Structural Stability of Linear Dynamically Varying (LDV) Controllers

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Abstract

LDV systems are linear systems with parameters that varying according to a nonlinear dynamical system. This paper examines the robust stability of such systems in the face of perturbations of the nonlinear system. Three classes of perturbation are examined; differentiable functions, Lipschitz continuous functions and continuous functions. It is found that in the first two cases the system remain stable. Whereas, if the perturbation are among continuous functions, the closed-loop may not be asymptotically stable, but, instead, is asymptotically bounded with the diameter of the residual set bounded by a function that is continuous in the size of the perturbation. It is also shown that in the case of differential perturbations, the resulting optimal LDV controller is continuous in the size of the perturbation. An example is presented that illustrates the continuity of the variation of the controller in the case of a non-structurally stable dynamical system.

1 Introduction

Linear parametrically varying (LPV) systems have been the focus of extensive research [8], [1], [14]. Essentially, a LPV system is a linear system with parameters that may vary over some set. This paper examines the specific case where the variation of the parameters is described by a given dynamical system. Such systems are known as *linear dynamically varying* (LDV) systems and have applications in nonlinear tracking. A LDV system can be decomposed into two subsystems; a linear system and a nonlinear system, where the nonlinear system drives the parameters of the linear system. While both linear quadratic [2] and H^{∞} [3] controllers have been developed for LDV systems, some questions regarding the robustness of the closed loop system remain unanswered. In the case of time-invariant linear systems, robust stability refers to the stability in the face of some uncertain parameter. While such concerns are valid for LDV systems, they can be handled in much the same way as they are in the case of time-invariant linear systems. However, stability in the face of uncertainty in the nonlinear subsystem is a unique concern to LDV systems and is the subject of this paper. Specifically, the variation of the parameters of the linear system is given by nonlinear system $\theta(k+1) = f(\theta(k))$. It will be shown that a stable LDV system remains stable in the face of small perturbation of f. Three cases are examined; where the perturbations are over C^1functions, Lipschitz continuous functions and C^0 functions. In the first two cases it is shown that stability is maintained. In the case of continuous perturbations, it is not possible to guarantee asymptotic stability. Instead, it is shown that the system is asymptotically bounded, i.e. the state of the linear system converges to a small neighborhood of the origin.

A related issue is the variation of the optimal LDV controller due to variation of the dynamical system f. This issue is the structural stability of the LDV controller. It will shown that the linear quadratic controller is structurally stable. This type of stability has applications in the development of computational methods for LDV controllers. As discussed in [2], an efficient method to compute the controller relies on making a small perturbation in the dynamical system f and finding the controller for this perturbed system. The controller for this nearby system is easily found. The question as to whether the controller of nearby system is an approximation of the controller for the original system is answered affirmatively in this paper.

The paper proceeds as follows: Section 2 formalizes LDV systems, briefly reviews previous results necessary for this paper and state some required definitions. Section 3 discusses conjugacy. Section 4 presents the main results of the paper. Finally, Section 5 presents an example of a structurally unstable system with a structurally stable LDV controller.

2 Background

An LDV system is defined as

$$\begin{aligned} x(k+1) &= A_{\theta(k)}x(k) + B_{\theta(k)}u(k) \\ z(k) &= \begin{bmatrix} C_{\theta(k)}x(k) \\ D_{\theta(k)}u(k) \end{bmatrix} \\ \theta(k+1) &= f(\theta(k)) \qquad \text{with } x(0) = x_o,\ \theta(0) = \theta_o \end{aligned} \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous map with $f(S) = S$, with S a compact set, $A : \mathbb{R}^n \to \mathbb{R}^{n\times n}$, $B : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, $C : \mathbb{R}^n \to \mathbb{R}^{p\times n}$, and $D : \mathbb{R}^n \to \mathbb{R}^{p\times m}$. A continuous LDV is an LDV where the maps A, B, C and D are continuous. This paper only considers continuous LDV systems. The pair (A, f) is *exponentially stable* if system (1) is exponentially stable. That is, for $u = 0$ and $\theta_o \in S$ there exists an $\alpha(\theta_o) < 1$ and a $\beta(\theta_o) < \infty$ such that if $\theta(0) = \theta_o$, then $\|x(k)\| < \beta(\theta_o)\alpha(\theta_o)^k \|x(0)\|$. Similarly, the pair (A, f) is *uniformly exponentially stable* if the pair (A, f) is exponentially stable and α and β can be chosen independent of $\theta(0)$. The triple (A, B, f) is *stabilizable* if there exists a bounded feedback $F : S \to \mathbb{R}^{m\times n}$ such that $(A + BF, f)$ is exponentially stable. The triple

(A,C,f) is *uniformly detectable* if there is a uniformly bounded map $H : S \to R^{n\times p}$ such that $(A+HC,f)$ is uniformly exponentially stable, that is, there exists $\alpha_d < 1$ and $\beta_d < \infty$ such that $\|x(k)\| < \beta\alpha^k \|x(0)\|$ where $x(k+1) = \left(A_{f^k(\theta_o)} + H_{f^k(\theta_o)} C_{f^k(\theta_o)}\right) x(k)$.

LDV systems naturally arise when controlling nonlinear dynamical systems. Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $f(S,0) \subset S$, $f \in C^1$ and let S be compact. Consider the nonlinear tracking problem

$$\begin{aligned} \varphi(k+1) &= f(\varphi(k),u(k)), \qquad & \varphi(0) &= \varphi_o, \\ \theta(k+1) &= f(\theta(k),0), \qquad & \theta(0) &= \theta_o \in S. \end{aligned} \tag{2}$$

The objective is to find a control u such that $\|\varphi(k) - \theta(k)\| \to 0$. In this context $\theta(k)$ is the desired trajectory and φ is the state of the system under control. Define the tracking error $x(k) = \varphi(k) - \theta(k)$. Then system (2) becomes

$$\begin{aligned} & x(k+1) = f(\varphi(k),u(k)) - f(\theta(k),0)) = \\ & A_{\theta(k)}x(k) + B_{\theta(k)}u(k) + \eta_x(x(k),u(k),\theta(k))\,x(k) + \eta_u(x(k),u(k),\theta(k))\,u(k), \end{aligned} \tag{3}$$

where $(A_\theta)_{i,j} = \frac{\partial f_i}{\partial \theta_j}(\theta,0)$, $(B_\theta)_{i,j} = \frac{\partial f_i}{\partial u_j}(\theta,0)$ and $\eta_x(x,u,\theta)\,x + \eta_u(x,u,\theta)\,u$ accounts for the high order terms. Since $f \in C^1$, if x and u are small, then $\eta_x(x,u,\theta)$ and $\eta_u(x,u,\theta)$ are small. Thus, when x and u are small, system (3) is well approximated by system (1) with $f(\theta) := f(\theta,0)$. In this case the LDV is said to be induced by f. It was shown in [2] and [3] that if the LDV system (1) induced by f with control u is uniformly exponentially stable, then the nonlinear system (2), with control u, is locally uniformly exponentially stable. By definition *locally uniformly exponentially stable* means that there exist $\alpha < 1$, $\beta < \infty$ and $\gamma > 0$ such that if $\|x(0)\| = \|\varphi(0) - \theta(0)\| < \gamma$ then $\|x(k)\| < \beta\alpha^k \|x(0)\|$ where α, β and γ can be taken independent of the initial condition θ_o, i.e. uniformly in θ_o and locally in x. Thus, if the LDV system induced by the nonlinear dynamically system f is LDV stabilizable, then the LDV system $\left(\frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial u}, f\right)$ is stabilizable.

The following theorems are needed in the sequel:

Suppose (A,C,f) is uniformly detectable. Then (A,f) is uniformly exponentially stable if and only if there exists a bounded function $X : S \to \mathbb{R}^{n\times n}$ with $X'_\theta = X_\theta \geq 0$ such that $A'_\theta X_{f(\theta)} A_\theta - X_\theta \leq -C'_\theta C_\theta$. Moreover, α and β in the definition of uniformly exponentially stable can be chosen to depend only on the bound on X and α_d and β_d in the definition of detectability.

