Nonlinear Tracking over Compact Sets with Linear Dynamically Varying H_∞ Control*

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Abstract

Linear dynamically varying H_∞ controllers are developed for tracking natural trajectories of a broad class of nonlinear systems defined over compact sets. It is shown that the existence of a suboptimal H_∞ controller is related to existence of a bounded solution to a functional algebraic Riccati equation. Even though nonlinear systems running over compact sets could exhibit sensitive dependence on initial conditions, the Riccati solution is continuous in the suboptimal case, but it may be discontinuous in the optimal case.

1 introduction

Nonlinear tracking has been thoroughly investigated. A popular approach is to linearize the system around an operating point, generate a linear controller for each operating point, and "schedule" the controllers in such a way that the closed-loop system remains stable as the operating point changes. In this approach, the nonlinear tracking error system is modeled, approximately, as a linear system with parameters that vary as the operating point varies. Such systems have been extensively studied [3], [4], [5], [6], [25], [28], [33] and are known as *linear parametrically varying* (LPV) systems.

For purpose of comparing the various LPV concepts, it is convenient to introduce *linear set-valued dynamically varying* (LSVDV) systems [11]:

$$\begin{bmatrix} x(k+1) \\ z(k) \end{bmatrix} = \begin{bmatrix} A_{\theta(k)} & B_{1_{\theta(k)}} & B_{2_{\theta(k)}} \\ C_{\theta(k)} & D_{1_{\theta(k)}} & D_{2_{\theta(k)}} \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \\ u(k) \end{bmatrix}, \tag{1}$$
$$\theta(k+1) \in \mathcal{F}(\theta(k)) \subseteq \Theta,$$
$$\text{with } \theta(0) = \theta_o \text{ and } x(0) = x_o.$$

Here, the parameters vector θ varies according to a set-valued dynamical system, continuous for the Hausdorff metric. w is the disturbance input, u the control, and z the controlled output.

In the most traditional LPV approach [4], [6], [5], [23], [25], all that is known about the parameter dynamics is that $\mathcal{F}(\theta) = \Theta$. The advantage of this model is that, if Θ is a convex polytope, then there are many computationally efficient controller synthesis methods [14]. Most of these approaches generate a suboptimal solution via a *linear matrix inequality* (LMI). However, these approaches can be conservative.

A slight refinement of the above LPV method consists in putting bounds on the rate at which the system parameters vary, i.e., $\mathcal{F}(\theta(k)) = \mathcal{B}_{\theta(k)}(\Delta)$, the ball with radius Δ and with its center at $\theta(k)$. There are efficient methods based on functional LMI's for designing controllers for

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these modified LPV systems [16], [33], [34], [35]. However, these design methods could fail when the parameters vary drastically; for example, when the controller needs to account for failures which lead to sudden changes in the system parameters [21]. Also, typically, these methods are conservative. Non-conservative LPV approaches are pursued in [11] and [29].

Another popular type of LPV systems are *jump linear* (JL) systems [15], [19]. Here $\Theta = \{\Theta_1, \Theta_2, ...\}$ is discrete and $\mathcal{F}(\theta(k))$ is equipped with a probability measure depending on $\theta(k)$ only, so that the transition among the Θ_i's is a Markov chain. The jump linear method for designing a controller for a such system is optimal (hence, nonconservative). The controller is provided by the solution to a system of coupled Riccati equations. Furthermore, there are efficient methods to compute the optimal controller [1], [2], [12].

A *linear dynamically varying* (LDV) system is a LSVDV system in which the parameter dynamics is completely known, that is, $\mathcal{F}(\theta(k))$ is reduced to a point $f(\theta(k))$. In [8], it was shown that a linear-quadratic controller for such a system (with $w = 0$) can be found by solving a *functional* algebraic Riccati equation (FARE). It should be noted that this functional algebraic Riccati *equation* is the LDV substitute for the functional linear matrix *inequality* of most other LPV approaches. Furthermore, the FARE of LDV design provides the *optimal* solution, while the functional LMI only provides a *suboptimal* solution. The mathematical difficulty with the LDV approach is to prove that the solution to the FARE is continuous, in which case the feedback gain matrix is a continuous function of the parameters. The LPV approaches described above avoid this continuity question by a priori assuming that the solution to the relevant functional LMI is continuous [33], polynomial [35], affine [16], [34], or even constant [4], [6], [5], [23], [25]. Since an arbitrary accuracy approximation of a discontinuous function has to duplicate the *exact* behavior at the discontinuity points, which are potentially uncountable in numbers, a discontinuous solution is numerically intractable, so that the continuity assumption is justifiable. However, it is important to know how constraining this continuity assumption is.

Tracking trajectories of the important class of hyperbolic nonlinear systems on compact sets can be accomplished by modeling the dynamics as a Markov chain [22] and resorting to JL methods. However, the resulting closed-loop system is only *stochastically* stable and it is not possible to directly show that the system is robustly stable. For this reason, the typical JL approach is not appropriate for the nonlinear tracking problem. The connection between JL and LDV control systems designs is examined in [10].

While in [8] LDV systems were stabilized using linear-quadratic methods, here, the same systems are stabilized by means of H^∞ methods. This paper shows that, if the parameter dynamics is completely known, then the existence of a suboptimal H^∞ controller is equivalent to existence of a continuous solution to the FARE. Of particular interest are LDV systems that arise as linearized versions of nonlinear tracking error dynamics. In this case, it can be shown that the linearization error is a bounded feedback around the linearized system, so that the H^∞ formulation is well-suited to minimize the effect of the error due to linearization and amplify the domain of attraction.

The paper proceeds as follows: The next section formalizes the tracking control problem of interest and shows how the tracking error dynamics can be approximated as an LDV system. Section 3 formally develops LDV systems. Section 4 develops the suboptimal H^∞ controller for this class of systems. Section 5 provides the proofs of the main technical results. Section 6 shows that these linear controllers are suitable for stabilization of nonlinear dynamical systems.

Notation: $|x(k)| := (x'(k)\, x(k))^{1/2}$, $\|x\|_{[k,j]} := \left(\sum_{i=k}^{j} x'(i)\, x(i)\right)^{1/2}$ and $\|x\|_{l_2} := \|x\|_{[0,\infty)}$. If $x \in \mathbb{R}^n$, then $\|x\|_\infty := \max_{i\le n} |x_i|$ and, if $x \in \mathbb{R}^{n\times\mathbb{Z}}$, $\|x\|_{l_\infty} := \sup_{k\in\mathbb{Z}} |x(k)|$. If A is a matrix, then $\|A\| := \sup_{|x|=1} |Ax|$, whereas, if $T : l_2 \to l_2$, then $\|T\| := \sup_{\|x\|_{l_2}=1} \|Tx\|_{l_2}$; the context in which these norms are used will resolve potential confusion. If $f : \Theta \times \mathbb{R}^m \to \Theta$ with $\Theta \subset \mathbb{R}^n$, denote $\frac{\partial f}{\partial \theta}(\theta, u)$ to be the Jacobian matrix of f where the derivatives are taken with respect to θ and are evaluated at $(\theta, u) \in \Theta \times \mathbb{R}^m$. Define $\frac{\partial f}{\partial u}(\theta, u)$ similarly. With reference to system (1), let $z_{\theta_o}(u, w, x_o)$ denote the output signal z due to initial conditions $\theta(0) = \theta_o$, $x(0) = x_o$ and input signals u and w. Let $z_{\theta_o}(u, w, x_o; k)$ denote this output at time k. Let $z_{\theta_o}(F, w, x_o)$ and $z_{\theta_o}(F, w, x_o; k)$ be defined similarly, except that the control u is replaced by the control law defined by F. For succinctness, we often write $f(\theta) := f(\theta, 0)$.

2 problem statement

A dynamical system, $\theta(k+1) = f(\theta(k))$ where $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, gives rise to a string of nested invariant subsets, $P(f) \subseteq \overline{P(f)} \subseteq \overline{R(f)} \subseteq NW(f)$, where $P(f)$ is the periodic set, $\overline{P(f)}$ its closure, $\overline{R(f)}$ the closure of the recurrent set, and $NW(f)$ is the nonwandering set [22]. We specifically consider systems where $NW(f)$ is bounded, in which case $\overline{P(f)}$, $\overline{R(f)}$ and $NW(f)$ are compact, and we choose the domain Θ to be any of those compact invariant sets. More generally, Θ could be taken to be any compact invariant subset. In particular, if f is an Axiom A diffeomorphism satisfying the strong transversality condition, then $NW(f)$ is a disjoint union of attractors [27], which by definition are compact and invariant and hence could be taken to be Θ. If the uniform hyperbolic conditions fails, f could still have an attractor, which could be taken to be Θ.

We take the control u to be a small perturbation of the parameters of the nominal dynamics f. More specifically, the nominal and perturbed dynamics are, respectively,

$$\theta(k+1) = f(\theta(k), 0) + v_1(k), \text{ with } \theta(0) = \theta_o, \tag{2}$$

$$\varphi(k+1) = f(\varphi(k), u(k)) + v_2(k), \text{ with } \varphi(0) = \varphi_o, \tag{3}$$

where

1. $$f \in C^1(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n). \tag{4}$$
2. $f(\Theta, 0) \subset \Theta$, i.e., Θ is f-invariant, and $f(\cdot, 0) : \Theta \to \Theta$.
3. Θ is a compact subset of $\mathbb{R}^n$.

