta)} A_\theta - A'_\theta X_{f(\theta)} B_\theta \left(D'_\theta D_\theta + B'_\theta X_{f(\theta)} B_\theta \right)^{-1} B'_\theta X_{f(\theta)} A_\theta + C'_\theta C_\theta . \tag{9}$$

such that $X_\theta = X'_\theta \geq 0$*, and the feedback*

$$u(k) = - \left(D'_{\theta(k)} D_{\theta(k)} + B'_{\theta(k)} X_{f(\theta(k))} B_{\theta(k)} \right)^{-1} B'_{\theta(k)} X_{f(\theta(k))} A_{\theta(k)} x(k) \tag{10}$$

uniformly exponentially stabilizes system (8). For $\|x(0)\| < \infty$*, this feedback minimizes*

$$\sum_{k=0}^{\infty} \|z(k)\|^2$$

and X *is a uniformly continuous function.*

Conversely, if assumptions 1 and 3 hold and if there exists a bounded solution, X*, to equation (9) such that* $X_\theta = X'_\theta \geq 0$*, then system (8) is stabilizable and* X *is continuous. In this case, if* $X(k, N, \theta)$ *solves the finite horizon Riccati equation, i.e.*

$$\begin{aligned} X(k, N, \theta) \;=\; & A'_{f^k(\theta)} X(k+1, N, \theta) A_{f^k(\theta)} + C'_{f^k(\theta)} C_{f^k(\theta)} \\ & - A'_{f^k(\theta)} X(k+1, N, \theta) B_{f^k(\theta)} \\ & \times \left(D'_{f^k(\theta)} D_{f^k(\theta)} + B'_{f^k(\theta)} X(k+1, N, \theta) B_{f^k(\theta)} \right)^{-1} \\ & \times B'_{f^k(\theta)} X(k+1, N, \theta) A_{f^k(\theta)} \end{aligned} \tag{11}$$

with

$$X(N, N, \theta) = C'_{f^N(\theta)} C_{f^N(\theta)},$$

then $X(0, N, \theta) \to X_\theta$ *uniformly in* θ*.*

Proof. The proof of this theorem is in section VIII.. ■

Remark 1 *Stabilizability is a rather weak assumption. Indeed, stabilizability merely assumes that every trajectory is stabilizable. Given this obviously necessary condition, it is interesting to observe that continuity and compactness are all that is needed to prove the existence of a continuous and uniformly stabilizing controller.*

Remark 2 *The continuity of the cost X is counter-intuitive in the case where f is sensitively dependent on initial conditions. It is easily seen that the general time-varying infinite horizon optimal quadratic cost is continuous with respect to the uniform topology, that is, if V_0 the is optimal infinite horizon quadratic cost associated with $\{A_k, B_k, C_k, D_k : k \geq 0\}$ and $\tilde{V}_0$ is the optimal cost associated with $\left\{\tilde{A}_k, \tilde{B}_k, \tilde{C}_k, \tilde{D}_k : k \geq 0\right\}$ and if*

$$\begin{aligned} & d\left(\{A_k, B_k, C_k, D_k : k \geq 0\}, \left\{\tilde{A}_k, \tilde{B}_k, \tilde{C}_k, \tilde{D}_k : k \geq 0\right\}\right) \\ = & \sup_k \left(\left\|A_k - \hat{A}_k\right\| + \left\|B_k - \hat{B}_k\right\| + \left\|C_k - \hat{C}_k\right\| + \left\|D_k - \hat{D}_k\right\|\right) \end{aligned}$$

is small, then $\left\|V_0 - \tilde{V}_0\right\|$ is small. We say that f has sensitive dependence on initial conditions if there exists an $r > 0$ such that for all $\theta \in \Theta$ and $\varepsilon > 0$, there exist a $\varphi \in \Theta$, $\|\varphi - \theta\| < \varepsilon$, and a $K < \infty$ such that $\left\|f^K(\theta) - f^K(\varphi)\right\| > r$. Hence, if f has sensitivity to initial conditions, then

$$d\left(\left\{A_{f^k(\theta)}, B_{f^k(\theta)}, C_{f^k(\theta)}, D_{f^k(\theta)} : k \geq 0\right\}, \left\{A_{f^k(\varphi)}, B_{f^k(\varphi)}, C_{f^k(\varphi)}, D_{f^k(\varphi)} : k \geq 0\right\}\right)$$

may remain bounded from below for arbitrarily small $\|\theta - \varphi\|$. Thus the time-varying system

$$\left\{A_{f^k(\theta)}, B_{f^k(\theta)}, C_{f^k(\theta)}, D_{f^k(\theta)} : k \geq 0\right\}$$

is discontinuous in θ and standard continuity results from time-varying control cannot be applied. Surprisingly, despite the fact that the distance between the time-varying systems indexed by θ and φ remains bounded from below, theorem 5 implies that $\|X_\varphi - X_\theta\|$ can be made small by taking $\|\varphi - \theta\|$ small enough. In particular, this continuity implies that the cost to stabilize a periodic orbit is nearly the same as the cost to stabilize an aperiodic one, whereas general time-varying results seem to imply that the cost of stabilizing these different orbits may be very different.

Remark 3 *When evaluated along a particular trajectory, the FARE and hence the controller become time-invariant, periodically varying or time-varying depending on whether the trajectory is fixed, periodic or aperiodic, respectively.*

IV. STABILITY AND ROBUSTNESS OF NONLINEAR SYSTEMS WITH LDV CONTROLLERS

Here, we address the issue as to whether the LDV quadratic controller, guaranteed to stabilize the LDV system, also stabilizes the nonlinear system.

With the feedback given by theorem 5 in place, the nonlinear system (3) becomes

$$\begin{aligned} x(k+1) &= \left(A_{\theta(k)} + B_{\theta(k)}F_{\theta(k)}\right)x(k) \\ &\quad + \left(\eta_x\left(x(k), u(k), \theta(k)\right) + \eta_u\left(x(k), u(k), \theta(k)\right)F_{\theta(k)}\right)x(k) \\ \theta(k+1) &= f\left(\theta(k)\right) \end{aligned} \tag{12}$$

where η_x and η_u are given by equations (5) and (6). By theorem 5,

$$\begin{aligned} x(k+1) &= \left(A_{\theta(k)} + B_{\theta(k)}F_{\theta(k)}\right)x(k) \\ \theta(k+1) &= f\left(\theta(k)\right) \end{aligned}$$

is uniformly exponentially stable. Thus, by theorem 24.7 in [22], there exists an $\bar{\eta} > 0$ such that if $\left\|\eta_x\left(x(k), u(k), \theta(k)\right) + \eta_u\left(x(k), u(k), \theta(k)\right)F_{\theta(k)}\right\| \leq \bar{\eta}$, then system (12) is uniformly exponentially stable. If we define

$$\begin{aligned} &\tilde{\eta}(x, u, \theta) \\ &:= \begin{cases} \eta_x(x, u, \theta) + \eta_u(x, u, \theta)F_\theta & \text{if } \|\eta_x(x, u, \theta) + \eta_u(x, u, \theta)F_\theta\| < \bar{\eta} \\ \bar{\eta}I & \text{otherwise} \end{cases}, \end{aligned}$$

then

$$\tilde{x}(k+1) = \left(A_{\theta(k)} + B_{\theta(k)}F_{\theta(k)} + \tilde{\eta}\left(\tilde{x}(k), F_{\theta(k)}\tilde{x}(k), \theta(k)\right)\right)\tilde{x}(k)$$

is uniformly exponentially stable. In this case, there exists a $G < \infty$ such that

$$\|\tilde{x}(k)\| < G\|\tilde{x}(0)\| \tag{13}$$

for all $\theta(0) \in \Theta$. It was shown in section II. that $\eta_x\left(x(k), u(k), \theta(k)\right)$ and $\eta_u\left(x(k), u(k), \theta(k)\right)$ can be made arbitrarily small by limiting the size of $x(k)$ and $u(k)$. Since X_θ is continuous, F_θ is continuous, and since Θ is compact, there exists a bound $\bar{F} < \infty$ such that $\|F_\theta\| < \bar{F}$. Thus there exists an $\bar{x} > 0$ such that if $\|x(k)\| < \bar{x}$ and $\|u(k)\| = \left\|F_{\theta(k)}x(k)\right\| < \bar{F}\bar{x}$, then

$$\left\|\eta_x\left(x(k), u(k), \theta(k)\right) + \eta_u\left(x(k), u(k), \theta(k)\right)F_{\theta(k)}\right\| < \bar{\eta}.$$