This is a simple extension of theorem 7.1 in [4]

Suppose (1) is a continuous LDV system. If (A,B,f) is stabilizable, (A,C,f) is uniformly detectable and $D'_\theta D_\theta > 0$ for $\theta \in S$, then there exists a bounded and continuous function $X : S \to$

$\mathbb{R}^{n\times n}$ with $X_\theta' = X_\theta \geq 0$ and satisfying the functional discrete time algebraic Riccati equation

$$X_\theta = A_\theta' X_{f(\theta)} A_\theta + C_\theta' C_\theta - A_\theta' X_{f(\theta)} B_\theta \left(D_\theta' D_\theta + B_\theta' X_{f(\theta)} B_\theta\right)^{-1} B_\theta' X_{f(\theta)} A_\theta \tag{4}$$

The control $u_{LQ}(k) = F_{\theta(k)} x\left(k\right) := -\left(D'_{\theta(k)} D_{\theta(k)} + B'_{\theta(k)} X_{f(\theta(k))} B_{\theta(k)}\right)^{-1} B'_{\theta(k)} X_{f(\theta(k))} A_{\theta(k)} x\left(k\right)$ is optimal in the sense that it minimizes the quadratic cost

$$V(\theta_o, u, x_o) = \sum_{k=0}^{\infty} x(k)' C'_{f^k(\theta_o)} C_{f^k(\theta_o)} x(k) + u(k)' D'_{f^k(\theta_o)} D_{f^k(\theta_o)} u(k).$$

Furthermore, this control uniformly exponentially stabilizes the system and $x_o' X_{\theta_o} x_o = \min_u V(\theta_o, u, x_o)$.

See [2].

Given a stabilizable map f, we will study maps close to f in the following topologies:

Let $f, \hat{f} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ with $f, \hat{f} \in C^1$. The C^1 topology is generated by the metric

$$\begin{aligned} d_{C^1}\left(f, \hat{f}\right) = & \sup_{x\in\mathbb{R}^n, u\in\mathbb{R}^m} \left\| f(x,u) - \hat{f}(x,u) \right\| + \sup_{x\in\mathbb{R}^n, u\in\mathbb{R}^m} \left\| \frac{\partial f}{\partial x}(x,u) - \frac{\partial \hat{f}}{\partial x}(x,u) \right\| \\ & + \sup_{x\in\mathbb{R}^n, u\in\mathbb{R}^m} \left\| \frac{\partial f}{\partial u}(x,u) - \frac{\partial \hat{f}}{\partial u}(x,u) \right\| \end{aligned}$$

where $\frac{\partial f}{\partial x}(x,u)$ is the Jacobian matrix of f with respect to x and $\|\cdot\|$ is the l_2 induced matrix norm.

Let $f, \hat{f} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, with $f, \hat{f} \in LC$, where LC denotes the set of Lipschitz continuous functions. The LC topology is generated by the metric

$$d_{LC}\left(f, \hat{f}\right) = \sup_{x\in\mathbb{R}^n, u\in\mathbb{R}^m} \left\| f(x,u) - \hat{f}\left(x,u\right) \right\| + \sup_{\substack{x,y\in\mathbb{R}^n \\ u,v\in\mathbb{R}^m \\ i\in[1,n]}} \left\{ \frac{\left| f_i\left(x,u\right) - f_i\left(y,v\right) - \left(\hat{f}_i\left(x,u\right) - \hat{f}_i\left(y,v\right)\right) \right|}{\sqrt{\|x-y\|^2 + \|u-v\|^2}} \right\}.$$

Let $f, \hat{f} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, with $f, \hat{f} \in C^0$. The C^0 topology is generated by the metric

$$d_{C^0}\left(f, \hat{f}\right) = \sup_{x\in\mathbb{R}^n, u\in\mathbb{R}^m} \left\| f(x,u) - \hat{f}\left(x,u\right) \right\|.$$

The supremums in the last three definitions are over $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. This can be eased to the supremum over $\mathcal{N}$ where $\mathcal{N} \subset \mathbb{R}^n \times \mathbb{R}^m$ is a tubular neighborhood of $S \times \{0\}$. This modification has no effect on the development that follows if $\mathcal{N}$ is large enough, i.e. for all $x \in S$, $\sup_{y\in E_x} \|y - x\|$ is large enough, where $E_x := \{y \in \mathcal{N} : x := \arg\min_{v\in S} \|y - v\|\}$.

In the definition of system (1), the set S is invariant, i.e. $f\left(S\right) = S$. As the map f varies it is likely that S is no longer invariant. Indeed, it is possible that arbitrarily small variations in the map f lead to drastic changes in invariant sets. This is problematic since X, the solution to the Riccati equation (4), is only defined on S where S is invariant. It is difficult to discuss the dependence of X on f if as f varies the domain of X greatly varies. Thus we will restrict our attention to variations in f such that S only varies slightly. That is, we will require

$$d.\left(f,\hat{f}\right)+H\left(S,\hat{S}\right)<\varepsilon,\ \text{with } f\left(S\right)=S,\ \text{and } \hat{f}\left(\hat{S}\right)=\hat{S} \tag{5}$$

where $H\left(\cdot,\cdot\right)$ is the Hausdorff metric, i.e.

$$H\left(S,\hat{S}\right):=\max\left(\sup_{\theta\in\hat{S}}\inf\left\{\left\|\theta-\hat{\theta}\right\|:\hat{\theta}\in\hat{S}\right\},\ \sup_{\hat{\theta}\in\hat{S}}\inf\left\{\left\|\theta-\hat{\theta}\right\|:\theta\in S\right\}\right).$$

In the sequel it will be understood that S is an invariant set of f and $\hat{S}$ is an invariant set of $\hat{f}$.

Next we extend the feedback $F:S\to\mathbb{R}^{n\times m}$ defined by equation (**??**) to all of $\mathbb{R}^n$ by $\tilde{F}_{\hat{\theta}}:=F_{\theta(\hat{\theta})}$where

$$\theta\left(\hat{\theta}\right)=\arg\min\left\{\left\|\theta-\hat{\theta}\right\|:\theta\in S\right\}. \tag{6}$$

The cost quadratic X can be extended to $\tilde{X}$ in the same fashion. Note that $\theta\left(\hat{\theta}\right)$ is not necessarily well defined and $\tilde{X}$ might not be continuous. However, by perhaps invoking the axiom of choice, one can properly define $\theta\left(\hat{\theta}\right)$. Furthermore, $\tilde{X}$ is continuous on S and if $\left\|\hat{\theta}-\hat{\varphi}\right\|$ is small and $\hat{\theta},\hat{\varphi}\in\hat{S}\cup S$ with $H\left(S,\hat{S}\right)$ small, then $\left\|\tilde{X}_{\hat{\theta}}-\tilde{X}_{\hat{\varphi}}\right\|$ is small. Finally, let $X:S\to\mathbb{R}^n$ and $\hat{X}:\hat{S}\to\mathbb{R}^n$; define

$$d_{S,\hat{S}}\left(X,\hat{X}\right):=\max\left(\sup_{\theta\in S}\left\|X_\theta-\hat{X}_{\hat{\theta}(\theta)}\right\|,\sup_{\hat{\theta}\in\hat{S}}\left\|X_{\theta(\hat{\theta})}-\hat{X}_{\hat{\theta}}\right\|\right).$$