Here $\{\theta(k) : k \geq 0\}$ is the desired trajectory and $\varphi(k)$ is the state of the system under control. The exogenous inputs $v_1(k)$ and $v_2(k)$ are typically small with $\theta(k+1) \in \Theta$. The purpose of v_1 is to allow the desired trajectory to occasionally jump from a point on one orbit to a nearby point on another orbit [9]. On the other hand, v_2 is to allow for some modeling inaccuracies. At time k, it is assumed that both $\theta(k)$ and $\varphi(k)$ are known. The basic objective is to find a control u such that, when $v_1 = v_2 = 0$ and for $|\theta(0) - \varphi(0)|$ small enough, we have $\lim_{k\to\infty} |\varphi(k) - \theta(k)| = 0$.

A distinguishing feature of the present approach is that the tracking controller takes the form of a *spatially varying* gain $F : \Theta \to \mathbb{R}^{m\times n}$, guaranteed to be continuous under suitable conditions. As the first and most generic application, given an *arbitrary* desired trajectory $\{\theta(k) : k = 0, \cdots\}$, evaluating the controller F along the trajectory $\{\theta(k) : k = 0, \cdots\}$ yields the time-varying controller $F_{\theta(k)}$ that makes the nonlinear system $\varphi(k+1) = f\left(\varphi(k), F_{\theta(k)}(\varphi(k) - \theta(k))\right)$ asymptotically track $\theta(k+1) = f(\theta(k))$. More importantly, the *globally* defined controller F becomes fully motivated in those specialized applications where there is a need to quickly adapt the tracking controller to a new reference trajectory without recomputing a new time-varying controller along the new trajectory [9], [13], [20], [21].

If $v_1 = v_2 = 0$, then stability of the closed-loop system, which implies asymptotic tracking, is guaranteed if $|\varphi(0) - \theta(0)| < R_{Capture}$ where $R_{Capture} > 0$. If $v_1 \neq 0$ and/or $v_2 \neq 0$, then asymptotic tracking can still be guaranteed if $\|v_1 - v_2\|_{l_\infty}$ and $|\varphi(0) - \theta(0)|$ are small enough and $v_1(k) - v_2(k)$ is intermittent enough. If $v_1 - v_2$ is persistent, then one cannot expect asymptotic tracking; however, under suitable conditions, the gain $\frac{\|\theta-\varphi\|_{\ell_\infty}}{\|v_1-v_2\|_{\ell_\infty}}$ can easily be shown to be bounded [7]. Besides, the effect of the model uncertainty v_2 can be minimized using standard H^∞ methods. Therefore, we shall no further pursue the investigation of the effect of v_1, v_2.

The tracking controller design relies on linearizing the tracking error dynamics as follows: Define the tracking error

$$x(k) := \varphi(k) - \theta(k).$$

Then

$$x(k+1) = f(\varphi(k), u(k)) - f(\theta(k), 0).$$

The first degree Taylor approximation of $f(\varphi(k), u(k))$ around $\varphi(k) = \theta(k)$ and $u(k) = 0$ yields

$$\begin{aligned} f(\varphi(k), u(k)) &= f(\theta(k), 0) + A_{\theta(k)}(\varphi(k) - \theta(k)) \\ &\quad + B_{2_{\theta(k)}} u(k) + \eta(x(k), u(k), \theta(k)), \end{aligned}$$

where

$$A_\theta := \frac{\partial f}{\partial \theta}(\theta, 0), \quad B_{2_\theta} := \frac{\partial f}{\partial u}(\theta, 0) \tag{5}$$

and $\eta(x(k), u(k), \theta(k))$ accounts for nonlinear terms. Thus

$$x(k+1) = A_{\theta(k)} x(k) + B_{2_{\theta(k)}} u(k) + \eta(x(k), u(k), \theta(k)). \tag{6}$$

Since $f \in C^1$, η can be decomposed as

$$\eta(x(k), u(k), \theta(k)) = \eta_x(x(k), u(k), \theta(k))\, x(k) + \eta_u(x(k), u(k), \theta(k))\, u(k), \tag{7}$$

where

$$\eta_x(x, u, \theta)_{i,j} = \int_0^1 \left(\frac{\partial f_i}{\partial x_j}(tx + \theta, tu) - \frac{\partial f_i}{\partial x_j}(\theta, 0) \right) dt \tag{8}$$

and

$$\eta_u(x, u, \theta)_{i,j} = \int_0^1 \left(\frac{\partial f_i}{\partial u_j}(tx + \theta, tu) - \frac{\partial f_i}{\partial u_j}(\theta, 0) \right) dt. \tag{9}$$

Since $f \in C^1$ and Θ is compact, if x and u are bounded, then $\frac{\partial f_i}{\partial x_j}(tx + \theta, tu) - \frac{\partial f_i}{\partial x_j}(\theta, 0)$ is uniformly continuous. In particular, for any $\varepsilon > 0$, there is a $\delta > 0$ such that, if $|x|, |u| < \delta$, then $\left| \frac{\partial f_i}{\partial x_j}(tx + \theta, tu) - \frac{\partial f_i}{\partial x_j}(\theta, 0) \right| < \varepsilon$. Therefore,

$$\lim_{\bar{x} \to 0, \bar{u} \to 0} \sup \{ \|\eta_x(x, u, \theta)\| : |x| < \bar{x},\ |u| < \bar{u},\ \theta \in \Theta \} = 0 \tag{10}$$

and

$$\lim_{\bar{x} \to 0, \bar{u} \to 0} \sup \{ \|\eta_u(x, u, \theta)\| : |x| < \bar{x},\ |u| < \bar{u},\ \theta \in \Theta \} = 0. \tag{11}$$

If u and x are small, we can approximate the error dynamics as

$$\begin{aligned} x(k+1) &= A_{\theta(k)} x(k) + B_{2_{\theta(k)}} u(k), \\ \theta(k+1) &= f(\theta(k), 0). \end{aligned} \tag{12}$$

This systems is linear in the tracking error x, but the coefficient matrices A and B vary (in general in a nonlinear way) as θ varies. Since $\theta(k)$ varies according to (2), the system described in (12) is a *Linear Dynamically Varying (LDV)* system. Before controllers can be developed for such systems, linear systems with dynamically varying parameters must be formalized.

3 linear dynamically varying systems and LQ control

Motivated by the preceding considerations, a general LDV system is defined as

$$\begin{bmatrix} x(k+1) \\ z(k) \end{bmatrix} = \begin{bmatrix} A_{\theta(k)} & B_{1_{\theta(k)}} & B_{2_{\theta(k)}} \\ C_{\theta(k)} & D_{1_{\theta(k)}} & D_{2_{\theta(k)}} \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \\ u(k) \end{bmatrix}, \tag{13}$$

$$\begin{aligned} &\theta(k+1) = f(\theta(k)), \\ &\text{with } \theta(0) = \theta_o \text{ and } x(0) = x_o, \end{aligned} \tag{14}$$

subject to the following general conditions:

1. $\Theta \subset \mathbb{R}^n$ is compact and $f : \Theta \to \Theta$ is a continuous function.

2. $A : \Theta \to \mathbb{R}^{n\times n}$, $B_1 : \Theta \to \mathbb{R}^{n\times l}$, $B_2 : \Theta \to \mathbb{R}^{n\times m}$, $C : \Theta \to \mathbb{R}^{p\times n}$, $D_1 : \Theta \to \mathbb{R}^{p\times l}$, and $D_2 : \Theta \to \mathbb{R}^{p\times m}$ are functions that need not be continuous.

In the above, $\theta(k) \in \Theta$ is the state of the dynamic system, $x(k) \in \mathbb{R}^n$ is the state of the linear system, $u(k) \in \mathbb{R}^m$ is the control input, $w(k) \in \mathbb{R}^l$ is the disturbance input, and $z(k) \in \mathbb{R}^p$ is the output to be controlled.

It is often assumed that the system coefficient matrices A, B_1, B_2, C, D_1, and D_2 are continuous. We will refer to such systems as *continuous* LDV systems. In Section 2, it was assumed that $f \in C^1$, and since A and B are matrices of partial derivatives of f, A and B are indeed continuous. Thus the tracking error system associated with (2) and (3) can be approximated by a *continuous* LDV system. However, if a feedback $F : \Theta \to \mathbb{R}^{m\times n}$ is used to stabilize a continuous LDV system, then the resulting closed-loop system is a continuous LDV system if and only if F is continuous. Although this paper will focus on stabilizing continuous LDV systems, we cannot a priori assume that the feedback is continuous. Therefore, the definition of an LDV system must allow for possibly discontinuous coefficient matrices.

Since an LDV system is an uncountable collection of linear time-varying systems indexed by $\theta(0)$, the concept of stability is slightly more complex in the dynamically varying case than it is in the time-varying case.