Therefore, if

$$\|\tilde{x}(0)\| < \frac{\bar{x}}{G} =: R_{Capture}$$

then $\|\tilde{x}(k)\| < \bar{x}$ and $\left\|F_{\theta(k)}\tilde{x}(k)\right\| < \bar{F}\bar{x}$ and thus

$$\left\|\eta_x\left(\tilde{x}(k), F_{\theta(k)}\tilde{x}(k), \theta(k)\right) + \eta_u\left(\tilde{x}(k), F_{\theta(k)}\tilde{x}(k), \theta(k)\right) F_{\theta(k)}\right\| < \bar{\eta}$$

and

$$\tilde{\eta}\left(\tilde{x}(k), F_{\theta(k)}\tilde{x}(k), \theta(k)\right) = \eta\left(\tilde{x}(k), F_{\theta(k)}\tilde{x}(k), \theta(k)\right).$$

By uniqueness of the solution to a difference equation, we conclude that if $\tilde{x}(0) = x(0)$, with

$$\|x(0)\| < R_{Capture}, \tag{14}$$

then $x(k) = \tilde{x}(k)$ and system (12) is uniformly exponentially stable.

Remark 4 *For x and u bounded, the nonlinearity term η in (12) is clearly a bounded feedback wrapped around the LDV plant, so that the natural way to reduce the effect of the effect of η — and amplify the domain of attraction — is an H^∞ design. This approach is pursed in [5] and [6].*

V. EXPLOITING ERGODICITY TO SOLVE THE FUNCTIONAL ALGEBRAIC RICCATI EQUATION (FARE)

Many methods can be devised to solve the FARE. Here we investigate, in detail, a method based on the ergodic property of recurrence and we give glimpses at two other methods: one based on a jump linear approximation and another based on iterating a Riccati recursion.

A. solving FARE over recurrent set

For notational convenience, define the Riccati map

$$\rho_\theta(X) = A_\theta' X A_\theta + C_\theta' C_\theta - A_\theta' X B_\theta \left(D_\theta' D_\theta + B_\theta' X B_\theta\right)^{-1} B_\theta' X A_\theta.$$

With this notation, the FARE can be written $X_\theta = \rho_\theta\left(X_{f(\theta)}\right)$.

By definition, the point θ is *recurrent* if for an arbitrary $\delta > 0$, there exists an $N(\delta) < \infty$ such that $\left\|\theta - f^{N(\delta)}(\theta)\right\| < \delta$. Let $R(f)$ denote the set of recurrent points. Since $R(f)$ is invariant ($f(R(f)) \subseteq R(f)$), the FARE can be restricted to $R(f)$ and furthermore, the solution X_θ can

be extended by continuity from $R(f)$ to $\overline{R(f)}$. Recall that for a generic class of diffeomorphisms, $\overline{R(f)}$ is the largest set where nontransient behavior occurs [16].

Take $\theta \in R(f)$. Since X is continuous, for $\varepsilon > 0$, there exists a $\delta > 0$ such that $\left\|X_\theta - X_{f^{N(\delta)}(\theta)}\right\| < \varepsilon$. It follows that

$$\left\|\rho_\theta \circ \rho_{f(\theta)} \circ \cdots \circ \rho_{f^{N(\delta)-1}(\theta)} \left(X_{f^{N(\delta)}(\theta)}\right) - X_{f^{N(\delta)}(\theta)}\right\| < \varepsilon.$$

If we define

$$\tilde{\rho}_\theta (Y) = \rho_\theta (Y) + X_{f^{N(\delta)}(\theta)} - X_\theta.$$

we have

$$\tilde{\rho}_\theta \circ \rho_{f^1(\theta)} \circ \cdots \rho_{f^{N(\delta)-1}(\theta)} \left(X_{f^{N(\delta)}(\theta)}\right) = X_{f^{N(\delta)}(\theta)}$$

and $\|\tilde{\rho}_\theta - \rho_\theta\| < \varepsilon$. Replacing $\tilde{\rho}_\theta$ with ρ_θ in the above yields the approximate equation for $X_{f^{N(\delta)}(\theta)}$

$$\rho_\theta \circ \rho_{f^1(\theta)} \circ \cdots \rho_{f^{N(\delta)-1}(\theta)} (Y) = Y. \tag{15}$$

It turns out that Y, $X_{f^{N(\delta)}(\theta)}$ and X_θ are "close."

Theorem 6 *Assume that f is a diffeomorphism and $R(f)$ is structurally stable [7]. Let θ be a recurrent point and let $\varepsilon > 0$. Then there exists a $\delta > 0$ such that $\left\|\theta - f^{N(\delta)}(\theta)\right\| < \delta$ implies that $\|X_\theta - Y\| < \varepsilon$ where X solves the FARE and Y solves (15).*

Proof. The definition of the structural stability of $R(f)$ and the proof of this theorem are given Section VIII.C. ■

Clearly, an approximate solution to the FARE is given by the fixed point of (15), which can be found as follows. Since f is a diffeomorphism, A_θ is invertible for all $\theta \in \Theta$, and if we define

$$\begin{aligned}
\alpha_\theta &= A_\theta^{-1} \\
\beta_\theta &= A_\theta^{-1} B_\theta \left(D_\theta' D_\theta\right)^{-1} B_\theta', \\
\gamma_\theta &= C_\theta' C_\theta A_\theta^{-1}, \\
\delta_\theta &= A_\theta' + C_\theta' C_\theta A_\theta^{-1} B_\theta \left(D_\theta' D_\theta\right)^{-1} B_\theta';
\end{aligned}$$

then it can be shown [10] that

$$\rho_\theta(Y) = (\gamma_\theta + \delta_\theta Y)(\alpha_\theta + \beta_\theta Y)^{-1}$$

and

$$\rho_\theta \circ \rho_{f^1(\theta)} \circ \cdots \rho_{f^{N-1}(\theta)}(Y) = \left(\tilde{\gamma} + \tilde{\delta} Y\right)\left(\tilde{\alpha} + \tilde{\beta} Y\right)^{-1} \tag{16}$$

where

$$\begin{bmatrix} \tilde{\alpha} & \tilde{\beta} \\ \tilde{\gamma} & \tilde{\delta} \end{bmatrix} = \left(\prod_{k=0}^{N-1} \begin{bmatrix} \alpha_{f^k(\theta)} & \beta_{f^k(\theta)} \\ \gamma_{f^k(\theta)} & \delta_{f^k(\theta)} \end{bmatrix} \right)$$

Equation (16) can be solved by finding a Y such that

$$\left(\prod_{k=0}^{N-1} \begin{bmatrix} \alpha_{f^k(\theta)} & \beta_{f^k(\theta)} \\ \gamma_{f^k(\theta)} & \delta_{f^k(\theta)} \end{bmatrix} \right) \begin{bmatrix} I \\ Y \end{bmatrix} \subset \left\{ \begin{bmatrix} I \\ Y \end{bmatrix} \right\}$$

where $\left\{ \begin{bmatrix} I \\ Y \end{bmatrix} \right\}$ is the span of $\begin{bmatrix} I \\ Y \end{bmatrix}$. This invariant subspace problem can be solved by the usual methods.

VI. EXAMPLE

In the following, an LDV controller is devised for the Hénon system. The Hénon system is defined as

$$\begin{bmatrix} \theta_1(k+1) \\ \theta_2(k+1) \end{bmatrix} = \begin{bmatrix} f_1(\theta(k), u(k)) \\ f_2(\theta(k), u(k)) \end{bmatrix} = \begin{bmatrix} 1 - (a + u(k))\theta_1(k)^2 + \theta_2(k) \\ b\theta_1(k) \end{bmatrix} \tag{17}$$

where u is the control input. In this example, $a = 1.4$ and $b = 0.3$. For these parameter values and $u \equiv 0$, it is known that the Hénon map has an attractor Θ, that is, these exists an open set $V \supsetneq \Theta$ such that $\lim_{k\to\infty} d\left(f^k(\theta_o), \Theta\right) = 0$ for all $\theta_0 \in V$. This attractor is the crescent shaped object shown in figure 1.