Now, if f is a hyperbolic on S (see section 3), then $H\left(S,\hat{S}\right)$ is small whenever $d_{C^1}\left(f,\hat{f}\right)$ is small [7]. Furthermore, if $f:S\to S$ is hyperbolic on S and S is a manifold (i.e. f is an Anosov diffeomorphism) and if $d_{C^1}\left(f,\hat{f}\right)$ is small enough, $\hat{f}$ is hyperbolic on S and S is invariant. Therefore, if f is an Anosov diffeomorphism, $H\left(S,\hat{S}\right)=0$ and $d_{S,\hat{S}}\left(X,\hat{X}\right)=d_{C^0}\left(X,\hat{X}\right)$. On the other hand, suppose that S is an attractor for f, that is for all $\varphi\in\mathcal{N}\left(S\right)$, we have $\lim_{k\to\infty}f^k\left(\varphi\right)\subset S$, where $\mathcal{N}\left(S\right)$ is any small enough neighborhood of S. Then it is a generic property for such diffeomorphisms that $H\left(S,\hat{S}\right)$ is small whenever $d_{C^0}\left(f,\hat{f}\right)$ is small [9], where $\hat{S}$ is an attractor for $\hat{f}$. Hence, in these three cases $d_{C^1}\left(f,\hat{f}\right)$ small implies $H\left(S,\hat{S}\right)$ is small, and therefore condition (5) is repetitious. Nonetheless, to maintain generality, condition (5) will be assumed. On the other hand, for dynamical systems that are not structurally stable, it is possible that $H\left(S,\hat{S}\right)\nrightarrow 0$ as $d_C\left(f,\hat{f}\right)\to 0$. Clearly, the LDV controller is not structurally stable in these situations, therefore it is assumed that f is such that $d_C\left(f,\hat{f}\right)\to 0$ implies that $H\left(S,\hat{S}\right)\to 0$.

3 Conjugacy

Conjugacy provides an equivalence relationship between dynamical systems. In this section $C^1$and C^0 conjugacy are examined. It will be shown that C^1 conjugacy preserves LDV stabilizability and the conjugacy maps can be used to transform the controller. However, in the case of C^0 conjugacy, LDV stabilizability is not preserved. Since structural stability of dynamical systems implies C^0 conjugacy with nearby systems [7], structural stability of the dynamical system alone does not imply the structural stability of LDV stabilizaility or of the LDV controller.

Let f be LDV stabilizable and let f and $\hat{f}$ be C^1 conjugate. That is, there exists diffeomorphisms

$$g:\mathbb{R}^n\to\mathbb{R}^n,\ g\left(S\right)=\hat{S},\qquad\text{and}\qquad h:\mathbb{R}^m\to\mathbb{R}^m$$

such that $f(\theta,u)=g^{-1}\left(\hat{f}\left(g\left(\theta\right),h\left(u\right)\right)\right)$. Define $G\left(\theta\right)=\frac{\partial g}{\partial\theta}\left(\theta\right)$, $H\left(\theta\right)=\frac{\partial h}{\partial\theta}$. Since g and h are diffeomorphisms, G and H are invertible. Therefore the following diagram commutes:

$$\begin{tikzcd}
TS\times\mathbb{R}^m \arrow[r, "(A,B)"] \arrow[d] \arrow[ddd, bend right=60, leftrightarrow, "{(G,H)}"'] & TS \arrow[d] \arrow[ddd, bend left=60, leftrightarrow, "G"] \\
S\times\mathbb{R}^m \arrow[r, "f"] \arrow[d, leftrightarrow, "{(g,h)}"'] & S \arrow[d, leftrightarrow, "g"'] \\
\hat{S}\times\mathbb{R}^m \arrow[r, "\hat{f}"] & \hat{S} \\
T\hat{S}\times\mathbb{R}^m \arrow[r, "(\hat{A},\hat{B})"] \arrow[u] & T\hat{S} \arrow[u]
\end{tikzcd}$$

Thus, if (A,B,f) is the LDV system induced by f and $\left(\hat{A},\hat{B},\hat{f}\right)$ is the LDV system induced by $\hat{f}$, then $\hat{A}_{g(\theta)}=G_{g(\theta)}A_\theta G_{g(\theta)}^{-1}$, $\hat{B}_{g(\theta)}=G_{g(\theta)}B_\theta H_{g(\theta)}^{-1}$. If $F:S\to\mathbb{R}^{m\times n}$ uniformly exponentially stabilizes (A,B,f) then $\hat{F}_{g(\theta)}:=H_\theta F_\theta G_{g(\theta)}^{-1}$ uniformly exponentially stabilizes $\left(\hat{A},\hat{B},\hat{f}\right)$. Thus, LDV stabilizability is preserved under C^1 conjugacy. Similarly, LDV uniformly detectability is preserved under C^1 conjugacy.

Now, suppose (A,B,f) is stabilizable, (A,C,f) is uniformly detectable, $D_\theta'D_\theta>0$ for $\theta\in S$, $\hat{C}_{g(\theta)}=C_\theta G_{g(\theta)}^{-1}$ and $\hat{D}_{g(\theta)}=D_\theta H_\theta^{-1}$. Since $f\in C^1$, the LDV system induced by f is continuous and Theorem 2 implies that there exists a continuous function $X:S\to\mathbb{R}^{n\times n}$ that solves equation (4). It is clear that $\hat{X}_{g(\theta)}:=\left(G_{g(\theta)}^{-1}\right)'X_\theta G_{g(\theta)}$ solves the Riccati equation associated with the

LDV system $\left(\hat{A}, \hat{B}, \hat{C}, \hat{D}, \hat{f}\right)$. Therefore, LDV systems and quadratic controllers are well defined in a coordinate free approach. The topological and geometric issues associated with LDV systems on a non-orientable or non-parallelizable manifolds S are addressed in [6].

Now, if f is hyperbolic and $\hat{f}$ is C^1 close to f, then $\hat{f}$ is hyperbolic and $\hat{f}$ is topologically conjugate to f (for exact result see [7]). However, f and $\hat{f}$ are not necessarily C^1 conjugate. If hyperbolicity implied $\hat{f}$ and f are C^1 conjugate, then LDV would clearly be structurally stable in the hyperbolic case. Note that in the case of topological conjugacy, the nonlinear control $\hat{u}(k) = h^{-1}\left(F_{\theta(k)}\left(g^{-1}\left(\varphi(k)\right) - g^{-1}\left(\theta\left(k\right)\right)\right)\right)$ might not exponentially stabilize $\hat{f}$. Consider, for example, a system with no input:

$$f\left(\varphi\right) = \frac{1}{2}\varphi, \qquad S = \left[0, \frac{1}{2}\right]$$

and the induced LDV system

$$x\left(k+1\right) = \frac{1}{2}x\left(k\right). \tag{7}$$

Under homeomorphisms $g\left(\varphi\right) = -\left(\log\left(\varphi\right)\right)^{-1}$ and $g^{-1}\left(\hat{\varphi}\right) = e^{-\frac{1}{\hat{\varphi}}}$, the conjugate system becomes

$$\hat{f}\left(\hat{\theta}\right) = \frac{1}{1-\hat{\theta}\log\left(\frac{1}{2}\right)}\hat{\theta}, \qquad \hat{S} = \left[0, \frac{-1}{\log\left(\frac{1}{2}\right)}\right]$$

and the induced LDV system is

$$\hat{x}\left(k+1\right) = \left(\frac{1}{\left(1+\hat{\theta}\left(k\right)\ln 2\right)^2}\right)\hat{x}\left(k\right), \qquad \hat{\theta}\left(k+1\right) = \hat{f}\left(\hat{\theta}\left(k\right)\right). \tag{8}$$

The above system is not uniformly exponentially stable because for every $\beta < \infty$ and $\alpha < 1$, $\left\|\hat{x}\left(k\right)\right\| > \beta\alpha^k\left\|\hat{x}\left(0\right)\right\|$ for some k. Of course $\hat{f}^k\left(\theta\right) \to 0$, just not exponentially fast. Thus, f and $\hat{f}$ are topologically conjugate, yet (8) is LDV stabilizable and (7) is not. Therefore, since hyperbolicity only leads to topological conjugacy, hyperbolicity will not help to prove the structural stability of LDV stabilizability. We must rely on the fact that f and $\hat{f}$ are C^1 close to infer LDV stabilizability of $\hat{f}$.