The LDV system (13) is uniformly exponentially stable if, for $u(k) = 0$ and $w(k) = 0$, there exist an $\alpha \in [0, 1)$ and a $\beta < \infty$ such that, for all $\theta(0) \in \Theta$,

$$|x(k)| \leq \beta\alpha^k |x(0)| .$$

System (13) is exponentially stable if, for $u(k) = 0$, $w(k) = 0$ and for each $\theta(0) \in \Theta$, there exist an $\alpha_{\theta(0)} \in [0, 1)$ and a $\beta_{\theta(0)} < \infty$ such that, for all $x(j)$ and $j \leq k$,

$$|x(k)| \leq \beta_{\theta(0)}\alpha_{\theta(0)}^{k-j} |x(j)| .$$

System (13) is asymptotically stable if, for $u(k) = 0$, $w(k) = 0$, any $|x(0)| < \infty$, and any $\theta(0) \in \Theta$,

$$|x(k)| \to 0 \text{ as } k \to \infty.$$

Note that an exponentially stable system is stable uniformly in time k, but not necessarily uniformly in the initial condition $\theta(0)$. That is, along any given positive trajectory $\{f^k(\theta(0)) : k \geq 0\}$, an exponentially stable system is (uniformly in time) exponentially stable; however, if $\{\theta(0)_i : i \geq 0\}$ is a convergent sequence, with $\theta(0) = \lim_{i\to\infty} \theta(0)_i$, it is possible that $\alpha_{\theta(0)_i} \to 1$ while $\alpha_{\theta(0)} < 1$, in which case the system is exponentially stable, but not $\theta(0)$-uniformly exponentially stable. To emphasize the difference between exponential and uniformly exponential stability, exponential stability will occasionally be referred to as uniform *in time* exponential stability.

In the case of continuous LDV systems, asymptotic, exponential and uniform exponential stability are equivalent (Proposition 2 in [8]). Since uniformly exponentially stable systems are inherently more robust than exponentially stable systems, it is preferable to remain within the confines of continuous LDV systems. Thus, when synthesizing a feedback for controlling a continuous LDV system, it is important to ensure that the feedback is not only asymptotically stabilizing, but also continuous. However, to maintain generality, an LDV system is considered stabilizable if there exists an exponentially stabilizing feedback, that is,

System (13) is stabilizable if there exists a, not necessarily continuous, function $F : \mathbb{Z} \times \Theta \to \mathbb{R}^{m\times n}$ with bound $\overline{F}_{\theta(0)} < \infty$ such that, for all $\theta(0) \in \Theta$ and for all $k \geq 0$, we have $\left\|F_{\theta(0)}(k)\right\| \leq \overline{F}_{\theta(0)}$ and the system

$$\begin{aligned} x(k+1) &= \left(A_{\theta(k)} + B_{2_{\theta(k)}} F_{\theta(0)}(k)\right) x(k), \\ \theta(k) &= f^k(\theta(0)) \end{aligned}$$

is exponentially stable. That is, there exist $\alpha_{\theta(0)} \in [0,1)$ and $\beta_{\theta(0)} < \infty$ such that, for any $\theta(0) \in \Theta$, there exists a time-varying, bounded feedback $F_{\theta(0)}(k)$, which may depend on $\theta(0)$, such that

$$\left\| \prod_{i=j}^{k-1} \left(A_{f^i(\theta(0))} + B_{2_{f^i(\theta(0))}} F_{\theta(0)}(i) \right) \right\| \leq \beta_{\theta(0)} \alpha_{\theta(0)}^{k-j},$$

where the factors of the matrix product are taken in the proper order.

Therefore, along every trajectory $\left\{ f^k(\theta(0)) : k \geq 0 \right\}$, the time-varying system is (uniformly in time) exponentially stabilizable by means of a function F which, as defined in Definition 3, depends on the initial condition $\theta(0)$. In this sense, the control is not quite "closed-loop" and more importantly there are no assumptions about the global properties of the feedback F. In particular, the feedback may not be a continuous nor even a uniformly bounded function of $\theta(0)$. However, in the case of continuous LDV systems, it was shown in [8] that a stabilizable system has a continuous, uniformly exponentially stabilizing feedback $F : \Theta \rightarrow \mathbb{R}^{m \times n}$. In this case, the feedback gain takes the form $F_{\theta(k)}$ and does not depend on the initial condition $\theta(0)$, but only the current state $\theta(k)$. Hence the controller is "closed-loop."

The dual concept of detectability has two versions. The first one is *uniform* detectability.

System (13) is uniformly detectable if there exists a, not necessarily continuous, function $H : \Theta \rightarrow \mathbb{R}^{n \times p}$ with uniform bound $\bar{H} < \infty$ such that, for all $\theta \in \Theta$, we have $\|H_\theta\| \leq \bar{H}$ and the system

$$\begin{aligned} x(k+1) &= \left(A_{\theta(k)} + H_{\theta(k)} C_{\theta(k)} \right) x(k), \\ \theta(k) &= f^k(\theta(0)) \end{aligned}$$

is uniformly exponentially stable. That is, there exist an $\alpha_d \in [0,1)$ and a $\beta_d < \infty$ such that, for all $\theta(0) \in \Theta$,

$$\|x(k)\| \leq \beta_d \alpha_d^k \|x(0)\|.$$

System (13) is detectable if there exists a, not necessarily continuous, function $H : \mathbb{Z} \times \Theta \rightarrow \mathbb{R}^{n \times p}$ with bound $\bar{H}_{\theta(0)} < \infty$ such that, for all $\theta(0) \in \Theta$ and all k, we have $\left\| H_{\theta(0)}(k) \right\| \leq \overline{H}_{\theta(0)} < \infty$ and the system

$$\begin{aligned} x(k+1) &= \left(A_{\theta(k)} + H_{\theta(0)}(k) C_{\theta(k)} \right) x(k), \\ \theta(k) &= f^k(\theta(0)) \end{aligned}$$

is exponentially stable.

If f is invertible, the LDV system has an adjoint system running backwards in time. If a continuous LDV is detectable and f is invertible, then the adjoint system is stabilizable. It is easily shown that this implies that the adjoint LDV is in fact *uniformly* stabilizable and therefore that the LDV system is *uniformly* detectable. Thus, if f is invertible and the LDV system is continuous, then uniform detectability and detectability are equivalent. Although stabilizability and uniform detectability are slightly asymmetric, to avoid putting extra assumptions on f, stabilizable and uniformly detectable continuous LDV systems will be considered.

Since stabilizability only depends on A, B_2 and f, we will say that the triple (A, B_2, f) is stabilizable to mean that system (13) is stabilizable. Similarly, we say that the triple (A, C, f) is uniformly detectable to mean that system (13) is uniformly detectable.

Since an LDV system is a collection of time-varying systems, the following time-varying Lyapunov stability theorem is useful:

Assume that system (13) is uniformly detectable and $w \equiv 0$. Then there exist an $\alpha_{\theta_o} \in [0,1)$ and a $\beta_{\theta_o} < \infty$ such that, for $\theta(0) = \theta_o$ and any $x(j) \in \mathbb{R}^n$,

$$|x(k)| \leq \beta_{\theta_o} \alpha_{\theta_o}^{k-j} |x(j)|$$

if and only if there exists a sequence $\left\{X_{f^k(\theta_o)} : k \geq 0\right\}$ with bound $\bar{X} : \Theta \to \mathbb{R}$ such that $\left\|X_{f^k(\theta_o)}\right\| \leq \bar{X}_{\theta_o} < \infty$, $X_{f^k(\theta_o)} = X'_{f^k(\theta_o)} \geq 0$, and

$$A'_{f^k(\theta_o)} X_{f^{k+1}(\theta_o)} A_{f^k(\theta_o)} - X_{f^k(\theta_o)} \leq -C'_{f^k(\theta_o)} C_{f^k(\theta_o)}. \tag{15}$$

Furthermore, if equation (15) is satisfied, then α_{θ_o} and β_{θ_o} can be taken to only depend on the bound $\bar{X}_{\theta_o}$ and on α_d and β_d in the definition of detectability.

For $\theta(0)$ fixed, the system is a time-varying system. Thus the theorem is simply a statement about the stability of linear time-varying systems and can be found on page 41 in [18].

Assume that system (13) is uniformly detectable and $w \equiv 0$. Then there exists an $\alpha \in [0, 1)$ and a $\beta < \infty$ such that

$$|x(k)| \leq \beta \alpha^k |x(0)|$$

if and only if there exists a uniformly bounded function $X : \Theta \to \mathbb{R}^{n \times n}$ with $X_\theta = X'_\theta \geq 0$ such that, for all $\theta_o \in \Theta$,

$$A'_{f^k(\theta_o)} X_{f^{k+1}(\theta_o)} A_{f^k(\theta_o)} - X_{f^k(\theta_o)} \leq -C'_{f^k(\theta_o)} C_{f^k(\theta_o)}. \tag{16}$$

Since X_θ is uniformly bounded and the system is uniformly detectable, Theorem 3 can be applied at each $\theta_o \in \Theta$.