Define the associated LDV system by

$$A_{\theta(k)} = \left.\frac{df}{d\theta}\right|_{\theta(k),0} = \begin{bmatrix} -2a\,\theta_1(k) & 1 \\ b & 0 \end{bmatrix}, \tag{18}$$

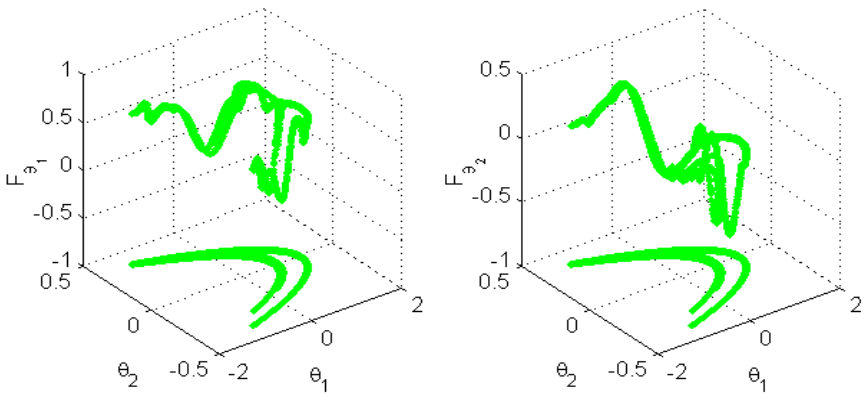

Figure 1: Feedback $F_\theta = \begin{bmatrix} F_{\theta_1} & F_{\theta_2} \end{bmatrix}$ for the Hénon map. The plots show that the feedback is continuous in θ. A plot of the attractor of the Hénon map in the $(\theta_1, \theta_2, -6)$ plane is included for reference.

$$B_{\theta(k)} = \left.\frac{df}{du}\right|_{\theta(k),0} = \begin{bmatrix} -\theta_1^2(k) \\ 0 \end{bmatrix}. \tag{19}$$

Numerical simulation indicates that Θ is transitive, that is, for almost every $\theta_o \in \Theta$, the trajectory $\{f^k(\theta_o)\}$ enters every ε-neighborhood of every point $\theta \in \Theta$ for every $\varepsilon > 0$. Therefore, the numerical methods developed in Section V.B.2 for solving the FARE apply. In this way, the gain $F : \Theta \to \mathbb{R}^{1\times 2}$ is obtained and the closed-loop tracking error dynamics becomes

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \left(A_{\theta(k)} + B_{\theta(k)} F_{\theta(k)}\right) \begin{bmatrix} x_1(k) \\ x_2(k)) \end{bmatrix} + \begin{bmatrix} \eta_1(k) \\ \eta_2(k) \end{bmatrix} \tag{20}$$

where

$$\eta_1(k) = \begin{bmatrix} \left((-\varphi_1(k) - \theta_1(k)) F_{1_{\theta(k)}} - a\right) x_1(k) \\ -\left(\varphi_1(k) + x_{1_{traject}}(k)\right) F_{2_{\theta(k)}} x_1(k) \end{bmatrix}' \begin{bmatrix} x_1(k)) \\ x_2(k) \end{bmatrix} \tag{21}$$

$$\eta_2(k) = 0 \tag{22}$$

$$F_\theta = \begin{bmatrix} F_{1_\theta} & F_{2_\theta} \end{bmatrix}$$

Figure 1 shows the feedback gain, F, for the LDV system (20). Note that the feedback is continuous on Θ, the attractor of the Hénon map.

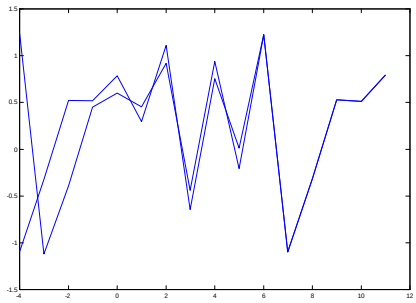

Figure 2: Tracking: φ_1 (solid line) and θ_1 (dotted line) are shown. At time $k = 0$, $\|\varphi(0) - \theta(0)\|$ is small enough for the control to be safely turned on. Once the control is turned on, φ tracks θ.

The objective in this example is to control the Hénon system so as to follow an aperiodic orbit described by $\theta(k+1) = f(\theta(k), 0)$ with $\theta(0) = (0.6961, 0.2088)$. Since the controlled system is only locally stable, control cannot begin until time $k = T$ when $\|\varphi(T) - \theta(T)\| < R_{Capture}$ and $R_{Capture} > 0$ is the initial tracking error bound that ensures stability as defined in section IV.. Computer simulations indicate that the Hénon map is not distal and that $f \times f$ is ergodic [19]. Since $\{(\varphi, \theta) \in \Theta \times \Theta : \|\varphi - \theta\| < R_{Capture}\}$ has positive measure, the Poincaré recurrence theorem [16] on $\Theta \times \Theta$ implies that for almost every initial condition (θ_o, φ_o), we have $\|\varphi(T) - \theta(T)\| < R_{Capture}$ for some T. For $k > T$, control force is applied via the control law $u(k) = F_{\theta(k)} x(k)$, where F is given by the LDV quadratic control method of section III..

Figures 2 and 3 show the controlled trajectory, $\varphi_1(k)$, the desired trajectory, $\theta_1(k)$, and the tracking error, $x_1(k) = \varphi_1(k) - \theta_1(k)$. At time index 0, the error is small enough to safely turn the control on (i.e. $T = 0$) and φ_1 tracks the desired trajectory. After the control is applied the error may increase beyond $R_{Capture}$. Extensive simulation imply that if $\sqrt{\left(\frac{1}{1.3}\right)^2 (\varphi_1(T) - \theta_1(T))^2 + \left(\frac{1}{0.4}\right)^2 (\varphi_2(T) - \theta_2(T))^2} < 0.15$, the system remains stable, where the scaling factors account for that fact that $\theta_1 \in [-1.3, 1.3]$ and $\theta_2 \in [-.4, .4]$.

An example of controlling the Hénon map to avoid its fixed point is available in [5].

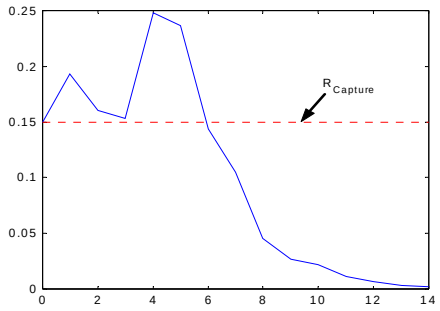

Figure 3: Tracking Error: At time $k = 0$, the tracking error (solid line) is small enough, i.e. $\|\varphi(0) - \theta(0)\| < R_{Capture}$, where $R_{Capture} = 0.15$. Hence control is initiated at $k = 0$. However, the tracking error exceeds $R_{Capture}$ for the next few time steps. This is acceptable, and the tracking error converges to zero.

VII. CONCLUSION

Linear Dynamically Varying controllers for tracking natural and composite trajectories of nonlinear dynamical systems running over compact sets have been developed. The necessary and sufficient conditions for the existence of such controllers are rather weak and are equivalent to the existence of a bounded positive semi-definite solution to a functional algebraic Riccati equation (FARE). If the dynamical system has adequate ergodic properties, there are many techniques for computing the solution to the FARE.

The LDV theory complements the popular LPV/gain scheduling theory by focusing on the extreme case of *known* parameter dynamics. Mathematically, the LMI of LPV design is pushed to the extreme situation of an equation linking the values of the solution for two successive values of the parameters, with the inescapable problem of proving continuity of the solution. Such equations, referred to as functional, are indeed notorious for generating badly behaved solutions, so that the LDV limit to the LPV theory was due to involve some mathematical difficulties.

It is hoped that the LSVDV theory, along with its continuous-time and H^∞ counterparts [14], [5], [6]will emerge as a unification of the various gain scheduling concepts.

VIII. APPENDIX

Since a linear dynamically varying system is a collection of time-varying systems, the following time-varying Lyapunov stability theorem will prove useful.