The example above illustrated the weakness of the LDV approach compared to nonlinear methods. The LDV approach implies that $\hat{f}$ is not stable, when, in fact, it is stable.

4 Structural Stability

In this section it will be shown that if $\hat{f}$ is near an LDV stabilizable f in the C^1 topology and $H\left(S, \hat{S}\right)$ is small, then $\hat{f}$ is LDV stabilizable (proposition 4). In fact, the LDV optimal quadratic cost varies continuously with $d_{C^1}\left(f,\hat{f}\right)+H\left(S,\hat{S}\right)$ - that is the map $(f,S)\mapsto X$ is continuous where X is the positive semi-definite function which solves equation (4) (proposition 4). Furthermore, if $\hat{f}$ is near f in the Lipschitz topology and $H\left(S,\hat{S}\right)$ is small, then $\hat{f}$ is also LDV stabilizable (proposition 4). Finally, if $\hat{f}$ is near f in the C^0 topology and $H\left(S,\hat{S}\right)$ is small, then an LDV controller may only stabilize $\hat{f}$ in the sense that $\limsup_k \|x(k)\|<\varepsilon$, where the control objective is $x(k)\to 0$ (proposition 4).

Let system (1) be a continuous LDV. If the pair (A,f) is uniformly exponentially stable, then there exists an $\varepsilon>0$ such that if $d_{C^0}(f,\hat{f})+d_{C^0}\left(A,\hat{A}\right)+H\left(S,\hat{S}\right)<\varepsilon$, then the pair $\left(\hat{A},\hat{f}\right)$ is uniformly exponentially stable. Furthermore, the α and β in the definition of uniform exponential stability of $\left(\hat{A},\hat{f}\right)$ can be taken to only depend on A, f and ε.

Note, this lemma is only examining LDV systems and therefore does not require that $A=\frac{\partial f}{\partial x}$.

Fix $K\supset\mathcal{N}(S)$, where K is compact and $\mathcal{N}(S)$ a tubular neighborhood of S. Define ε_1 such that $H\left(S,\hat{S}\right)<\varepsilon_1$ implies that $\hat{S}\subset K$. Since (A,f) is uniformly exponentially stable, Theorem 2 implies that there exists a continuous positive semi-definite function X such that for $\hat{\theta}\in K$,

$$A'_{\theta(\hat{\theta})}X_{f(\theta(\hat{\theta}))}A_{\theta(\hat{\theta})}=X_{\theta(\hat{\theta})}-I, \tag{9}$$

where $\theta\left(\hat{\theta}\right)$ is defined by (6). Since X is continuous, there exists a $\delta>0$ such that $\|\varphi-\theta\|<\delta$ implies $\|X_\varphi-X_\theta\|<\frac{1}{8}\frac{1}{\bar{A}^2+1}$, where $\bar{A}=\sup\{A_\theta:\theta\in K\}$. Furthermore, since f is uniformly continuous over K, there exists a $\gamma>0$ such that for $\theta,\hat{\theta}\in K$, and $\left\|\theta-\hat{\theta}\right\|<\gamma$, we have $\left\|f(\theta)-f\left(\hat{\theta}\right)\right\|<\frac{\delta}{3}$. Therefore, if $d_{C^0}\left(f,\hat{f}\right)+H\left(S,\hat{S}\right)<\min\left(\frac{\delta}{3},\gamma,\varepsilon_1\right):=\varepsilon_2$, then for $\hat{\theta}\in\hat{S}$, we have

$$\begin{aligned}&\left\|\theta\left(\hat{f}\left(\hat{\theta}\right)\right)-f\left(\theta\left(\hat{\theta}\right)\right)\right\|\leq\\&\left\|\theta\left(\hat{f}\left(\hat{\theta}\right)\right)-\hat{f}\left(\hat{\theta}\right)\right\|+\left\|\hat{f}\left(\hat{\theta}\right)-f\left(\hat{\theta}\right)\right\|+\left\|f\left(\hat{\theta}\right)-f\left(\theta\left(\hat{\theta}\right)\right)\right\|<\delta\end{aligned}$$

and thus $\left\|X_{\theta(\hat{f}(\hat{\theta}))}-X_{f(\theta(\hat{\theta}))}\right\|<\frac{1}{8}\frac{1}{\bar{A}^2+1}$. Hence, if $\hat{\theta}\in\hat{S}$ and $d_{C^0}\left(A,\hat{A}\right)<\min\left(\frac{1}{8}\frac{1}{\bar{X}}\frac{1}{\bar{A}},\sqrt{\frac{1}{8}\frac{1}{\bar{X}}},1\right)=:\delta_1$ and $d\left(f,\hat{f}\right)+H\left(S,\hat{S}\right)<\varepsilon_2$, then with a bit of elementary manipulation it can be shown that

$$\begin{aligned}&\hat{A}'_{\hat{\theta}}X_{\theta(\hat{f}(\hat{\theta}))}\hat{A}_{\hat{\theta}}=A'_\theta X_{f(\theta)}A_\theta+\left(A_\theta-\hat{A}_\theta\right)'X_{f(\theta)}\left(A_\theta-\hat{A}_\theta\right)\\&-A'_\theta X_{f(\theta)}\left(A_\theta-\hat{A}_\theta\right)-\left(A_\theta-\hat{A}_\theta\right)'X_{f(\theta)}A_\theta+\hat{A}'_\theta\left(X_{\hat{f}(\theta)}-X_{f(\theta)}\right)\hat{A}_\theta\\&\leq X_{\theta(\hat{\theta})}-\frac{1}{2}I.\end{aligned}$$

Set $\alpha=1-\frac{1}{2\min_{\theta\in S}(\underline{\lambda}(X_\theta))}$ and $\beta=\frac{\max_{\theta\in S}(\bar{\lambda}(X_\theta))}{\min_{\theta\in S}(\underline{\lambda}(X_\theta))}$. Since X is bounded and continuous, S compact and $X > 0$, α and β are finite. Since $X_{\theta(\cdot)}$ solves equation (9), it is not hard to show (for example see [12]) that $\alpha < 1$ and if $x(k+1) = \hat{A}_{\hat{f}^k(\hat{\theta}_o)} x(k)$, then $\|x(k)\| < \beta \alpha^k \|x(0)\|$.

Assume $f \in C^1$ induces a stabilizable LDV system, that is, there exists a continuous map $F : S \to \mathbb{R}^{m \times n}$ such that $(A + BF, f)$ is uniformly exponentially stable. Then there exists a $\delta > 0$ such that if $d_{C^1}(f, \hat{f}) + H\left(S, \hat{S}\right) < \delta$, then $\hat{f}$ is LDV stabilizable and is stabilized by the feedback F. Furthermore, with this feedback, the α, β and γ in the definition of locally uniformly exponentially stability can be chosen to depend only on f, F and δ. Thus LDV stabilizability is a structurally stable.

Define $\tilde{A}_{\hat{\theta}} = \frac{\partial \hat{f}}{\partial \hat{\theta}}\left(\hat{\theta}, 0\right) + \frac{\partial \hat{f}}{\partial u}\left(\hat{\theta}, 0\right) F_{\theta(\hat{\theta})}$. Then $d_{C^0}\left(\tilde{A}, A\right) < C d_{C^1}\left(\hat{f}, f\right)$ where C is a constant that depends on F. Lemma 4 implies that there exists an $\varepsilon > 0$ such that if $d_{C^0}(f, \hat{f}) + d_{C^0}\left(A, \tilde{A}\right) + H\left(S, \hat{S}\right) < \varepsilon$, then $\left(\tilde{A}, \hat{f}\right)$ is stable. Hence, there exists a $\delta > 0$ such that if $d_{C^1}(f, \hat{f}) + H\left(S, \hat{S}\right) < \delta$, then $\left(\frac{\partial \hat{f}}{\partial \hat{\theta}}\left(\hat{\theta}, 0\right), \frac{\partial \hat{f}}{\partial u}\left(\hat{\theta}, 0\right), \hat{f}\right)$ is stabilizable. Lemma 4 further states that the parameters of stability, α and β, only depend on $A + BF$, f and ε. Since ε and $A + BF$ depend on F, and A and B are the partial derivative of f, we conclude that α and β can be taken to only depend on f, F and ε.