The main result of [8] is the following:

Suppose that

1. $f : \Theta \to \Theta$ is continuous and Θ is compact.
2. The functions A, B_2, C, D_2 are continuous.
3. $D'_{2_\theta} D_{2_\theta} > 0$.
4. $C'_\theta D_{2_\theta} = 0$ for all $\theta \in \Theta$ and (A, C, f) is uniformly detectable.

Then the triple (A, B_2, f) is stabilizable if and only if there exists a unique, uniformly bounded solution $X_2 : \Theta \to \mathbb{R}^{n \times n}$ such that

1. X_2 satisfies the functional algebraic Riccati equation (FARE)

$$X_{2_\theta} = A'_\theta X_{2_{f(\theta)}} A_\theta - A'_\theta X_{2_{f(\theta)}} B_{2_\theta} \left(D'_{2_\theta} D_{2_\theta} + B'_{2_\theta} X_{2_{f(\theta)}} B_{2_\theta}\right)^{-1} B'_{2_\theta} X_{2_{f(\theta)}} A_\theta + C'_\theta C_\theta. \tag{17}$$

2. $X_{2_\theta} \geq 0$.

In this case, the closed-loop control

$$u_{LQ}(k) := -\left(D'_{2_{\theta(k)}} D_{2_{\theta(k)}} + B'_{2_{\theta(k)}} X_{2_{f(\theta(k))}} B_{2_{\theta(k)}}\right)^{-1} B'_{2_{\theta(k)}} X_{2_{f(\theta(k))}} A_{\theta(k)} x(k) \tag{18}$$

uniformly exponentially stabilizes system (13). Moreover, for $|x(0)| < \infty$ and $w \equiv 0$,

$$x'(0) X_{2_{\theta(0)}} x(0) = \inf \left\{ \sum_{k=0}^{\infty} |z(k)|^2 : u \in l_2 \right\}, \tag{19}$$

where the is infimum is attained for $u = u_{LQ}$. Furthermore, X_2 is a uniformly continuous function. Finally, if $X_{2_\theta}(k, N+1)$ solves the finite horizon Riccati equation, i.e.,

$$\begin{aligned} X_{2_\theta}(k, N+1) = {} & A'_{f^k(\theta)} X_{2_\theta}(k+1, N+1) A_{f^k(\theta)} + C'_{f^k(\theta)} C_{f^k(\theta)} \\ & - A'_{f^k(\theta)} X_{2_\theta}(k+1, N+1) B_{2_{f^k(\theta)}} \\ & \times \left(D'_{2_{f^k(\theta)}} D_{2_{f^k(\theta)}} + B'_{2_{f^k(\theta)}} X_{2_\theta}(k+1, N+1) B_{2_{f^k(\theta)}}\right)^{-1} \\ & \times B'_{2_{f^k(\theta)}} X_{2_\theta}(k+1, N+1) A_{f^k(\theta)} \end{aligned} \tag{20}$$

with

$$X_{2_\theta}(N+1, N+1) = C'_{f^{N+1}(\theta)} C_{f^{N+1}(\theta)},$$

then $X_{2_\theta}(0, N+1) \to X_{2_\theta}$ uniformly in θ.

4 linear dynamically varying H^∞ control

In the following, the H^∞ control problem for LDV systems of the general form (13) will be formulated and the solution will be provided. There are two related problems.

The first is the finite horizon problem. For all $\theta \in \Theta$, let the terminal weighting $X_\theta(N+1,N+1) \geq 0$ be given. The objective in this problem is to find a controller F_u such that, if

$$u(k) = F_{u_{\theta_o}}(k, N+1) \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} \text{ for } k \leq N,$$

then

Objective A For $x(0) = 0$, there exists an $\varepsilon > 0$ such that, for $w \in l_2[0,N]$ and $\theta_o \in \Theta$,

$$\|z\|^2_{[0,N]} - \gamma^2 \|w\|^2_{[0,N]} + x'(N+1) X_{\theta_o}(N+1,N+1) x(N+1) \leq -\varepsilon \|w\|^2_{[0,N]}.$$

The second problem is the infinite horizon problem where the objective is to find a uniformly exponentially stabilizing controller F_u such that, if

$$u(k) = F_{u_{\theta_o}}(k) \begin{bmatrix} x(k) \\ w(k) \end{bmatrix},$$

then

Objective B For $x(0) = 0$, there exists an $\varepsilon > 0$ such that, for $w \in l_2$ and $\theta_o \in \Theta$,

$$\|z\|^2_{l_2} - \gamma^2 \|w\|^2_{l_2} \leq -\varepsilon \|w\|^2_{l_2}$$

and if $w = 0$ and $x(0) \neq 0$, then $x(k) \to 0$.

If Objective B is achieved, then

$$\frac{\|z\|_{l_2}}{\|w\|_{l_2}} < \gamma.$$

It will be shown that the solution to Objective B is the limit as $N \to \infty$ of solutions to Objective A.

4.1 finite horizon full information controller

For notational simplicity define $\begin{bmatrix} A_\theta & \bar{B}_\theta \\ \bar{C}_\theta & \bar{D}_\theta \end{bmatrix} := \begin{bmatrix} A_\theta & B_{1_\theta} & B_{2_\theta} \\ C_\theta & D_{1_\theta} & D_{2_\theta} \\ 0 & I_l & 0 \end{bmatrix}$ and $J =: \begin{bmatrix} I_p & 0 \\ 0 & -\gamma^2 I_l \end{bmatrix}$.

Let $X_{\theta_o}(N+1,N+1) \geq 0$ be given. In a recursive manner, define

$$\begin{aligned} X_{\theta_o}(k,N+1) &= A'_{f^k(\theta_o)} X_{\theta_o}(k+1,N+1) A_{f^k(\theta_o)} + C'_{f^k(\theta_o)} C_{f^k(\theta_o)} \\ &\quad - L_{\theta_o}(k,N+1)' R^{-1}_{\theta_o}(k,N+1) L_{\theta_o}(k,N+1), \end{aligned} \tag{21}$$

where

$$R_{\theta_o}(k,N+1) := \bar{D}'_{f^k(\theta_o)} J \bar{D}_{f^k(\theta_o)} + \bar{B}'_{f^k(\theta_o)} X_{\theta_o}(k+1,N+1) \bar{B}_{f^k(\theta_o)}, \tag{22}$$

$$L_{\theta_o}(k,N+1) := \bar{D}'_{f^k(\theta_o)} J \bar{C}_{f^k(\theta_o)} + \bar{B}'_{f^k(\theta_o)} X_{\theta_o}(k+1,N+1) A_{f^k(\theta_o)}. \tag{23}$$

We partition $R = \begin{bmatrix} R_1 & R_2' \\ R_2 & R_3 \end{bmatrix}$ and $L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$ such that $R_3 \in \mathbb{R}^{m \times m}$ and $L_2 \in \mathbb{R}^{m \times n}$. With the assumption that $R_{\theta_o}(k,N+1)$ is nonsingular, the Schur decomposition yields

$$\begin{aligned} &R_{\theta_o}(k,N+1) \\ &= \begin{bmatrix} I & R'_{2_{\theta_o}}(k,N+1,\theta_o) R^{-1}_{3_{\theta_o}}(k,N+1) \\ 0 & I \end{bmatrix} \begin{bmatrix} \nabla_{\theta_o}(k,N+1) & 0 \\ 0 & R_{3_{\theta_o}}(k,N+1) \end{bmatrix} \\ &\times \begin{bmatrix} I & 0 \\ R^{-1}_{3_{\theta_o}}(k,N+1) R_{2_{\theta_o}}(k,N+1) & I \end{bmatrix}, \end{aligned}$$

where

$$\nabla_{\theta_o}(k, N+1) := R_{1_{\theta_o}}(k, N+1) - R'_{2_{\theta_o}}(k, N+1)\, R^{-1}_{3_{\theta_o}}(k, N+1)\, R_{2_{\theta_o}}(k, N+1). \tag{24}$$

Note that since $R_{3_{\theta_o}}(k, N+1) = D'_{2_{f^k(\theta_o)}} D_{2_{f^k(\theta_o)}} + B'_{2_{f^k(\theta_o)}} X_{\theta_o}(k, N+1)\, B_{2_{f^k(\theta_o)}}$ and $D'_{2_{f^k(\theta_o)}} D_{2_{f^k(\theta_o)}} > 0$, we have

$$R_{3_{\theta_o}}(k, N+1) > 0 \tag{25}$$

whenever $X_{\theta_o}(k, N+1) \geq 0$. Hence, if

$$X_{\theta_o}(k, N+1) \geq 0, \tag{26}$$
$$\nabla_{\theta_o}(k, N+1) \leq -\varrho I, \tag{27}$$

then $R_{\theta_o}(k, N+1)$ is nonsingular.