Proposition 7 *Assume (A, C, f) is uniformly detectable. Then there exists an $\alpha(\theta_o) \in [0, 1)$ and a $\beta(\theta_o) < \infty$ such that for $\theta(0) = \theta_o$, and any $x(k) \in \mathbb{R}^n$,*

$$\|x(k+j)\| \leq \beta(\theta_o)\,\alpha(\theta_o)^j\,\|x(k)\|$$

if and only if there exists a sequence $\left\{X_{f^k(\theta_o)} : k \geq 0\right\}$ with $\left\|X_{f^k(\theta_o)}\right\| \leq \bar{X}(\theta_o) < \infty$ and $X_{f^k(\theta_o)} = X'_{f^k(\theta_o)} \geq 0$ such that

$$A'_{f^k(\theta_o)} X_{f^{k+1}(\theta_o)} A_{f^k(\theta_o)} - X_{f^k(\theta_o)} \leq -C'_{f^k(\theta_o)} C_{f^k(\theta_o)}. \tag{23}$$

Furthermore, if equation (23) is satisfied, then $\alpha(\theta)$ and $\beta(\theta)$ can be taken to only depend on the bound $\bar{X}(\theta_o)$ and on α_d and β_d in the definition of uniform detectability.

Proof. For $\theta(0)$ fixed, the resulting LDV system is a linear time varying system. Thus the theorem is simply a statement about the stability of linear time varying systems and can be found in [12]. ■

Corollary 8 *Assume (A, C, f) is uniformly detectable. Then there exist an $\alpha < 1$ and a $\beta < \infty$ such that*

$$\|x(k)\| \leq \beta\alpha^k\,\|x(0)\|$$

if and only if there exists a uniformly bounded function $X : \Theta \to \mathbb{R}^{n \times n}$ with $X_\theta = X'_\theta \geq 0$ such that for all $\theta \in \Theta$

$$A'_\theta X_{f(\theta)} A_\theta - X_\theta \leq -C'_\theta C_\theta. \tag{24}$$

Proof. Since X_θ is uniformly bounded and the system is uniformly detectable, proposition 7 can be applied at each $\theta \in \Theta$. ■

A. proof of proposition 2

Proof. Clearly, uniformly exponential stability implies exponential stability, which implies asymptotic stability. It remains to be shown that asymptotic stability implies uniformly exponential stability. Define $T(k,\theta(0))$ such that

$$x(k) = A_{\theta(k-1)}A_{\theta(k-2)}\cdots A_{\theta(0)}x(0) = T(k,\theta(0))\,x(0).$$

We claim that

$$\gamma = \sup_k \sup_{\theta\in\Theta} \|T(k,\theta)\| < \infty. \tag{25}$$

Suppose this is not true, i.e. $\gamma = \infty$. Define $T(N) = \sup_{\theta\in\Theta}\|T(N,\theta)\|$. Then $T(N) \to \infty$ as $N \to \infty$ and there is a sequence N_n such that $T(N_n) > T(M)$ for $M < N_n$ and $T(N_n)$ is monotone increasing to infinity. Since Θ is compact and $T(N_n,\theta)$ is continuous in θ, there exists a θ_n such that $\|T(N_n,\theta_n)\| = T(N_n)$. Since Θ is compact, $\{\theta_n\}$ contains a convergent subsequence $\theta_{n_m} \to \theta_o \in \Theta$. Thus,

$$\|T(N_{n_m},\theta_{n_m})\| = T(N_{n_m}) \geq T(M) = \sup_{\theta\in\Theta}\|T(M,\theta)\| \geq \|T(M,\theta)\| \text{ for all } M \leq N_{n_m},\ \theta\in\Theta \tag{26}$$

Since the system is asymptotically stable, if $\theta(0) = \theta_o$, then $\|x(k)\| \to 0$. Thus there exists a $P = N_k < \infty$ such that $\|T(P,\theta_o)\| < 1/4$. Since $P < \infty$, $T(P,\theta)$ is continuous in θ. Thus there is a $\delta > 0$ such that $\|\theta_o - \theta\| < \delta$ implies that $\|T(P,\theta_o) - T(P,\theta)\| < 1/4$. Since $\theta_{n_m} \to \theta_o$, there exists a $\overline{m} < \infty$ such that if $m \geq \overline{m}$ then $\|\theta_o - \theta_{n_m}\| < \delta$, $N_{n_m} > P$ and $\|T(P,\theta_o) - T(P,\theta_{n_m})\| < 1/4$. It follows that $\|T(P,\theta_{n_m})\| \leq 1/2$. However, $T(N_{n_m},\theta_{n_m}) = T(N_{n_m} - P, f^P(\theta_{n_m}))T(P,\theta_{n_m})$ so that

$$\|T(N_{n_m},\theta_{n_m})\| \leq \left\|T(N_{n_m} - P, f^P(\theta_{n_m}))\right\| \|T(P,\theta_{n_m})\|,$$

which implies that

$$2\,\|T(N_{n_m},\theta_{n_m})\| \leq \left\|T(N_{n_m} - P, f^P(\theta_{n_m}))\right\|.$$

This contradicts equation (26). Thus the claim (25) is true.

Since the system is asymptotically stable, for any $\theta_o \in \Theta$, if $\theta(0) = \theta_o$ then $\|x(k)\| \to 0$. Thus there is a $N(\theta_o) < \infty$ such that if $N \geq N(\theta_o)$, then $\|T(N,\theta_o)\| < \frac{1}{4\gamma}$, where γ is

given by equation (25). With $N(\theta_o) < \infty$, $T(N(\theta_o), \theta)$ is continuous in θ. Thus there exists a $\delta(\theta_o, N(\theta_o)) > 0$ such that if $\|\theta_o - \theta\| < \delta(\theta_o, N(\theta_o))$, then $\|T(N(\theta_o), \theta_o) - T(N(\theta_o), \theta)\| < \frac{1}{4\gamma}$. Therefore, $\|T(N(\theta_o), \theta)\| < \frac{1}{2\gamma}$, which implies that for $N \geq N(\theta_o)$

$$\|T(N, \theta)\| \leq \left\|T(N - N(\theta_o), f^{N(\theta_o)}(\theta))\right\| \|T(N(\theta_o), \theta)\| < \gamma \frac{1}{2\gamma} = \frac{1}{2}. \tag{27}$$

Since $\delta(\theta, N(\theta)) > 0$, the set $\{B(\theta, \delta(\theta, N(\theta))) \bigcap \Theta : \theta \in \Theta\}$ is an open covering of Θ. Since Θ is compact, there exists a finite subcovering, i.e. $\{B(\theta, \delta(\theta, N(\theta))) \bigcap \Theta : \theta \in I\}$ is a open covering and I is a finite set. Set $\overline{N} = \max_{\theta \in I} N(\theta) < \infty$. For all $\varphi \in \Theta$, there exists a $\theta \in I$ such that $\|\varphi - \theta\| < \delta(\theta, N(\theta))$; thus by (27)

$$\|T(N, \varphi)\| < \frac{1}{2} \text{ for } N \geq \overline{N}$$

and

$$\|T(mN, \varphi)\| < \left(\frac{1}{2}\right)^m \text{ for } N \geq \overline{N}$$

Define α and β by

$$\begin{aligned} 0 &< \alpha = \left(\frac{1}{2}\right)^{1/\overline{N}} < 1, \\ \beta &= \frac{1}{\alpha^{\overline{N}}} \max_{k \leq \overline{N}} \sup_{\theta \in \Theta} \|T(k, \theta)\| < \infty. \end{aligned}$$

Let $k = m\overline{N} + l$ with $l < \overline{N}$, then

$$\begin{aligned} \|T(k, \theta)\| &\leq \|T(m\overline{N}, \theta)\| \left\|T\left(l, f^{m\overline{N}}(\theta)\right)\right\| \\ &\leq \left(\frac{1}{2}\right)^m \left\|T\left(l, f^{m\bar{N}}(\theta)\right)\right\| \\ &\leq \left(\frac{1}{2}\right)^{\frac{m\overline{N}}{\bar{N}}} \beta\alpha^{\bar{N}} \leq \alpha^{m\overline{N}}\beta\alpha^l = \beta\alpha^k \end{aligned}$$

■

B. proof of theorem 5

The first, and most difficult, problem is to show that stabilizability implies that there exists a uniformly bounded $X \geq 0$ that solves the functional Riccati equation (9). The second problem is

to show that uniform detectability and a uniformly bounded $X \geq 0$ that solves (9) imply that the optimal control (10) is uniformly stabilizing and X is continuous.

To show that stabilizability implies existence of a solution to (9), the finite horizon time varying linear quadratic controller will be examined. By stabilizability, for each initial condition θ_0, this controller will be shown to exist and to be bounded along the trajectory $\left\{f^k(\theta_o)\right\}$. This in turn will imply that the infinite horizon time varying Riccati equation is actually of the form (9). Finally, it will be shown that the solution to (9) is uniformly bounded.