Thus, if the feedback F stabilizes f, then F also stabilizes any function $\hat{f}$ near f in the C^1 topology. A natural question is, how good of a controller is F? For instance, if F is the LDV quadratic controller for the LDV system induced by f, how far is it from the LDV quadratic controller for $\hat{f}$. That is, are LDV quadratic controllers structurally stable? First, note that detectability is a structurally stable property. That is:

Assume that A, C and f are continuous and (A, C, f) is uniformly detectable. In this case there exists a $\delta > 0$, such that if $d_{C^0}\left(f, \hat{f}\right) + d_{C^0}\left(A, \hat{A}\right) + d_{C^0}\left(C, \hat{C}\right) + H\left(S, \hat{S}\right) < \delta$, then $\left(\hat{A}, \hat{C}, \hat{f}\right)$ is uniformly detectable. That is, there exists a $\hat{\alpha}_d < 1$ and $\hat{\beta}_d < \infty$ such that if $d_{C^0}\left(f, \hat{f}\right) + d_{C^0}\left(A, \hat{A}\right) + d_{C^0}\left(C, \hat{C}\right) + H\left(S, \hat{S}\right) < \delta$, then there exists a feedback $\hat{L}$ such that $\|\xi(k)\| < \hat{\beta}_d \hat{\alpha}_d^k \|\xi(0)\|$ where $\xi(k+1) = \left(\hat{A} + \hat{L}\hat{C}\right)\xi(k)$. Furthermore, $\hat{\alpha}_d$ and $\hat{\beta}_d$ only depend on A, C, f, and δ.

The proof is nearly identical to the proof of Lemma 4.

Let $f \in C^1$. Assume that the LDV induced by f is stabilizable, (A, C, f) is uniformly detectable and $D_\theta' D_\theta > 0$ for all $\theta \in \mathbb{R}^n$. Then for all $\varepsilon > 0$, there exists a $\delta > 0$ such that if $d_{C^1}\left(f, \hat{f}\right) + d_{C^0}\left(C, \hat{C}\right) + d_{C^0}\left(D, \hat{D}\right) + H\left(S, \hat{S}\right) < \delta$, then $d_{S, \hat{S}}\left(X, \hat{X}\right) < \varepsilon$, where X is the positive semi-definite solution to the Riccati equation (4) induced by (C, D, f) and $\hat{X}$ is the positive semi-definite solution induced by $\left(\hat{C}, \hat{D}, \hat{f}\right)$.

Let $\varepsilon > 0$. By Proposition 4, there exists α, β and δ_1 such that if $d_{C^1}\left(f,\hat{f}\right)+H\left(S,\hat{S}\right)<\delta_1$ and $\hat{\theta}_o \in \hat{S}$, then $\|\hat{x}(k)\| \leq \beta\alpha^k \|\hat{x}(0)\|$ where $\hat{x}(k+1) = \left(\hat{A}_{\hat{f}^k(\hat{\theta}_o)} + \hat{B}_{\hat{f}^k(\hat{\theta}_o)} F_{\theta(\hat{f}^k(\hat{\theta}_o))}\right)\hat{x}(k)$ and F is the optimal LQ feedback gain for (A,B,C,D,f). Therefore, it is possible to show that if $d_{C^1}\left(f,\hat{f}\right)+d_{C^0}\left(C,\hat{C}\right)+d_{C^0}\left(D,\hat{D}\right)+H\left(S,\hat{S}\right)<\delta_1$ and

$$
\begin{aligned}
x_o'\hat{X}_{\theta_o}x_o &:= \min_{u\in l_2}\sum_{k=0}^{\infty}\hat{x}(k)'\hat{C}'_{\hat{f}^k(\theta_o)}\hat{C}_{\hat{f}^k(\theta_o)}\hat{x}(k)+u(k)'\hat{D}'_{\hat{f}^k(\theta_o)}\hat{D}_{\hat{f}^k(\theta_o)}u(k)\\
&\text{subject to } \hat{x}(k+1)=\hat{A}_{\hat{f}^k(\theta_o)}\hat{x}(k)+\hat{B}_{\hat{f}^k(\theta_o)}u(k)\ ,
\end{aligned}
$$

then $\left\|\hat{X}_{\hat{\theta}}\right\| < \bar{X}$, where $\bar{X} := \beta^2\frac{1}{1-\alpha^2}\left(\left(\bar{C}+\delta_1\right)^2+\bar{F}^2\left(\bar{D}+\delta_1\right)^2\right)$, $\bar{C} := \max_{\theta\in K}\|C_\theta\|$, $\bar{D} := \max_{\theta\in K}\|D_\theta\|$, $\bar{F} := \max_{\theta\in S}\|F_\theta\|$.

Lemma 4 shows that if $d_{C^1}\left(f,\hat{f}\right)+d_{C^0}\left(C,\hat{C}\right)+H\left(S,\hat{S}\right)<\delta_2$ then $\left(\hat{A},\hat{C},\hat{f}\right)$ is detectable with paramters $\hat{\alpha}$ and $\hat{\beta}$ that do not depend only on δ_2, C, and f.

Theorem 2 can then be applied to show that the closed-loop system $\left(\hat{A}+\hat{B}\hat{F},f\right)$ is uniformly exponentially stable with stability parameters that only depend on δ_1, δ_2, C, D and f. Therefore there eixsts a $N<\infty$ such that

$$
\|\hat{x}(N+1)\| < \sqrt{\frac{1}{16}\frac{\varepsilon}{\bar{X}}}\,\|\hat{x}(0)\|\,, \tag{10}
$$

where $\hat{x}$ is given by $\hat{x}(k+1) = \left(\hat{A}_{\hat{f}^k(\hat{\theta}_o)}+\hat{B}_{\hat{f}^k(\hat{\theta}_o)}\hat{F}_{\hat{f}^k(\hat{\theta}_o)}\right)\hat{x}(k)$.

Let $\hat{u}^*_{\theta_o,x_0}$ denote the optimal control due to initial conditions θ_o, $x(0)$ for the system with parameters $\left(\hat{A},\hat{B},\hat{C},\hat{D},\hat{f}\right)$. Define $u^*_{\theta_o,x(0)}$ similarly, but for the system with parameters (A,B,C,D,f). Define

$$
U_N := \left\{u\in l_2[0,N] : \|u\|^2_{[0,N]} \leq \frac{\bar{X}}{\frac{1}{2}\inf_{\theta\in K}\underline{\sigma}\left(D_\theta'D_\theta\right)}\right\},
$$

where $\underline{\sigma}\left(D_\theta'D_\theta\right)$ is the minimum singular value of $D_\theta'D_\theta$. Therefore, there is a $\delta_3>0$ such that if $d_{C^1}\left(f,\hat{f}\right)+d_{C^0}\left(C,\hat{C}\right)+d_{C^0}\left(D,\hat{D}\right)+H\left(S,\hat{S}\right)<\delta_3$, then $\left\{\hat{u}^*_{\theta_o,x(0)}(k) : k\leq N\right\}\in U_N$. Note that U_N is compact since $N<\infty$, where N is such that equation (10) holds.