For X, R, and ∇ defined as above, it is possible to show by completion of squares (see [17] page 485) that, for all $x(k)$ and all $u, w \in l_2[0, N]$, we have

$$\begin{aligned} & \|z\|^2_{[k,N]} - \gamma^2 \|w\|^2_{[k,N]} + x'(N+1)\, X_{\theta_o}(N+1, N+1)\, x(N+1) \\ & = x'(k)\, X_{\theta_o}(k, N+1)\, x(k) + \\ & + \sum_{j=k}^{N} (u(j) - u_N(j))'\, R_{3_{\theta_o}}(j, N+1)\,(u(j) - u_N(j)) \\ & + \sum_{j=k}^{N} (w(j) - w_N(j))'\, \nabla_{\theta_o}(j, N+1)\,(w(j) - w_N(j)), \end{aligned} \tag{28}$$

with

$$\begin{aligned} w_N(k) &:= -\nabla^{-1}_{\theta_o}(k, N+1)\, L_{\nabla_{\theta_o}}(k, N+1)\, x(k), \\ u_N(k) &:= -R^{-1}_{3_{\theta_o}}(k, N+1) \begin{bmatrix} L_{2_{\theta_o}}(k, N+1) & R_{2_{\theta_o}}(k, N+1) \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix}, \end{aligned} \tag{29}$$

and where

$$L_{\nabla_{\theta_o}}(k, N+1) := L_{1_{\theta_o}}(k, N+1) - R'_{2_{\theta_o}}(k, N+1)\, R^{-1}_{3_{\theta_o}}(k, N+1)\, L_{2_{\theta_o}}(k, N+1). \tag{30}$$

From (25), (27) and (28), it is clear that, for $\theta(0) = \theta_o$,

$$\begin{aligned} & x'_o X_{\theta_o}(0, N+1)\, x_o \\ & = \sup_{w \in l_2[0,N]} \inf_{u \in l_2[0,N]} \left\{ \|z\|^2_{[0,N]} - \gamma^2 \|w\|^2_{[0,N]} + x'(N+1)\, X_{\theta_o}(N+1, N+1)\, x(N+1) \right\}. \end{aligned} \tag{31}$$

The above is summarized by the following theorem which is a straightforward extension of [17], page 484.

Suppose $D'_{2_{f^k(\theta_o)}} D_{2_{f^k(\theta_o)}} > 0$ for all $k \leq N$ and $X_{\theta_o}(N+1, N+1) \geq 0$. In this case, there exists a causal full information control $u(k) = F_{u_{\theta_o}}(k, N+1) \begin{bmatrix} x(k) \\ w(k) \end{bmatrix}$ that satisfies Objective A if and only if, for $0 \leq k \leq N+1$, the following conditions hold:

1. $X_{\theta_o}(k, N+1)$ satisfies the time-varying Riccati recursion (21).

2. For some $\varrho > 0$, (26) and (27) hold.

In this case, the control given by (29) achieves Objective A.

4.2 infinite horizon full information controller

The second problem is the infinite horizon problem where the objective is to find a (uniformly in time) exponentially stabilizing controller F such that, if

$$u(k) = F_{\theta_o}(k) \begin{bmatrix} x(k) \\ w(k) \end{bmatrix},$$

then Objective B can be achieved. The following assumptions on system (13) are needed:

1. $f : \Theta \to \Theta$ is continuous and Θ is compact.
2. The system parameters A, B_1, B_2, C, D_1 and D_2 are matrix-valued continuous functions of θ.
3. $D'_{2_\theta} D_{2_\theta} > 0$ for all $\theta \in \Theta$.
4. For all $\theta \in \Theta$, we have $D'_{2_\theta} \begin{bmatrix} C_\theta & D_{1_\theta} \end{bmatrix} = 0$ and the triple (A, C, f) is uniformly detectable.
5. The triple (A, B_2, f) is stabilizable.

Assumption 4 is equivalent to:

4′ The triple $\left(A - B_2 (D'_2 D_2)^{-1} D'_2 C, \left(I - D_2 (D'_2 D_2)^{-1} D'_2\right) C, f\right)$ is uniformly detectable.

Indeed, if Assumption 4′ holds, then the feedback

$$\begin{aligned} u(k) = &-\left(D'_{2_{f^k(\theta)}} D_{2_{f^k(\theta)}}\right)^{-1} D'_{2_{f^k(\theta)}} C_{f^k(\theta)} x(k) \\ &-\left(D'_{2_{f^k(\theta)}} D_{2_{f^k(\theta)}}\right)^{-1} D'_{2_{f^k(\theta)}} D_{1_{f^k(\theta)}} w(k) + r(k) \end{aligned}$$

converts it to Assumption 4. Perhaps these assumptions could be weakened (for example, see [30]), but they are common.

The main result of the paper is the following:

Suppose Assumptions 1-5 hold. There exists a (uniformly in time) exponentially stabilizing controller $u(k) = F_{u_{\theta_o}}(k) \begin{bmatrix} x(k) \\ w(k) \end{bmatrix}$ such that Objective B can be achieved if and only if there exists a uniformly bounded map $X_\infty : \Theta \to \mathbb{R}^{n \times n}$ such that

1. X_∞ satisfies the FARE

$$X_{\infty_\theta} = C'_\theta C_\theta + A'_\theta X_{\infty_{f(\theta)}} A_\theta - L'_\theta R_\theta^{-1} L_\theta, \tag{32}$$

   where

$$\begin{aligned} R_\theta &:= \overline{D}'_\theta J \overline{D}_\theta + \bar{B}'_\theta X_{\infty_{f(\theta)}} \bar{B}_\theta, \\ L_\theta &:= \overline{D}'_\theta J \bar{C}_\theta + \bar{B}'_\theta X_{\infty_{f(\theta)}} A_\theta. \end{aligned} \tag{33}$$

2. For some $\varrho > 0$ and all $\theta \in \Theta$,

$$\begin{aligned} &X_{\infty_\theta} \geq 0, \\ &\nabla_\theta := R_{1_\theta} - R'_{2_\theta} R_{3_\theta}^{-1} R_{2_\theta} \leq -\varrho I. \end{aligned} \tag{34}$$

3. The closed-loop system

$$x(k+1) = \left(A_{\theta(k)} - \bar{B}_{\theta(k)} R_{\theta(k)}^{-1} L_{\theta(k)}\right) x(k) \tag{35}$$

   is uniformly exponentially stable.

In this case, the control

$$u_{\infty}(k) := -R_{3_{\theta(k)}}^{-1} \begin{bmatrix} L_{2_{\theta(k)}} & R_{2_{\theta(k)}} \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} \tag{36}$$

achieves Objective B, X_{∞} is continuous, and the closed-loop system with control (36), that is,

$$\begin{aligned} x(k+1) = & \left(A_{f^k(\theta_o)} - B_{2_{f^k(\theta_o)}} R_{3_{f^k(\theta_o)}}^{-1} L_{2_{f^k(\theta_o)}}\right) x(k) \\ & + \left(B_{1_{f^k(\theta_o)}} - B_{2_{f^k(\theta_o)}} R_{3_{f^k(\theta_o)}}^{-1} L_{2_{f^k(\theta_o)}}\right) w(k), \end{aligned}$$

is a uniformly (in θ) exponentially stable system.

The proof of this theorem is withheld until Section 5. The proof entails the major difficulty of proving continuity relative to $\theta(0)$, an issue that does not exist in the traditional time-varying case of [18] and [26]. Even though our approach is inspired from [17], [30] and [31], the continuity issue of the LDV case makes it of interest in its own right.

The control $u(k)$ produced by Theorem (4.2) depends on $w(k)$. Since $w(k)$ is meant to model the linearization error (see Section 6), it will likely depend on $u(k)$. Thus $u(k)$ and $w(k)$ are linked by some algebraic relationship, which may not be easily solved. The following shows how to find a control $u(k)$ that depends on the information $x(k)$ only. This type of control is referred to as *strictly causal*.

Suppose the assumptions of Theorem 4.2 hold and there exists a controller as described. Suppose also that $R_{1_\theta} \leq -\varrho I$. Then the above control can be taken to be strictly proper. In particular, the control

$$\begin{aligned} u_*(k) := & -\left(R_{3_{\theta(k)}} - R_{2_{\theta(k)}} R_{1_{\theta(k)}}^{-1} R_{2_{\theta(k)}}'\right)^{-1} \\ & \times \left(L_{2_{\theta(k)}} - R_{2_{\theta(k)}} R_{1_{\theta(k)}}^{-1} L_{1_{\theta(k)}}\right) x(k) \\ = & -\Delta_{\theta(k)}^{-1} L_{\Delta_{\theta(k)}} x(k), \end{aligned}$$

where

$$\Delta_\theta := R_{3_\theta} - R_{2_\theta} R_{1_\theta}^{-1} R_{2_\theta}'$$

and

$$L_{\Delta_\theta} := L_{2_\theta} - R_{2_\theta} R_{1_\theta}^{-1} L_{1_\theta},$$

achieves Objective B.

This corollary follows as a minor variation of the proof of Theorem 4.2.

The above results show the importance of the functional algebraic Riccati equation (FARE) (32). Solving a functional equation may be computationally difficult. However, in [8], [10] several methods for solving the FARE associated with a linear-quadratic objective were developed. These methods can easily be extended to solving the FARE (32). Furthermore, the stability of (35) can be checked via their respective FARE's.