To show that a solution of (9) implies uniform stabilizability, standard techniques will be employed to show that the LDV controller is uniformly stabilizing. Lemma 13 will show that the positive semi-definite solution to (9) is unique. Finally, lemma 14 will show that the positive semi-definite solution to (9) is continuous and the finite horizon solution to the time varying Riccati equation (28) converges uniformly to the positive semi-definite solution to (9).

Lemma 9 *If assumptions 1 and 2 hold and C and D are bounded, then for each $\theta_o \in \Theta$ and $k < \infty$, there exists an optimal control $u(k) = F(k, \infty, \theta_o)x(k)$, where $F(k, \infty, \theta_o)$ is given by equations (33) and (34). Furthermore, this control is exponentially stabilizing and for each $\theta \in \Theta$, the cost of this feedback given by (33) is finite.*

Proof. Define the finite horizon cost to go

$$V(k, N, \theta_o, x_o, u) = x'(N)C'_{f^N(\theta_o)}C_{f^N(\theta_o)}x(N)$$
$$+\sum_{i=k}^{N-1} x'(i)C'_{f^i(\theta_o)}C_{f^i(\theta_o)}x(i) + u'(i)D'_{f^i(\theta_o)}D_{f^i(\theta_o)}u(i)$$

Dynamic programming arguments show that

$$\inf_u V(k, N, \theta_o, x_o, u) = V(k, N, \theta_o, x_o, u_N) = x'_k X(k, N, \theta_o)\, x_k.$$

where

$$X(k, N, \theta_o) = A'_{\theta(k)}X(k+1, N, \theta_o)\, A_{\theta(k)} + C'_{\theta(k)}C_{\theta(k)} \tag{28}$$
$$-A'_{\theta(k)}X(k+1, N, \theta_o)\, B_{\theta(k)} \left(D'_{\theta(k)}D_{\theta(k)} + B'_{\theta(k)}X(k+1, N, \theta_o)\, B_{\theta(k)}\right)^{-1}$$
$$\times B'_{\theta(k)}X(k+1, N, \theta_o)\, A_{\theta(k)}$$

subject to the terminal condition

$$X\left(N,N,\theta_o\right)=C'_{\theta(N)}C_{\theta(N)}. \tag{29}$$

By assumption 2, system (8) is stabilizable; thus, there exists a sequence of suboptimal feedback matrices $\left\{F_{\theta(k),\ldots\theta(1),\theta(0)}:k\geq 0\right\}$ such that, if

$$\overline{u}(k)=F_{\theta(k),\ldots\theta(1),\theta(0)}x(k), \tag{30}$$

then $\|x(k)\|\leq\beta(\theta(0))\left(\alpha(\theta(0))\right)^{k-j}\|x(j)\|$. The definition of stabilizability implies that $\|u(k)\|\leq\overline{F}(\theta(0))\|x(k)\|$, where $\overline{F}(\theta(0))<\infty$. Furthermore, C_θ and D_θ are continuous and therefore bounded as $\|C_\theta\|\leq\overline{C}<\infty$, $\|D_\theta\|\leq\bar{D}<\infty$. Thus, with the suboptimal feedback in place and $0\leq k<N$, it is not hard to show that the cost can be bounded as

$$\overline{V}(x_o,\theta_o)=\lim_{N\to\infty}V\left(0,N,\theta_o,x_o,\overline{u}\right)\leq\left(\overline{C}^2+\overline{F}^2(\theta_o))\overline{D}^2\right)\|x_o\|^2\frac{\beta^2(\theta_o)}{1-\alpha^2(\theta_o)}<\infty. \tag{31}$$

Similarly, the cost to go can be bounded as

$$V\left(k,N,\theta_o,x_o,u_N\right)\leq\lim_{N\to\infty}V\left(k,N,\theta_o,x_o,\overline{u}\right)\leq\left(\overline{C}^2+\overline{F}^2(\theta_o))\overline{D}^2\right)\|x\left(k\right)\|^2\frac{\beta^2(\theta_o)}{1-\alpha^2(\theta_o)} \tag{32}$$

where $\bar{u}$ is given by (30). Clearly $X\left(k,N,\theta_o\right)$ is monotone increasing and bounded as $N\to\infty$, so

$$X(k,\infty,\theta_o)=\lim_{N\to\infty}X(k,N,\theta_o)$$

exists. Therefore, allowing $N\to\infty$ in 28 yields

$$\begin{aligned}X\left(k,\infty,\theta_o\right) &= A'_{\theta(k)}X\left(k+1,\infty,\theta_o\right)A_{\theta(k)}+C'_{\theta(k)}C_{\theta(k)} \\ &\quad -A'_{\theta(k)}X\left(k+1,\infty,\theta_o\right)B_{\theta(k)}\left(D'_{\theta(k)}D_{\theta(k)}+B'_{\theta(k)}X\left(k+1,\infty,\theta_o\right)B_{\theta(k)}\right)^{-1} \\ &\quad \times B'_{\theta(k)}X\left(k+1,\infty,\theta_o\right)A_{\theta(k)}.\end{aligned} \tag{33}$$

For $k<\infty$, define

$$F(k,\infty,\theta_o)=-\left(D'_{\theta(k)}D_{\theta(k)}+B'_{\theta(k)}X(k,\infty,\theta_o)B_{\theta(k)}\right)^{-1}B'_{\theta(k)}X(k,\infty,\theta_o)A_{\theta(k)} \tag{34}$$

To prove that this feedback is exponentially stabilizing, observe that by standard manipulation we get

$$\left(A_{\theta(k)}+B_{\theta(k)}F_{\theta(k)}\right)'X\left(k+1,\infty,\theta_o\right)\left(A_{\theta(k)}+B_{\theta(k)}F_{\theta(k)}\right)-X\left(k,\infty,\theta_o\right) \tag{35}$$

$$= -\begin{bmatrix} C'_{\theta(k)} & F'_{\theta(k)} \left(D'_{\theta(k)} D_{\theta(k)}\right)^{1/2} \end{bmatrix} \begin{bmatrix} C_{\theta(k)} \\ \left(D'_{\theta(k)} D_{\theta(k)}\right)^{1/2} F_{\theta(k)} \end{bmatrix}$$

Furthermore, $\left(A + BF, \begin{bmatrix} C \\ (DD)^{1/2} F \end{bmatrix}, f \right)$ is uniformly detectable, since

$$A + BF + \begin{bmatrix} H & -B\left(D'D\right)^{-1/2} \end{bmatrix} \begin{bmatrix} C \\ (DD)^{1/2} F \end{bmatrix} = A + HC$$

is uniformly exponentially stable for the output injection feedback H given by the uniform detectability assumption. Since $X(k, \infty, \theta_o)$ is bounded, theorem 7 applied to (35) implies that the closed loop system is exponentially stable. ■

Remark 5 *This lemma is nothing more than the infinite horizon, time-varying linear quadratic control along a trajectory (for more details on time-varying optimal control see [1] or [11].) However, we have not shown that the closed loop system is* uniformly *exponentially stable.*

The above lemma implies that for all $\theta \in \Theta$,

$$X_\theta = \lim_{N \to \infty} X(0, N, \theta).$$

exists and is finite, although we have not yet proved that it is uniformly bounded. Therefore X is a function $X : \Theta \to \mathbb{R}^{n \times n}$. This function is a solution to a functional Riccati equation.

Lemma 10 *If assumptions 1 and 2 hold, the function*

$$X : \Theta \to \mathbb{R}^{n \times n}$$

$$\theta \mapsto X_\theta$$

satisfies

$$X_\theta = A'_\theta X_{f(\theta)} A_\theta - A'_\theta X_{f(\theta)} B_\theta \left(R_\theta + B'_\theta X_{f(\theta)} B_\theta\right)^{-1} B'_\theta X_{f(\theta)} A_\theta + C'_\theta C_\theta. \tag{36}$$

Proof. Direct manipulation of equation 33. ■

As we show below, there is a bound on $X(0, N, \theta)$ that does not depend on θ or N, that is, $X(0, N, \theta)$ is uniformly bounded. This feature of stabilizable continuous LDV systems is surprising.

It means that, although the stabilizability assumption only implies that there exists a controller with possibly unbounded cost, the compactness and continuity assumptions of LDV systems imply that there exists a control such that the cost to stabilize any trajectory is uniformly bounded.