Let $x_{u,\theta_0}(k+1) = A_{f^k(\theta_o)}x_{\hat{u}^*_{\theta_0,x_o}}(k+1)+B_{f^k(\theta_o)}u(k)$, and define $\hat{x}_{u,\hat{\theta}_o}$ similarly. Since $N<\infty$, if we fix (A,B,f), $u\in U_N$, $\theta_o\in S$ and $x(0)$ with $\|x(0)\|\leq 1$, then $x_{u,\theta_o}(N+1) - \hat{x}_{u,\hat{\theta}_o}(N+1)$ is continuous in $\hat{A}$, $\hat{B}$, $\hat{f}$ and $\hat{\theta}_o$ and since U_N, K and $\{x(0) : \|x(0)\|\leq 1\}$ are compact, there exists a $\delta_4>0$, which can be taken independently of u, θ_o and $x(0)$ such that if $d_{C^0}\left(f,\hat{f}\right)+d_{C^0}\left(A,\hat{A}\right)+d_{C^0}\left(B,\hat{B}\right)+H\left(S,\hat{S}\right)<\delta_4$, then

$$
\left\|x_{u,\theta(\hat{\theta}_o)}(N+1) - x_{u,\hat{\theta}_o}(N+1)\right\| < \sqrt{\frac{1}{16}\frac{\varepsilon}{\bar{X}}}. \tag{11}
$$

Therefore, if $d_{C^1}\left(f,\hat{f}\right)+d_{C^0}\left(C,\hat{C}\right)+d_{C^0}\left(D,\hat{D}\right)+H\left(S,\hat{S}\right) < \min\left(\delta_0,\delta_1,\delta_2,\delta_3,\delta_4\right)$, and $\|x\left(0\right)\| \leq 1$, then equations (10) and (11) yield,

$$\begin{aligned}
&\bar{X}\left\|x_{\hat{u}^*_{\hat{\theta},x_0},\theta\left(\hat{\theta}_o\right)}\left(N+1\right)\right\|^2 = \bar{X}\left\|x_{\hat{u}^*_{\hat{\theta},x_0},\theta\left(\hat{\theta}_o\right)}\left(N+1\right)-\hat{x}_{\hat{u}^*_{\hat{\theta},x_0},\hat{\theta}_o}\left(N+1\right)+\hat{x}_{\hat{u}^*_{\hat{\theta},x_0},\hat{\theta}_o}\left(N+1\right)\right\|^2 = \\
&\bar{X}\left\|x_{\hat{u}^*_{\hat{\theta},x_0},\theta\left(\hat{\theta}_o\right)}\left(N+1\right)-\hat{x}_{\hat{u}^*_{\hat{\theta},x_0},\hat{\theta}_o}\left(N+1\right)\right\|^2+\bar{X}\left\|\hat{x}_{\hat{u}^*_{\hat{\theta},x_0},\hat{\theta}_o}\left(N+1\right)\right\|^2 \\
&+2\bar{X}\left(x_{\hat{u}^*_{\hat{\theta},x_0},\theta\left(\hat{\theta}_o\right)}\left(N+1\right)-\hat{x}_{\hat{u}^*_{\hat{\theta},x_0},\hat{\theta}_o}\left(N+1\right)\right)'\hat{x}_{\hat{u}^*_{\hat{\theta},x_0},\hat{\theta}_o}\left(N+1\right) \leq \frac{\varepsilon}{4}
\end{aligned} \tag{12}$$

Likewise,

$$\bar{X}\left\|\hat{x}_{u^*_{\hat{\theta},x_0},\theta\left(\hat{\theta}_o\right)}\left(N+1\right)\right\|^2 \leq \frac{\varepsilon}{4} \tag{13}$$

Note that $x_{\hat{u}^*_{\hat{\theta},x_0},\theta\left(\hat{\theta}_o\right)}\left(N+1\right)$ is the state after the non-optimal control $\hat{u}^*_{\theta\left(\hat{\theta}_o\right),x(0)}$ is applied; the optimal control is $u^*_{\hat{\theta}_o,x(0)}$.

Define

$$x(0)'W_f(u,N,\theta_o)x(0) := \sum_{k=0}^{N} x(k)C'_{f^k(\theta_o)}C_{f^k(\theta_o)}x(k)+u(k)D'_{f^k(\theta_o)}D_{f^k(\theta_o)}u(k) \tag{14}$$

where $x(k+1)=A_{f^k(\theta_o)}x(k)+B_{f^k(\theta_o)}u(k)$ and $x\left(0\right)=x\left(0\right)$.

Define $\hat{x}(0)'\hat{W}_{\hat{f}}(\hat{u},N,\hat{\theta}_o)\hat{x}(0)$ similarly. Since $N<\infty$, $x(0)'W_f\left(u,N,\theta\right)x(0)-\hat{x}(0)'\hat{W}_{\hat{f}}\left(u,N,\hat{\theta}\right)\hat{x}(0)$ is continuous in $\left(\hat{A},\hat{B},\hat{C},\hat{D},\hat{f}\right)$ and $\hat{\theta}$. Furthermore, since U_N and K are compact, if $\|x\left(0\right)\|\leq 1$, there exists a $\delta_5>0$ such that if $d_{C^1}\left(f,\hat{f}\right)+d_{C^0}\left(C,\hat{C}\right)+d_{C^0}\left(D,\hat{D}\right)+H\left(S,\hat{S}\right)<\delta_5$, then

$$\left\|x(0)'W_f\left(u,N,\theta\left(\hat{\theta}_o\right)\right)x(0)-x(0)'\hat{W}_{\hat{f}}\left(u,N,\hat{\theta}_o\right)x(0)\right\| < \frac{\varepsilon}{2} \text{ for all } u\in U_N \text{ and } \hat{\theta}_o\in\hat{S}. \tag{15}$$

Let $d_{C^1}\left(f,\hat{f}\right)+d_{C^0}\left(C,\hat{C}\right)+d_{C^0}\left(D,\hat{D}\right)+H\left(S,\hat{S}\right)<\min\left(\delta_2,\delta_4,\delta_5\right)$. Then, for $\|x\left(0\right)\|\leq 1$, inequalities (12) and (15) yield,

$$\begin{aligned}
&x\left(0\right)X_{\theta\left(\hat{\theta}_o\right)}x\left(0\right)-x\left(0\right)\hat{X}_{\hat{\theta}_o}x\left(0\right) \\
&\leq x\left(0\right)W_f\left(\hat{u}^*_{\hat{\theta}_o,x(0)},N,\theta\left(\hat{\theta}_o\right)\right)x\left(0\right)+x_{\hat{u}^*_{\hat{\theta}_o,x(0)},\theta\left(\hat{\theta}_o\right)}\left(N+1\right)'\left(X_{f^{N+1}\left(\theta\left(\hat{\theta}_o\right)\right)}\right)x_{\hat{u}^*_{\hat{\theta}_o,x(0)},\theta\left(\hat{\theta}_o\right)}\left(N+1\right) \\
&-x\left(0\right)\hat{W}_{\hat{f}}\left(\hat{u}^*_{\hat{\theta}_o,x(0)},N,\hat{\theta}_o\right)x\left(0\right)-x_{\hat{u}^*_{\hat{\theta}_o,x(0)},\hat{\theta}_o}\left(N+1\right)'\left(\hat{X}_{\hat{f}^{N+1}\left(\hat{\theta}_o\right)}\right)x_{\hat{u}^*_{\hat{\theta}_o,x(0)},\hat{\theta}_o}\left(N+1\right) \\
&<\varepsilon.
\end{aligned}$$

Similarly, (13) and (15) yield

$$x\left(0\right)\hat{X}_{\hat{\theta}_o}x\left(0\right)-x\left(0\right)X_{\theta\left(\hat{\theta}_o\right)}x\left(0\right)<\varepsilon.$$

Therefore, if $\|x\left(0\right)\|\leq 1$, then $\sup_{\hat{\theta}\in\hat{S}}\left|x(0)X\left(\theta\left(\hat{\theta}_o\right)\right)x(0)-x(0)\hat{X}\left(\hat{\theta}_o\right)x(0)\right|<\varepsilon$. Similarly, it can be shown that $\sup_{\theta\in S}\left|x(0)X\left(\theta\right)x(0)-x(0)\hat{X}\left(\hat{\theta}\left(\theta\right)\right)x(0)\right|<\varepsilon$, and thus $d_{S,\hat{S}}\left(X,\hat{X}\right)<\varepsilon$.