The continuity of the solution to the FARE is crucial when numerically computing it. For example, suppose that $\Theta = [0, 1]$ and that there exists a jump discontinuity at some point $0 \leq \rho \leq 1$ and that $X_\theta := \begin{cases} 0 \text{ if } \theta \leq \rho \\ \delta \text{ otherwise} \end{cases}$. Consider the construction of $\hat{X}$, an approximation of X, with error $\varepsilon < \delta$, i.e. $\left\| X_\theta - \hat{X}_\theta \right\| < \varepsilon$ for all $\theta \in \Theta$. In general, the point ρ would be estimated via some search method. However, unless ρ is known *exactly* (which entails an infinite search), $\left\| X_{\theta^*} - \hat{X}_{\theta^*} \right\| > \varepsilon$ for some $\theta^* \in \Theta$. If θ^* is a fixed point of f, then $\left\| X_{f^k(\theta^*)} - \hat{X}_{f^k(\theta^*)} \right\| > \varepsilon$ for all k, and a similar problem occurs if θ^* is a recurrent point of f. In general, if $X : \Theta \to \mathbb{R}^{n \times n}$ is continuous and Θ is compact, then X can be estimated by its value at a finite number of points. If X is not continuous, such an estimate is not possible in general. It is this continuity issue, and hence the ability to numerically evaluate the Riccati solution, that is the main distinction between an LDV controller and a family of infinite-horizon, time-varying controllers.

Another difference between an LDV controller and a family of infinite-horizon, time-varying controllers is that the LDV controller guarantees that the closed-loop system is uniformly exponentially stable, whereas the family of time-varying controllers only guarantees stability along every trajectory $\{\theta(k) : k \geq 0\}$. One situation where this distinction is important is noise rejection. For example, suppose that the signal w in system (13) is bounded as $\|w\|_{l_\infty} \leq \bar{w}$. Such a situation arises when the f in (2) is different from the f in (3). Then it follows from Section 6 that the maximum allowable $\bar{w}$ depends on the parameters α_{θ_o} and β_{θ_o} in the definition of stability. Hence, we write $\bar{w}_{\theta_o}$. Now suppose that the system is not *uniformly* exponentially stable, i.e. there exists a sequence $\{\theta(0)_i : i \geq 0\}$ such that either $\lim_{i\to\infty} \alpha_{\theta(0)_i} = 1$ or $\lim_{i\to\infty} \beta_{\theta(0)_i} = \infty$. In this case, even though for each θ_o, $\bar{w}_{\theta_o} > 0$, we have $\lim_{i\to\infty} \bar{w}_{\theta(0)_i} = 0$, that is, there is no positive bound on the noise that results in a stable system for all initial conditions θ_o.

5 proof of main theorem

5.1 necessity

In the following, it is assumed that

Assumption A Assumptions 1-5 of Theorem 4.2 hold and there exists a stabilizing controller that achieves Objective B.

Since (A, C, f) is uniformly detectable, $D_{2_\theta}' D_{2_\theta} > 0$, and (A, B_2, f) is stabilizable, the optimal stabilizing LDV linear-quadratic controller exists (Theorem 3). That is, there exists a unique, continuous, bounded function $X_2 : \Theta \to \mathbb{R}^{n\times n}$ such that $X_{2_\theta}' = X_{2_\theta} \geq 0$ solves equation (17). Furthermore, for $w \equiv 0$,

$$\inf_{u\in l_2} \|z\|_{l_2}^2 = x_o' X_{2_\theta} x_o \tag{37}$$

and this infimum is attained for u given by (18).

Define $X_{\theta_o}(k, N+1)$ as in (21) with terminal cost $X_{2_{f^{N+1}(\theta_o)}}$. It will be shown (Lemma 5.1) that

$$X_{\infty_\theta} = \lim_{N\to\infty} X_\theta(0, N+1) \tag{38}$$

provides a solution to equation (32) (see Lemma 5.1) such that system (35) is uniformly exponentially stable (see Lemma 5.1) and inequalities (34) are satisfied (see inequalities (59) and (60)). Furthermore, the convergence in equation (38) is uniform in θ, and hence X_{∞_θ} is a continuous function in θ (see Lemma 5.1), and the control given by (36) satisfies Objective B (see Lemma 5.1).

The proof of the following lemma follows from an easy adaptation of the arguments of Section B.2.3 of [17] and [31].

If Assumption A holds and $X_{\theta_o}(k, N+1)$ is given by (21) and $\nabla_{\theta_o}(k, N+1)$ is given by (24), then

1. For $\theta_o = \theta(0) \in \Theta$, all $k \leq N+1$ and $N \geq 0$, we have $\nabla_{\theta_o}(k, N+1) \leq -\varrho I$ and $X_{\theta_o}(k, N+1) \geq 0$.

2. For all $\theta_o \in \Theta$, there exists a $\bar{X}_{\infty_{\theta_o}} < \infty$ such that $\|X_{\theta_o}(k, N+1)\| \leq \bar{X}_{\infty_{\theta_o}}$ for all $k \leq N+1$ and all $N \geq 0$.

3. $X_{\theta_o}(k, N+1)$ is monotone increasing in N.

The bound $\bar{X}_{\infty_{\theta_o}}$ depends on θ_o, so we cannot say that there exists a single bound on $X_{\theta_o}(k, N+1)$ for all $\theta_o \in \Theta$. Since Θ is compact, if $\bar{X}_\infty$ is continuous, then $\bar{X}_\infty$ is bounded. However, we have not yet shown that $\bar{X}_\infty$ is continuous.

For fixed θ_o, $X_{\theta_o}(k, N+1)$ exists, is bounded, and is nondecreasing in N. Thus

$$X_{\theta_o}(k) := \lim_{N\to\infty} X_{\theta_o}(k, N+1)$$

exists for $k < \infty$. Furthermore, $X_{\theta_o}(k)$ solves

$$X_{\theta_o}(k) = A'_{f^k(\theta_o)} X_{\theta_o}(k+1)\, A_{f^k(\theta_o)} + C'_{f^k(\theta_o)} C_{f^k(\theta_o)} - L'_{\theta_o}(k)\, R^{-1}_{\theta_o}(k)\, L_{\theta_o}(k), \tag{39}$$

where

$$\begin{aligned} R_{\theta_o}(k) &:= \overline{D}'_{f^k(\theta_o)} J \overline{D}_{f^k(\theta_o)} + \bar{B}'_{f^k(\theta_o)} X_{\theta_o}(k+1)\, \bar{B}_{f^k(\theta_o)}, \\ L_{\theta_o}(k) &:= \overline{D}'_{f^k(\theta_o)} J \bar{C}_{f^k(\theta_o)} + \bar{B}'_{f^k(\theta_o)} X_{\theta_o}(k+1)\, A_{f^k(\theta_o)}. \end{aligned}$$

This is simply the Riccati equation associated with the infinite horizon, time-varying H^∞ control problem. Note that, since $X_{\theta_o}(k, N+1) \geq 0$,

$$X_{\theta_o}(k) \geq 0. \tag{40}$$

Next, since $X_{\theta_o}(k, N+1)$ converges, $\nabla_{\theta_o}(k) := \lim_{N\to\infty} \nabla_{\theta_o}(k, N+1)$ exists. Furthermore, $\nabla_{\theta_o}(k, N+1) \leq -\varrho I$ (from finite horizon problem) implies that $\nabla_{\theta_o}(k) \leq -\varrho I$. Furthermore, since $X_{\theta_o}(k)$ is bounded and $D'_{2_\theta} D_{2_\theta} > 0$ for all $\theta \in \Theta$, it is clear from (24) that $\nabla_{\theta_o}(k)$ is bounded from below. Hence

$$-\infty < \nabla_{\theta_o}(k) \leq -\varrho I. \tag{41}$$

Similarly, define $L_{\nabla_{\theta_o}}(k)$ as the limit of equation (30).

Next define

$$u_\infty(k) := F_{u_{\theta_o}}(k) \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} := -R^{-1}_{3_{\theta_o}}(k) \begin{bmatrix} L_{2_{\theta_o}}(k) & R_{2_{\theta_o}}(k) \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} \tag{42}$$

and

$$w_\infty(k) := F_{w_{\theta_o}}(k)\, x(k) := -\nabla^{-1}_{\theta_o}(k)\, L_{\nabla_{\theta_o}}(k)\, x(k). \tag{43}$$

It will be shown that, with $\theta(0) = \theta_o$, (42) is the best control and (43) is the worst disturbance (in the sense of Objective B).

For $w = 0$, the control $u(k) = u_\infty(k)$ given by (42) makes the closed-loop system $x(k+1) = A_{u_{\theta_o}}(k) x(k)$, where

$$A_{u_{\theta_o}}(k) := A_{\theta(k)} - B_{2_{\theta(k)}} R^{-1}_{3_{\theta_o}}(k)\, L_{2_{\theta_o}}(k), \tag{44}$$

exponentially stable.