Lemma 11 *If assumptions 1,2 and 3 hold, then $X(0,N,\theta)$ is uniformly bounded.*

Proof. Suppose $X(0,N,\theta)$ is not uniformly bounded. Define $\overline{X}(N) = \sup_{\theta\in\Theta} \|X(0,N,\theta)\|$. Since $N < \infty$, $X(0,N,\theta)$ is continuous in θ. Since Θ is compact, there exists a $\theta(N)$ such that $\|X(0,N,\theta(N))\| = \overline{X}(N)$. Since, by our supposition, $\lim_{N\to\infty} \overline{X}(N) = \infty$, the sequence $\theta(N)$ is such that $\|X(0,N,\theta(N))\| \to \infty$. Since Θ is compact, there exists a subsequence, $\theta(N_n)$, such that $\theta(N_n) \to \theta_o \in \Theta$. Define $\theta_n = \theta(N_n)$. Then $\|X(0,N_n,\theta_n)\| = \overline{X}(N_n)$; thus for all $\theta \in \Theta$, $\|X(0,N_n,\theta_n)\| \geq \|X(0,N_n,\theta)\|$.

By lemma 9, $\|X(0,\infty,\theta)\| < \infty$ for all $\theta \in \Theta$. In particular, $\|X(0,\infty,\theta_o)\| < \infty$. Furthermore, it is assumed that $\|X(0,N_n,\theta_n)\| \to \infty$ monotonically as $n \to \infty$. Thus there exists a $\overline{n}_1 < \infty$ such that, if $n_1 \geq \overline{n}_1$ then

$$4\|X(0,\infty,\theta_o)\| \leq \|X(0,N_{n_1},\theta_{n_1})\| < \infty \tag{37}$$

Lemma 9 implies that the closed loop system is exponentially stable. Thus, with the feedback in place, there is a $P < \infty$ such that for $\theta(0) = \theta_o$, we have $\|x(P)\| < \frac{1}{4}\|x_o\|$. Define

$$U_{\theta_0}(k,\theta) = \prod_{l=0}^{k-1} \left(A_{f^l(\theta)} + B_{f^l(\theta)} F_{f^l(\theta_o)}\right) \tag{38}$$

where the factors of the product are taken in the correct order. Then $U_{\theta_o}(k,\theta)$ is the state transition matrix using the feedback $F_{f^k(\theta_o)}$. When $\theta(0) \neq \theta_o$ the feedback in not optimal, but when $\theta(0) = \theta_o$, the feedback is optimal and

$$\|U_{\theta_o}(P,\theta_o)\| < \frac{1}{4}.$$

Since $P < \infty$, $U_{\theta_o}(P,\theta)$ is continuous in θ. Since $\theta_n \to \theta_o$, there exists a $\overline{n}_2 < \infty$ such that if $n_2 \geq \overline{n}_2$, then

$$\|U_{\theta_o}(P,\theta_o) - U_{\theta_o}(P,\theta_{n_2})\| < \frac{1}{4}.$$

Thus

$$\|U_{\theta_o}(P,\theta_{n_2})\| < \frac{1}{2}. \tag{39}$$

Define

$$\begin{aligned} W_{\theta_o}(0,N,\theta) &= \sum_{k=0}^{N-1} \Big(U'_{\theta_o}(k,\theta) C'_{f^k(\theta)} C_{f^k(\theta)} U_{\theta_o}(k,\theta) \\ &\quad + U'_{\theta_o}(k,\theta) F'_{f^k(\theta_o)} D'_{f^k(\theta)} D_{f^k(\theta)} F_{f^k(\theta_o)} U_{\theta_o}(k,\theta) \Big). \end{aligned} \tag{40}$$

Thus $x'_o W_{\theta_0}(0,N,\theta)x_o + x'_o U'_{\theta_o}(N,\theta) C'_{f^N(\theta)} C_{f^N(\theta)} U_{\theta_o}(N,\theta)x_o$ is the finite horizon cost using the feedback matrix $F_{f^k(\theta_o)}$. This is the optimal cost when $\theta(0)=\theta_o$ and $N=\infty$, so that $x'_o W_{\theta_o}(0,\infty,\theta_o)x_o = x'_o X(0,\infty,\theta_o)x_o$. Since $P<\infty$, $W_{\theta_o}(0,P,\theta)$ is continuous in θ. Thus there exists a $\overline{n}_3 < \infty$ such that $n_3 \geq \overline{n}_3$ implies that

$$\|W_{\theta_o}(0,P,\theta_{n_3}) - W_{\theta_o}(0,P,\theta_o)\| \leq \|X(0,\infty,\theta_o)\|$$

which implies that

$$\begin{aligned} \|W_{\theta_o}(0,P,\theta_{n_3})\| &\leq \|W_{\theta_o}(0,P,\theta_{n_3}) - W_{\theta_o}(0,P,\theta_o)\| + \|W_{\theta_o}(0,P,\theta_o)\| \\ &\leq \|X(0,\infty,\theta_o)\| + \|X(0,\infty,\theta_o)\| = 2\|X(0,\infty,\theta_o)\| \end{aligned} \tag{41}$$

Set $n = \max(\overline{n}_1, \overline{n}_2, \overline{n}_3, P)$ and let x_o be such that $\|x_o\|=1$ and $\|X(0,N_n,\theta_n)\| = x'_o X(0,N_n,\theta_n)x_o$.

Equations (37) and (41) imply that

$$x'_o W_{\theta_o}(0,P,\theta_n)x_o \leq 2x'_o X(0,\infty,\theta_0)x_o \leq \frac{1}{2}x'_o X(0,N_n,\theta_n)x_o.$$

Thus

$$\begin{aligned} & x'_o X(0,N_n,\theta_n)x_o - x'_o W_{\theta_o}(0,P,\theta_n)x_o \\ \geq\ & x'_o X(0,N_n,\theta_n)x_o - \frac{1}{2}x'_o X(0,N_n,\theta_n)x_o \\ =\ & \frac{1}{2}x'_o X(0,N_n,\theta_n)x_o \end{aligned} \tag{42}$$

Now, if we don't use the optimal control for the first P steps, our cost must be at least as high as the optimal cost, that is, (recalling $V(0,N_n,\theta_n,x_o,u)$ is the cost with initial conditions θ_n and

x_o using control u for N_n time steps)

$$\begin{aligned} & \min_u V(0, N_n, \theta_n, x_o, u) \\ \leq \quad & x_o' W_{\theta_o}(0, P, \theta_n) x_o + \min_u V(0, N_n - P, f^P(\theta_n), U_{\theta_o}(P, \theta_n) x_o, u) \end{aligned}$$

or

$$\begin{aligned} & \min_u V(0, N_n, \theta_n, x_o, u) = x_o' X(0, N_n, \theta_n) x_o \\ \leq \quad & x_o' W_{\theta_o}(0, P, \theta_n) x_o + x_o' U_{\theta_o}'(P, \theta_n) X\left(0, N_n - P, f^P(\theta_n)\right) U_{\theta_o}(P, \theta_n) x_o \end{aligned}$$

or

$$\begin{aligned} & x_o' X(0, N_n, \theta_n) x_o - x_o' W_{\theta_o}(0, P, \theta_n) x_o \qquad (43) \\ \leq \quad & x_o' U_{\theta_o}'(P, \theta_n) X\left(0, N_n - P, f^P(\theta_n)\right) U_{\theta_o}(P, \theta_n) x_o \end{aligned}$$

Thus combining equations (39), (42) and (43) yields

$$\begin{aligned} & \frac{1}{2} x_o' X(0, N_n, \theta_n) x_o \qquad (44) \\ \leq \quad & x_o' X(0, N_n, \theta_n) x_o - x_o' W_{\theta_o}(0, P, \theta_n) x_o \\ \leq \quad & x_o' U_{\theta_o}'(P, \theta_n) X\left(0, N_n - P, f^P(\theta_n)\right) U_{\theta_o}(P, \theta_n) x_o \\ < \quad & \frac{1}{4} \left\| X\left(0, N_n - P, f^P\left(\theta_n\right)\right) \right\|. \end{aligned}$$

Thus the contracting hypothesis leads to

$$2 \left\| X(0, N_n, \theta_n) \right\| < \left\| X(0, N_n - P, f^P(\theta_n)) \right\|$$

On the other hand we obviously have

$$\left\| X(0, N_n, \theta_n) \right\| \geq \left\| X(0, N_n - P, f^P(\theta_n) \right\|.$$

Thus there is a clear contradiction between the above two inequalities. Therefore $X(0, N, \theta)$ is uniformly bounded. ■

Lemmas 9-11 show that stabilizability implies that there exists a uniformly bounded solution to functional equation (36).