Next we weaken the assumptions in Proposition 4. First, we examine the case where $\hat{f} \in LC$ with $d_{LC}\left(f,\hat{f}\right) < \varepsilon$, and find that the above results still hold. Then the case where $\hat{f} \in C^0$ with $d_{C^0}\left(f,\hat{f}\right) < \varepsilon$ is examined. It is shown that such system can be made stable, but not necessarily asymptotically stable.

Let $f \in C^1$ induce a stabilizable LDV. Then there exists an $\varepsilon > 0$ such that if $\hat{f} \in LC$ and $d_{LC}\left(f,\hat{f}\right) + H\left(S,\hat{S}\right) < \delta$, then the LDV controller induced by f locally uniformly exponentially stabilizes $\hat{f}$.

Since $f \in C^1$, defining the tracking error $\hat{x}(k) = \hat{\varphi}(k+1) - \hat{\theta}(k+1)$, we see that error dynamics are given by

$$\begin{aligned}\hat{\varphi}(k+1) - \hat{\theta}(k+1) &= \hat{f}(\hat{\varphi}(k),\hat{u}(k)) - \hat{f}\left(\hat{\theta}(k),0\right) \\ &= A_{\hat{f}^k(\hat{\theta}_o)}\hat{x}(k) + B_{\hat{f}^k(\hat{\theta}_o)}\hat{u}(k) + \eta(\hat{x}(k),\hat{u}(k),\hat{\theta}(k)) \\ &+ \left(\hat{f}(\hat{\varphi}(k),\hat{u}(k)) - \hat{f}\left(\hat{\theta}(k),0\right) - \left(f(\hat{\varphi}(k),\hat{u}(k)) - f\left(\hat{\theta}(k)\right)\right)\right),\end{aligned} \tag{16}$$

where $\eta(\hat{x}(k),\hat{u}(k),\hat{\theta}(k))$ accounts for the nonlinear parts neglected in linear approximation. By Lemma 4 it is clear that if $d_{LC}\left(f,\hat{f}\right) + H\left(S,\hat{S}\right)$ is small enough, the LDV system $\left(\hat{A}+\hat{B}\tilde{F},f\right)$ is uniformly exponetially stable. It can be shown (see [2]) that $\eta(x,u,\hat{\theta})$ can be decomposed as

$$\eta(x,u,\hat{\theta}) = \eta_x(x,u,\hat{\theta})x + \eta_u(x,u,\hat{\theta})u$$

and that $\left\|\eta_x(x,u,\hat{\theta}) + \eta_u(x,u,\hat{\theta})F_{\theta(\hat{\theta})}\right\| \to 0$ as $\bar{x},\bar{u} \to 0$. Define $\rho(x,F_\theta x,\theta) \in \mathbb{R}^{n\times n+m}$ by

$$\rho(x,F_\theta x,\theta)_{i,j} := \frac{\left(\hat{f}_i(x+\theta,F_\theta x) - \hat{f}_i(\theta,0) - (f_i(x+\theta,F_\theta x) - f_i(\theta,0))\right)}{\|x\|^2 + \|F_\theta x\|^2}\cdot\begin{cases} x_j & \text{for } j \le n \\ (F_\theta x)_{j-n} & \text{for } n < j < n+m \end{cases}.$$

Thus,

$$\rho\left(x,F_{\theta(\hat{\theta})}x,\hat{\theta}\right)\begin{bmatrix} x \\ F_{\theta(\hat{\theta})}x \end{bmatrix} = \hat{f}\left(x+\hat{\theta},F_{\theta(\hat{\theta})}x\right) - \hat{f}\left(\hat{\theta},0\right) - \left(f\left(x+\hat{\theta},F_{\theta(\hat{\theta})}u\right) - f\left(\hat{\theta}\right)\right) \tag{17}$$

and $\sup_{\|x\|<\bar{x},}\left\|\rho\left(x,F_{\theta(\hat{\theta})}x,\hat{\theta}\right)\right\| \le \sqrt{n}d_{LC}\left(f,\hat{f}\right)$. Therefore, we see that (16) can be written as a uniformly exponetially stable linear system, with a pertubation that can be bounded by a $O\left(x^2\right)$ function. It is well know that such systems are asymptoically stable. [13]).

If we only restrict $d_{C^0}\left(f,\hat{f}\right) < \varepsilon$ then asymptotic stability cannot be guaranteed. For example, consider the dynamical system $f(\varphi,u) = \frac{1}{2}\varphi$. Then f is globally uniformly exponentially stable with $|\varphi(k) - \theta(k)| \le \frac{1}{2}^k|\varphi(0) - \theta(0)|$. Define $\hat{f}(\varphi,u) = \frac{1}{2}\varphi + \varepsilon\sin\left(\frac{\varphi}{\varepsilon}\frac{\pi}{6}\right)$. Then $d_{C^0}\left(f,\hat{f}\right) \le \varepsilon$ and $\hat{f}$ has three fixed points corresponding to the solutions of $\sin\left(\frac{\varphi}{\varepsilon}\frac{\pi}{6}\right) = \frac{\varphi}{2\varepsilon}$. Hence, $\pm\varepsilon$ are stable fixed points and zero is an unstable fixed point. Thus, if $\varphi > 0$ and $\theta < 0$, then $|\varphi(k) - \theta(k)| \nrightarrow 0$. However, $\limsup |\varphi(k) - \theta(k)| < 2\varepsilon$. Therefore, $\hat{f}$ is stable, but not asymptotically stable, instead asymptotically $\varphi(k) - \theta(k)$ approaches a small set around zero. This form of stability is often referred to as asymptotically bounded and the attractive set that $\varphi(k) - \theta(k)$ enters is called a residual set.

Let $f \in C^1$ be LDV stabilizable via the feedback F. Then there exists an $\varepsilon > 0$ such that if $d_{C^0}\left(f, \hat{f}\right) + H\left(S, \hat{S}\right) < \varepsilon$, then the feedback F makes $\hat{f}$ asymptotically bounded, with the diameter of the residual set continuous in ε.

Since $(A + BF, f)$ is uniformly exponentially stable, Lemma 4 implies that there exists an $\varepsilon > 0$ such that $\left(A + B\tilde{F}, \hat{f}\right)$ is uniformly exponentially stable if $d_{C^0}\left(f, \hat{f}\right) + H\left(S, \hat{S}\right) < \varepsilon$, where $\tilde{F}_{\hat{\theta}} := F_{\theta(\hat{\theta})}$. Consider the system

$$
\begin{aligned}
&x(k+1) \qquad (18)\\
&= f\left(x(k) + \hat{f}^k(\theta_o), F_{\theta\left(\hat{f}^k(\theta_o)\right)} x(k)\right) - f\left(\hat{f}^k(\theta_o), 0\right)\\
&= \left(A_{\hat{f}(\theta_o)} + B_{\hat{f}(\theta_o)} F_{\theta\left(\hat{f}(\theta_o)\right)}\right) x(k)\\
&+ \eta_x\left(x(k), F_{\theta\left(\hat{f}^k(\theta)\right)} \hat{x}(k), \hat{f}^k(\theta_o)\right) x(k) + \eta_u\left(x(k), F_{\theta\left(\hat{f}^k(\theta)\right)} x(k), \hat{f}^k(\theta_o)\right) F_{\theta\left(\hat{f}^k(\theta_o)\right)} x(k).
\end{aligned}
$$