Since $u_\infty(k) = -R^{-1}_{3_{\theta_o}}(k)\, L_{2_{\theta_o}}(k)\, x(k) - R^{-1}_{3_{\theta_o}}(k)\, R_{2_{\theta_o}}(k)\, w(k)$, the closed-loop system with $w = 0$ and $u = u_\infty$ is

$$\begin{aligned} x(k+1) &= \left(A_{\theta(k)} - B_{2_{\theta(k)}} R^{-1}_{3_{\theta_o}}(k)\, L_{2_{\theta_o}}(k)\right) x(k) \\ &= A_{u_{\theta_o}}(k)\, x(k). \end{aligned}$$

Set $\Gamma_{\theta_o}(k) = X_{\theta_o}(k) - X_{2_{f^k(\theta_o)}}$. By Lemma 5.1, $X_{\theta_o}(k, N+1) \geq X_{\theta_o}(k,k) = X_{2_{f^k(\theta_o)}}$. Thus $X_{\theta_o}(k) = \lim_{N\to\infty} X_{\theta_o}(k, N+1) \geq X_{2_{f^k(\theta_o)}}$ and $\Gamma_{\theta_o}(k) \geq 0$. It is possible to show (see [17], Eq. (B.2.39)) that

$$\begin{aligned} \Gamma_{\theta_o}(k) \geq{}& A'_{u_{\theta_o}}(k)\, \Gamma_{\theta_o}(k+1)\, A_{u_{\theta_o}}(k) \\ &+ A'_{u_{\theta_o}}(k)\, \Gamma_{\theta_o}(k+1)\, B_{2_{\theta(k)}} \\ &\times \left(R_{3_{\theta_o}}(k) - B'_{2_{\theta(k)}} \Gamma_{\theta_o}(k+1)\, B_{2_{\theta(k)}}\right)^{-1} B'_{2_{\theta(k)}} \Gamma_{\theta_o}(k+1)\, A_{u_{\theta_o}}(k). \end{aligned} \tag{45}$$

However,

$$A_\theta - B_{2_\theta} \left(D'_{2_\theta} D_{2_\theta} + B'_{2_\theta} X_{2_{f(\theta)}} B_{2_\theta}\right)^{-1} B'_{2_\theta} X_{2_{f(\theta)}} A_\theta$$

is the closed-loop system if the linear-quadratic feedback is used, that is, if $w = 0$ and $u = u_{LQ}$ where u_{LQ} is given by (18). After some manipulation, we find

$$\begin{aligned}
&A_{f^k(\theta_o)} - B_{2_{f^k(\theta_o)}} \left(D'_{2_{f^k(\theta_o)}} D_{2_{f^k(\theta_o)}} + B'_{2_{f^k(\theta_o)}} X_{2_{f^{k+1}(\theta_o)}} B_{2_{f^k(\theta_o)}} \right)^{-1} B'_{2_{f^k(\theta_o)}} X_{2_{f^{k+1}(\theta_o)}} A_{f^k(\theta_o)} \\
&= A_{u_{\theta_o}}(k) \\
&+ B_{2_{f^k(\theta_o)}} \left(R_{3_{\theta_o}}(k) - B'_{2_{f^k(\theta_o)}} \Gamma_{\theta_o}(k+1) B_{2_{f^k(\theta_o)}} \right)^{-1} B'_{2_{f^k(\theta_o)}} \Gamma_{\theta_o}(k+1) A_{u_{\theta_o}}(k)
\end{aligned}$$

and

$$\begin{aligned}
&B_{2_{f^k(\theta)}} \left(R_{3_{\theta_o}}(k) - B'_{2_{f^k(\theta)}} \Gamma_{\theta_o}(k+1) B_{2_{f^k(\theta)}} \right)^{-1} \\
&= B_{2_{f^k(\theta)}} \left(D'_{2_{f^k(\theta)}} D_{2_{f^k(\theta)}} + B'_{2_{f^k(\theta)}} X_{2_{f^{k+1}(\theta_o)}} B_{2_{f^k(\theta)}} \right)^{-1},
\end{aligned}$$

which is bounded since $D'_{2_\theta} D_{2_\theta} > 0$ and $X_{2_\theta} \geq 0$, for all $\theta \in \Theta$. Therefore

$$\begin{aligned}
\xi(k+1) &= A_{u_{\theta_o}}(k) \xi(k), \\
\upsilon(k) &= B'_{2_{\theta(k)}} \Gamma_{\theta_o}(k+1) A_{u_{\theta_o}}(k) \xi(k)
\end{aligned}$$

is a uniformly detectable system. Moreover,

$$\left(R_{3_{\theta_o}}(k) - B'_{2_{f^k(\theta_o)}} \Gamma_{\theta_o}(k+1) B_{2_{f^k(\theta_o)}} \right)^{-1} > 0$$

and $\|\Gamma_{\theta_o}(k)\| = \left\| X_{\theta_o}(k) - X_{2_{f^k(\theta_o)}} \right\| \leq \|X_{\theta_o}(k)\| \leq \bar{X}_{\infty\theta_o} < \infty$. Therefore, equation (45) is a Lyapunov equation and Theorem 3 implies that

$$x(k+1) = A_{u_{\theta_o}}(k) x(k)$$

is a exponentially stable system.

We cannot as yet claim that A_u as defined by (44) is uniformly exponentially stable in the sense of Definition 3. To conclude uniformly exponential stability, $\Gamma_{\theta_o}(k)$, the solution to the Lyapunov equation (45), must be uniformly bounded for all $\theta_o \in \Theta$ and all k. $\Gamma_{\theta_o}(k)$ is uniformly bounded only if $X_{\theta_o}(k)$, the solution to equation (39), is uniformly bounded. Lemma 5.1 will show that $X_{\theta_o}(k)$ is uniformly bounded.

Let $u_\infty(k) = F_{u_{\theta_o}}(k) \begin{bmatrix} x(k) \\ w(k) \end{bmatrix}$ where F_u is given by (42). Let $w_\infty(k) = F_{w_{\theta_o}}(k) x(k)$ be defined as in (43). Then, for $w \in l_2$,

$$\|z_{\theta_o}(F_u, w, x_o)\|_{l_2}^2 - \gamma^2 \|w\|_{l_2}^2 = x'_o X_{\theta_o}(k) x_o + \sum_{k=0}^{\infty} (w(k) - w_\infty(k))' \nabla_{\theta_o}(k) (w(k) - w_\infty(k)). \tag{46}$$

(See the end of Section 1 for the definition of the notation $z_{\theta_o}(F_u, w, x_o)$.)

By the previous lemma, $u_\infty(k) = F_{u_{\theta_o}}(k) \begin{bmatrix} x(k) \\ w(k) \end{bmatrix}$ is exponentially stabilizing. Hence, if $w \in l_2$, then $x \in l_2$, and $x(k) \to 0$. Furthermore, $X_{\theta_o}(k)$ is bounded and, by equation (41), $-\infty < \nabla_{\theta_o}(k) \leq -\varrho I$. Thus $F_{w_{\theta_o}}(k)$ is bounded, where F_w is defined by (43). Hence $w_\infty \in l_2$. Thus, letting $N \to \infty$ in equation (28) yields (46).

Now, it is shown that the control u_∞ achieves Objective B.

If $x(0) = 0$ and $u(k) = u_\infty(k) = F_{u_{\theta_o}}(k) \begin{bmatrix} x(k) \\ w(k) \end{bmatrix}$, then, for all $w \in l_2$, $\|z\|_{l_2}^2 - \gamma^2 \|w\|_{l_2}^2 \leq -\varepsilon \|w\|_{l_2}^2$.

By Assumption A, there exists an exponentially stabilizing control u_* that satisfies Objective B, that is, if $u = u_*$, $x(0) = 0$ and $w \in l_2$, then $x \in l_2$ and $\|z_{\theta_o}(u_*, w, 0)\|_{l_2}^2 - \gamma^2 \|w\|_{l_2}^2 \leq -\varepsilon \|w\|_{l_2}^2$. Since $X_{\theta_o}(k)$ is bounded and $\nabla_{\theta_o}(k) \leq -\varrho I$, $F_{u_{\theta_o}}(k)$ and $F_{w_{\theta_o}}(k)$ are bounded. Therefore, $u_\infty(k) = F_{u_{\theta_o}}(k) \begin{bmatrix} x_{\theta_o}(u_*, w, 0; k) \\ w(k) \end{bmatrix} \in l_2$ and $w_\infty(k) = F_{w_{\theta_o}}(k) x_{\theta_o}(u_*, w, 0; k) \in l_2$. Thus we can take the limit of equation (28) as $N \to \infty$, which yields

$$\begin{aligned} -\varepsilon \|w\|_{l_2}^2 \geq & \qquad (47) \\ \|z_{\theta_o}(u_*, w, 0)\|_{l_2}^2 - \gamma^2 \|w\|_{l_2}^2 &= \sum_{k=0}^{\infty} (u_*(k) - u_\infty(k))' R_{3_{\theta_o}}(k) (u_*(k) - u_\infty(k)) \\ &+ \sum_{k=0}^{\infty} (w(k) - w_\infty(k))' \nabla_{\theta_o}(k) (w(k) - w_\infty(k)) \\ &\geq \sum_{k=0}^{\infty} (w(k) - w_\infty(k))' \nabla_{\theta_o}(k) (w(k) - w_\infty(k)) \\ &= \|z_{\theta_o}(F_u, w, 0)\|_{l_2}^2 - \gamma^2 \|w\|_{l_2}^2, \end{aligned}$$

where the last equality follows from Lemma 5.1.