Now we show the converse, that is, if $X \geq 0$ solves (36) and is bounded, then the LDV system (8) is stabilizable. In fact, it is uniformly stabilizable and X is continuous.

With the solution X to (36), we construct the optimal feedback

$$F_\theta = -\left(R_\theta + B_\theta' X_{f(\theta)} B_\theta\right)^{-1} B_\theta' X_{f(\theta)} A_\theta \tag{45}$$

and we prove the following

Lemma 12 *Suppose assumptions 1 and 3 hold, and $X : \Theta \to \mathbb{R}^{n\times n}$ is a uniformly bounded solution to the functional algebraic Riccati equation (36) such that $X_\theta = X_\theta' \geq 0$. If $u(k) = F_{\theta(k)}x(k)$, where F_θ is given by equation (45), then the resulting closed loop system is uniformly exponentially stable. Therefore, system (8) is stabilizable.*

Proof. Standard manipulation shows that

$$\begin{aligned} & \left(A_\theta + B_\theta F_\theta\right)' X_{f(\theta)} \left(A_\theta + B_\theta F_\theta\right) - X_\theta \\ = & -\left[\begin{array}{cc} C_\theta' & F_\theta' \left(D_\theta' D_\theta\right)^{1/2} \end{array}\right] \left[\begin{array}{c} C_\theta \\ \left(D_\theta' D_\theta\right)^{1/2} F_\theta \end{array}\right] \end{aligned} \tag{46}$$

where $\left(A + BF, \left[\begin{array}{c} C \\ \left(D'D\right)^{1/2} F \end{array}\right], f\right)$ is uniformly detectable. Since X is uniformly bounded, corollary 8 implies that the closed loop system is uniformly exponentially stable. ■

Remark 6 *If the assumptions 1,2 and 3 hold, lemmas 9-11 imply that there exists a solution X to equation (36) such that $X_\theta' = X_\theta \geq 0$. Thus lemma 12 implies that the resulting closed-loop system is uniformly exponentially stable.*

Lemma 13 *Suppose assumptions 1 and 3 hold. If a solution X to equation (36) exists with $X_\theta = X_\theta' \geq 0$, $\|X_\theta\| < \overline{X} < \infty$, then it is unique.*

Proof. By standard results of linear quadratic control of time-varying systems,the time-varying Riccati equation associated with the linear time-varying system

$$x\left(k+1\right) = A_{f^k(\theta_o)} x\left(k\right) + B_{f^k(\theta_o)} u\left(k\right)$$

$$z(k) = \begin{bmatrix} C_{f^k(\theta_o)} x(k) \\ D_{f^k(\theta_o)} u(k) \end{bmatrix}$$

is unique. Since the solution X to the functional Riccati equation (36) coincides with the solution to the time varying Riccati equation, X must also be unique. ■

Lemma 14 *Suppose assumptions 1 and 3 hold and there exists a solution X to equation (36) such that $X_\theta = X'_\theta \geq 0$. Suppose that there exists a $\overline{X} < \infty$ such that $\|X(0,k,\theta)\| \leq \overline{X}$ for all $\theta \in \Theta$ and k. Then X is uniformly continuous and $X(0,k,\theta) \to X_\theta$ uniformly.*

Proof. Suppose X is not continuous. Then there exists a $\theta_o \in \Theta$, an x_o with $\|x_o\| = 1$, an $\varepsilon > 0$ and a sequence $\{\theta_n\}$ with $\theta_n \to \theta_o$, such that

$$x'_o X_{\theta_o} x_o > x'_o X_{\theta_n} x_o + \varepsilon \text{ for all } n \tag{47}$$

or

$$x'_o X_{\theta_o} x_o < x'_o X_{\theta_n} x_o - \varepsilon \text{ for all } n \tag{48}$$

Suppose equation (48) is true. Lemma 12 implies that the closed loop system is asymptotically stable; thus there exists a $N < \infty$ such that $\|U_{\theta_o}(N,\theta_o)\| < \frac{1}{2}\sqrt{\frac{\varepsilon}{4\overline{X}}}$, where U is defined by equation (38). Since $N < \infty$, $U_{\theta_o}(N,\theta)$ is continuous in θ. Thus there is a $n_1 < \infty$ such that if $n \geq n_1$, then $\|U_{\theta_o}(N,\theta_n) - U_{\theta_o}(N,\theta_o)\| < \frac{1}{2}\sqrt{\frac{\varepsilon}{4\overline{X}}}$ thus $\|U_{\theta_o}(N,\theta_n)\| < \sqrt{\frac{\varepsilon}{4\overline{X}}}$. Likewise, $W_{\theta_o}(0,N,\theta)$ is continuous in θ, where W is defined by equation (40). Thus there is a $n_2 < \infty$ such that if $n \geq n_2$, then $\|W_{\theta_o}(0,N,\theta_o) - W_{\theta_o}(0,N,\theta_n)\| < \frac{\varepsilon}{4}$. Set $n \geq \max(n_1, n_2)$. If we use the feedback $u(k) = F_{f^k(\theta_o)} x(k)$ for $k \leq N$, and the optimal feedback for $k > N$, then the cost will be at least as large as the optimal cost. That is,

$$\begin{aligned} x'_o X_{\theta_n} x_o &\leq x'_o W_{\theta_o}(0,N,\theta_n) x_o + x'_o U'_{\theta_o}(N,\theta_n) X_{f^N(\theta_n)} U(N,\theta_n) x_o \\ &\leq x'_o W_{\theta_o}(0,N,\theta_o) x_o + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\ &\leq x'_o X_{\theta_o} x_o + \frac{\varepsilon}{2} < x'_o X_{\theta_n} x_o - \frac{\varepsilon}{2} \end{aligned}$$

where the last inequality follows from equation (48). Thus $x'_o X_{\theta_n} x_o < x'_o X_{\theta_n} x_o - \frac{\varepsilon}{2}$, which is impossible; thus equation (48) must not hold.

Similarly, suppose equation (47) holds. By lemma (12), $\|U_\theta(k,\theta)\| \leq \beta\alpha^k$ for some $\beta < \infty$ and $0 < \alpha < 1$. Thus there exists a $\overline{N} < \infty$ such that for all $\theta \in \Theta$ and $N \geq \overline{N}$, $\|U_\theta(N,\theta)\| \leq \sqrt{\frac{\varepsilon}{2\overline{X}}}$. By assumption 3, A and B are bounded from above and R is bounded from below. Thus, since $\|X_\theta\| \leq \overline{X}$, there exists a $\overline{F} < \infty$ such that $\|F_\theta\| \leq \overline{F}$ for all $\theta \in \Theta$. Therefore F_{θ_n} contains a convergent subsequence, $\left\{F_{\theta_{n_m}}\right\}$. Let $\tilde{F}_1 = \lim_{m\to\infty} F_{\theta_{n_m}}$. Similarly, there exists a sub-subsequence $\theta_{n_{m_l}} \to \theta_o$ such that $\tilde{F}_2 = \lim_{l\to\infty} F_{f(\theta_{n_{m_l}})}$ exists. In this fashion, define $\tilde{K}_k$ for $k \leq N$. Redefine $\{\theta_n\}$ to be the subsequence such that $\lim_{n\to\infty} F_{f^k(\theta_n)} = \tilde{K}_k$ for $k \leq N$. Let $P(k) = \prod_{j=0}^{k-1}\left(A_{f^j(\theta_o)} + B_{f^j(\theta_o)}\tilde{K}_j\right)$, then $U_{\theta_n}(k,\theta_n) \to P(k)$ for $k \leq N$. Since $\|U_{\theta_n}(N,\theta_n)\| \leq \sqrt{\frac{\varepsilon}{2\overline{X}}}$, $\|P(N)\| \leq \sqrt{\frac{\varepsilon}{2\overline{X}}}$. Furthermore, since the feedback $\tilde{K}_k$ is not necessarily optimal,

$$\begin{aligned}
x_o' X_{\theta_o} x_o &\leq x_o'\left(\sum_{k=0}^{N-1} P'(k)C'_{f^k(\theta_o)}C_{f^k(\theta_o)}P(k) + P'(k)\tilde{K}_k' R_{f^k(\theta_o)}\tilde{K}_k P(k)\right)x_o \\
&\quad + x_o' P'(N) X_{f^N(\theta_o)} P(N) x_o \\
&\leq \lim_{n\to\infty} x_o' \sum_{k=0}^{N-1}\Big(U'_{\theta_n}(k,\theta_n)C'_{f^k(\theta_o)}C_{f^k(\theta_o)}U_{\theta_n}(k,\theta_n) \\
&\quad + U'_{\theta_n}(k,\theta_n)F'_{f^k(\theta_n)}R_{f^k(\theta_o)}F_{f^k(\theta_n)}U_{\theta_n}(k,\theta_n)\Big)x_o + \frac{\varepsilon}{2} \\
&\leq \limsup_{n\to\infty} x_o' X_{\theta_n} x_o + \frac{\varepsilon}{2} < x_o' X_{\theta_o} x_o - \frac{\varepsilon}{2}
\end{aligned}$$

where the last inequality follows from equation (47). Thus equation (47) must be false. Therefore, X is continuous. Since Θ is compact, X is uniformly continuous.