where η_x and η_u account for the error due to linearization. Since $(A + BF, f)$ is uniformly exponentially stable, Lemma 4, $\|\eta_x\|$ and $\|\eta_u\| \to 0$ and x and u go to zero, and the fact that uniformly exponentially stable linear system remain stable under small gain perturbation, we conclude that the nonlinear system (18) is locally uniformly stable. Now, consider the system

$$
\begin{aligned}
&\hat{\varphi}(k+1) - \hat{\theta}(k+1) = \hat{f}\left(\hat{\varphi}(k), F_{\theta(\hat{\theta}(k))}\left(\hat{\varphi}(k) - \hat{\theta}(k)\right)\right) - \hat{f}\left(\hat{\theta}(k), 0\right) \qquad (19)\\
&= f\left(\hat{\varphi}(k), F_{\theta(\hat{\theta}(k))}\left(\hat{\varphi}(k) - \hat{\theta}(k)\right)\right) - f\left(\hat{\theta}(k), 0\right)\\
&+ \left(\hat{f}\left(\hat{\varphi}(k), F_{\theta(\hat{\theta}(k))}\left(\hat{\varphi}(k) - \hat{\theta}(k)\right)\right) - \hat{f}\left(\hat{\theta}(k), 0\right) - f\left(\hat{\varphi}(k), F_{\theta(\hat{\theta}(k))}\left(\hat{\varphi}(k) - \hat{\theta}(k)\right)\right) - f\left(\hat{\theta}(k), 0\right)\right).
\end{aligned}
$$

We see that (19) is a locally uniformly stable nonlinear system with a extraanous noise input. It is not difficult to show that such systems are asymptotically bounded

Typically one assumes that control methods of nonlinear maps based on linear approximation are only applicable to differentiable maps. However, the last two propositions show that this assumption is not necessary.

5 Structural Stability of the Optimal LDV Controller of the Hénon Map

The Hénon map is defined as

$$\varphi(k+1) = \begin{bmatrix} 1-(a+u(k))(\varphi_1(k))^2 + \varphi_2(k) \\ b\varphi_1(k) \end{bmatrix}.$$

The LDV approximation of this system is:

$$x(k+1) = \begin{bmatrix} -2a\theta_1(k) & 1 \\ b & 0 \end{bmatrix} x(k) + \begin{bmatrix} (\varphi_1(k))^2 \\ 0 \end{bmatrix} u(k), \qquad \theta(k+1) = f(\theta(k), 0).$$

The Hénon map has been studied for a wide variety of parameters. It was first introduced with $a = 1.4$ and $b = 0.3$ [5]. With these parameters it is not yet known if this system is chaotic. However, computer simulations show that the system has an attractor. An LDV controller for the Hénon map was found in [2]. Since the Hénon map has an attractor [11], one can expect that the attractor does not change for small perturbation of the parameters [9]. In this case, Proposition 4 implies that the optimal controller should not change too drastically for small changes in the parameters if the attractor does not change to much. However, the Hénon map is not structurally stable. For example, if $a = 1.392$ and $b = 0.3$, then the attractor S is non-trivial with aperiodic orbits and the map appears to be chaotic. However, there are parameters $\hat{a}$ arbitrarily close to 1.392 such that the attractor is simply an infinite set of periodic orbits [10]. Thus, for arbitrarily small changes in the parameters, the dynamics of the system drastically change. However, according the results presented here, for small changes in the parameters, the closed loop system will remain stable and the optimal controller should only slightly vary. Figure 1 confirms this fact and shows $\log\left(d_{C^1}\left(f,\hat{f}\right) + H\left(S,\hat{S}\right)\right)$ versus $\log\left(d_{S,\hat{S}}\left(X,\hat{X}\right)\right)$ where X is the solution to the functional algebraic Riccati equation (4) with $a = 1.4$ and $b = 0.3$ and $\hat{X}$ is the solution for other values of a and b.

6 Conclusion

It has been shown that LDV systems are well behaved under small perturbations of the nonlinear subsystem. In particular, for C^1 or Lipschitz perturbations, a stabilized system remains stable. For continuous perturbations, the system merely remains ultimately bounded. However, the size

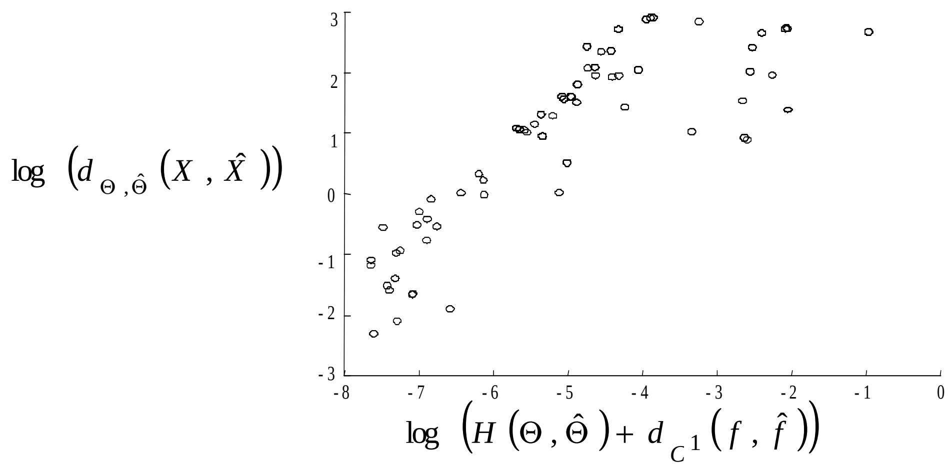

Figure 1: Optimal quadratic cost versus system perturbation

of the residual set is continuous in the perturbation. Therefore, it is not required to have perfect knowledge of the nonlinear system when designing the controller. Indeed, the actual system may not even be differentiable and yet methods based on linear approximation will be successful. An important result is that in the case of C^1 perturbations the optimal LDV controller is continuous in the perturbation. This feature of LDV controllers is utilized in efficient schemes to compute the solution to the functional algebraic Riccati equation [2].

References

[1] P. Apkarian. Advanced gain-scheduling techniques for uncertain systems. *IEEE Transactions on Control Systems Technology*, 6(1):21–32, 1998.

[2] S. Bohacek and E. Jonckheere. Linear dynamically varying LQ control of nonlinear systems over compact sets. *To appear in IEEE Transactions on Automatic Control.*

[3] S. Bohacek and E. Jonckheere. Nonlinear tracking over compact sets with linear dynamically varying H^∞ control. *Siam Journal of Control and Optimization*, (Submitted).

[4] A. Halanay and V. Ionescu. *Time-Varying Discrete Linear Systems.* Birkhauser Verlag, 1994.

[5] M. Henon. A two-dimensional mapping with a strange attractor. *Comm. Math. Phys.*, 50:69–77, 1976.

[6] E. A. Jonckheere and S. K. Bohacek. Ergodic and topological aspects of linear dynamically varying (LDV) control. *MTNS98*, 1998.

[7] A. Katok and B. Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems.* Cambridge University Press, 1995.

[8] A. Packard. Gain scheduling via linear fractional transformations. *Systems and Control Letters*, 22:79–92, 1994.

[9] S. Y. Pilyugin. *The Space of Dynamical Systems with the C^0-Topology.* Springer-Verlag, 1991.

[10] C. Robinson. Bifurcation to infinitely many sinks. *Comm. Math. Phys.*, 90:433–459, 1983.

[11] C. Robinson. *Dynamical Systems.* CRC Press, 1995.

[12] W. Rugh. *Linear System Theory.* Prentice Hall, New Jersey, 1996.

[13] M. Vidyasagar. *Nonlinear Systems Analysis.* Prentice Hall, 1993.

[14] F. Wu, X. H. Yang, A. Packard, and G. Becker. Induced l-2 norm control for LPV systems with bounded parameter variation rates. *International Journal of Robust and Nonlinear Control*, 6:983–998, 1996.