Since X is continuous, $X(0,N,\theta)$ is continuous, $X(0,N,\theta) \to X_\theta$ monotonically and Θ is compact, it follows that $X(0,N,\theta) \to X_\theta$ uniformly. For details see [17]. ■

C. proof of theorem 6

We prove that for $\varepsilon > 0$ there exists a $\delta > 0$ such that if $\|f^N(\theta) - \theta\| < \delta$ for some N, then $\|X_\theta - Y\| < \varepsilon$ where X solves the FARE and Y solves (15). Since θ is recurrent, given $\delta > 0$, there exists an $N < \infty$ such that $\|f^N(\theta) - \theta\| < \delta$. It is assumed that f is a diffeomorphism and that $\overline{R(f)}$ is structurally stable, i.e., for $\nu > 0$ there exists a $\delta > 0$ such that if $d_{C^0}\left(f,\tilde{f}\right) < \delta$ then $d_H\left(R(f), R\left(\tilde{f}\right)\right) < \nu$, where $d_H(\cdot,\cdot)$ is the Hausdorff metric and $d_{C^0}\left(f,\tilde{f}\right) = \sup_\theta \left\|f(\theta) - \tilde{f}(\theta)\right\|$. Note that if $\overline{R(f)}$ is attracting, then structural stability of

$\overline{R(f)}$ is a generic property [20]. Furthermore, the closure of the recurrent set of a hyperbolic system is structurally stable [16].

Lemma 15 *Let f be a diffeomorphism and θ be a recurrent point. For any $\delta > 0$ there exist an $N < \infty$ and a continuous $\tilde{f}$ such that $\tilde{f}^N(\theta) = \theta$ and $d_{C^0}\left(f, \tilde{f}\right) < \delta$.*

Proof. Let K be a connected compact set with $\Theta \subsetneqq K$. $f \in C^0$ implies that there exists a γ such that $\delta > \gamma > 0$, and for $a, b \in K$,

$$\|a - b\| < \gamma \text{ implies that } \|f(a) - f(b)\| < \delta. \tag{49}$$

Similarly, $f^{-1} \in C^0$ implies that there exists a ν with $\delta > \nu > 0$ such that for $\theta, \varphi \in K$

$$\|\theta - \varphi\| < \nu \text{ implies that } \left\|f^{-1}(\theta) - f^{-1}(\varphi)\right\| < \frac{\gamma}{3}. \tag{50}$$

Now, since θ is recurrent, there exists an $N < \infty$ such that $\left\|f^N(\theta) - \theta\right\| < \nu$. Thus relation (50) implies that $\left\|f^{-1}(\theta) - f^{N-1}(\theta)\right\| < \frac{\gamma}{3}$. Let M be a smooth curve connecting $f^{-1}(\theta)$ and $f^{N-1}(\theta)$ such that if $a, b \in M$, then $\|a - b\| < \gamma\frac{2}{3}$ and $\min_{k=0,\cdots N-2} d\left(f^k(\theta), M\right) > \gamma_2 > 0$. Let $\gamma_3 = \min\left(\frac{\gamma}{3}\frac{1}{2}, \frac{\gamma_2}{2}\right)$. The C^0 closing lemma [20] implies that there exists a $\tilde{f} \in C^0$ such that:

1. $\tilde{f}(\varphi) = f(\varphi)$ for $\varphi \notin B(M, \gamma_3)$ where $B(M, \gamma_3)$ is the open γ_3-ball around M.
2. $\tilde{f}\left(f^{N-1}(\theta)\right) = \theta$ thus $\tilde{f}^N(\theta) = \theta$.
3. $d_{C^0}\left(\tilde{f}, f\right) < \max\left(diam\left(B(M, \gamma_3)\right), diam\left(f\left(B(M, \gamma_3)\right)\right)\right)$.

Since $diam(M) < \gamma\frac{2}{3}$, and $\gamma_3 \leq \frac{\gamma}{3}\frac{1}{2}$, we have $diam\left(B(M, \gamma_3)\right) < \gamma < \delta$. And since $diam\left(B(M, \gamma_3)\right) < \gamma$, relation (49) implies that $diam\left(f\left(B(M, \gamma_3)\right)\right) < \delta$. It follows that $d_{C^0}\left(\tilde{f}, f\right) < \delta$ as desired.

■

Proposition 16 *Let $f \in C^1$. Assume that the LDV system induced by f is stabilizable, (A, C, f) is uniformly detectable, $D_\theta' D_\theta > 0$ for all $\theta \in \Theta$, and C and D are continuous. Then for all $\varepsilon > 0$, there exists a $\delta > 0$ such that if*

$$d_{C^0}\left(f, \tilde{f}\right) + d_H\left(\Theta, \tilde{\Theta}\right) < \delta,$$

then

$$\left\| X_\theta - \tilde{X}_{\arg\min\{\|\tilde{\theta}-\theta\|:\tilde{\theta}\in\tilde{\Theta}\}} \right\| < \varepsilon \text{ for all } \theta \in \Theta$$

and

$$\left\| \tilde{X}_{\tilde{\theta}} - X_{\arg\min\{\|\tilde{\theta}-\theta\|:\theta\in\Theta\}} \right\| < \varepsilon \text{ for all } \tilde{\theta} \in \tilde{\Theta},$$

where X is the positive semi-definite solution to the FARE induced by (A, B, C, D, f) and $\tilde{X}$ is the solution induced by $\left(A, B, C, D, \tilde{f}\right)$ and $\tilde{\Theta}$ is an $\tilde{f}$-invariant set.

Proof. See [7]. ■

The above proposition implies that if $\varepsilon > 0$, $\overline{R(f)} = \Theta$ and $\overline{R(f)}$ is structurally stable, there exists a $\delta > 0$ such that if $d_{C^0}\left(f, \tilde{f}\right) < \delta$ then $\left\| X_\theta - \tilde{X}_{\arg\min\{\|\tilde{\theta}-\theta\|:\tilde{\theta}\in\tilde{\Theta}\}} \right\| < \varepsilon$. Setting $\theta \in R(f)$, lemma 15 provides an $N < \infty$ such that there exists an $\tilde{f}$ such that $d_{C^0}\left(f, \tilde{f}\right) < \delta$ with $\tilde{f}^N(\theta) = \theta$. Therefore, $\tilde{X}_\theta = \rho_\theta \circ \rho_{\tilde{f}^1(\theta)} \circ \cdots \rho_{\tilde{f}^{N-1}}\left(\tilde{X}_\theta\right)$ where $\tilde{X}$ solves the FARE associated with the LDV system $\left(A, B, C, D, \tilde{f}\right)$. But since $\tilde{f}^k(\theta) = f^k(\theta)$ for $k \leq N-1$, we have $\tilde{X}_\theta = \rho_\theta \circ \rho_{f^1(\theta)} \circ \cdots \rho_{f^{N-1}(\theta)}\left(\tilde{X}_\theta\right)$, that is, $\tilde{X}_\theta$ solves (15) and $\left\| X_\theta - \tilde{X}_\theta \right\| < \varepsilon$. ■

Remark 7 *In numerical analysis language, lemma 15 is the numerical stability of the algorithm - the computed fixed point θ is the exact fixed point for a nearby function. Proposition 16 is the numerical conditioning of the algorithm - a small perturbation of the function f results in a small perturbation of the Riccati solution.*

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