From equation (47), it is clear that u_∞ is the best control and w_∞ is the worst disturbance in the sense of Objective B.

$\sup_{w \in l_2} \inf_{u \in l_2} \|z_{\theta_o}(u, w, x_o)\|_{l_2}^2 - \gamma^2 \|w\|_{l_2}^2 = \|z_{\theta_o}(F_u, w_\infty, x_o)\|_{l_2}^2 - \gamma^2 \|w_\infty\|_{l_2}^2 = x_o' X_{\theta_o}(0) x_o$.

Since $\nabla_{\theta_o}(k) \leq -\varrho I$, if $u(k) = u_\infty(k) = F_{u_{\theta_o}}(k) \begin{bmatrix} x(k) \\ w(k) \end{bmatrix}$, then (46) implies that

$$\begin{aligned} &\sup_{w \in l_2} \|z_{\theta_o}(F_u, w, x_o)\|_{l_2}^2 - \gamma^2 \|w\|_{l_2}^2 \\ &= x_o' X_{\theta_o}(0) x_o + \sup_{w \in l_2} \sum_{k=0}^{\infty} (w(k) - w_\infty(k))' \nabla_{\theta_o}(k) (w(k) - w_\infty(k)) \\ &= x_o' X_{\theta_o}(k) x_o, \end{aligned}$$

where $w_\infty(k) = F_{w_{\theta_o}}(k) x(k)$. Therefore,

$$\begin{aligned} \sup_{w \in l_2} \inf_{u \in l_2} \|z_{\theta_o}(u, w, x_o)\|_{l_2}^2 - \gamma^2 \|w\|_{l_2}^2 &\leq \sup_{w \in l_2} \|z_{\theta_o}(F_u, w, x_o)\|_{l_2}^2 - \gamma^2 \|w\|_{l_2}^2 \\ &= x_o' X_{\theta_o}(k) x_o. \end{aligned} \qquad (48)$$

Similarly, if $w = w_\infty$, then $\inf_{u \in l_2} \|z_{\theta_o}(u, w_\infty, x_o)\|_{l_2}^2 - \gamma^2 \|w_\infty\|_{l_2}^2 = \|z_{\theta_o}(F_u, w_\infty, x_o)\|_{l_2}^2 - \gamma^2 \|w_\infty\|_{l_2}^2 = x_o' X_{\theta_o}(k) x_o$. Therefore,

$$\sup_{w \in l_2} \inf_{u \in l_2} \|z_{\theta_o}(u, w, x_o)\|_{l_2}^2 - \gamma^2 \|w\|_{l_2}^2 \geq \inf_{u \in l_2} \|z_{\theta_o}(u, w_\infty, x_o)\|_{l_2}^2 - \gamma^2 \|w_\infty\|_{l_2}^2 = x_o' X_{\theta_o}(k) x_o. \qquad (49)$$

Combining inequalities (48) and (49) yields the desired result.

Up to this point, $\theta(0) = \theta_o \in \Theta$ has been fixed. $X_{\theta_o}(k)$ is nothing more than the stabilizing solution of the time-varying Riccati equation associated with the time-varying system

$$\begin{aligned} x(k+1) &= A_{f^k(\theta_o)} x(k) + B_{1_{f^k(\theta_o)}} w(k) + B_{2_{f^k(\theta_o)}} u(k), \\ z(k) &= C_{f^k(\theta_o)} x(k) + D_{1_{f^k(\theta_o)}} w(k) + D_{2_{f^k(\theta_o)}} u(k). \end{aligned}$$

Since θ_o is arbitrary, for all $\theta \in \Theta$, define $X_\infty : \Theta \to \mathbb{R}^{n \times n}$ by

$$X_{\infty_\theta} := \lim_{N \to \infty} X_\theta(0, N+1). \qquad (50)$$

The function

$$X_\infty : \Theta \to \mathbb{R}^{n\times n},$$
$$\theta \mapsto X_{\infty_\theta}$$

satisfies the FARE (32)-(33), viz.,

$$\begin{aligned} X_{\infty_\theta} &= A'_\theta X_{\infty_{f(\theta)}} A_\theta + C'_\theta C_\theta \\ &\quad - \left(\bar{D}'_\theta J\bar{C}_\theta + \bar{B}'_\theta X_{\infty_{f(\theta)}} A_\theta\right)' \left(\bar{D}'_\theta J\bar{D}_\theta + \bar{B}'_\theta X_{\infty_{f(\theta)}} \bar{B}_\theta\right)^{-1} \left(\bar{D}'_\theta J\bar{C}_\theta + \bar{B}'_\theta X_{\infty_{f(\theta)}} A_\theta\right). \end{aligned} \tag{51}$$

Let $f(\theta_1) = \theta_2$. Clearly,

$$X_{\theta_1}(N+1, N+1) = X_{2_{f^{N+1}(\theta_1)}} = X_{2_{f^N(\theta_2)}} = X_{\theta_2}(N, N). \tag{52}$$

Next, by equation (21),

$$\begin{aligned} & X_{\theta_1}(N, N+1) \\ &= A'_{f^N(\theta_1)} X_{\theta_1}(N+1, N+1) A_{f^N(\theta_1)} + C'_{f^N(\theta_1)} C_{f^N(\theta_1)} \\ &- \left(\bar{D}'_{f^N(\theta_1)} J\bar{C}_{f^N(\theta_1)} + \bar{B}'_{f^N(\theta_1)} X_{\theta_1}(N+1, N+1) A_{f(\theta_1)}\right)' \\ &\times \left(\bar{D}'_{f^N(\theta_1)} J\bar{D}_{f^N(\theta_1)} + \bar{B}'_{f^N(\theta_1)} X_{\theta_1}(N+1, N+1) \bar{B}_{f^N(\theta_1)}\right)^{-1} \\ &\times \left(\bar{D}'_{f^N(\theta_1)} J\bar{C}_{f^N(\theta_1)} + \bar{B}'_{f^N(\theta_1)} X_{\theta_1}(N+1, N+1) A_{f^N(\theta_1)}\right) \\ &= A'_{f^{N-1}(\theta_2)} X_{\theta_2}(N, N) A_{f^{N-1}(\theta_2)} + C'_{f^{N-1}(\theta_2)} C_{f^{N-1}(\theta_2)} \\ &- \left(\bar{D}'_{f^{N-1}(\theta_2)} J\bar{C}_{f^{N-1}(\theta_2)} + \bar{B}'_{f^{N-1}(\theta_2)} X_{\theta_2}(N, N) A_{f^{N-1}(\theta_2)}\right)' \\ &\times \left(\bar{D}'_{f^{N-1}(\theta_2)} J\bar{D}_{f^{N-1}(\theta_2)} + \bar{B}'_{f^{N-1}(\theta_2)} X_{\theta_2}(N, N) \bar{B}_{f^{N-1}(\theta_2)}\right)^{-1} \\ &\times \left(\bar{D}'_{f^{N-1}(\theta_2)} J\bar{C}_{f^{N-1}(\theta_2)} + \bar{B}'_{f^{N-1}(\theta_2)} X_{\theta_2}(N, N) A_{f^{N-1}(\theta_2)}\right) \\ &= X_{\theta_2}(N-1, N). \end{aligned} \tag{53}$$

Repeating the above, we reach the result:

$$X_{\theta_1}(k, N+1) = X_{\theta_2}(k-1, N). \tag{54}$$

Setting $k = 0$ and $\theta = \theta_1$ in equation (21) and substituting $X_{\theta_2}(0, N)$ for $X_{\theta_1}(1, N+1)$ into the right-hand side yields

$$\begin{aligned} X_{\theta_1}(0, N+1) &= A'_{\theta_1} X_{\theta_2}(0, N) A_{\theta_1} + C'_{\theta_1} C_{\theta_1} \\ &\quad - \left(\bar{D}'_{\theta_1} J\bar{C}_{\theta_1} + \bar{B}'_{\theta_1} X_{\theta_2}(0, N) A_{\theta_1}\right)' \left(\bar{D}'_{\theta_1} J\bar{D}_{\theta_1} + \bar{B}'_{\theta_1} X_{\theta_2}(0, N) \bar{B}_{\theta_1}\right)^{-1} \\ &\quad \times \left(\bar{D}'_{\theta_1} J\bar{C}_{\theta_1} + \bar{B}'_{\theta_1} X_{\theta_2}(0, N) A_{\theta_1}\right). \end{aligned} \tag{55}$$

Next, we take the limit as $N \to \infty$. In order to take this limit, we must ensure that the left-hand side is continuous in $X_{\theta_2}(0, N)$. Since $R_{3_{\theta_2}}(0, N) \geq D'_{2_{\theta_2}} D_{2_{\theta_2}} > 0$ and $\nabla_{\theta_2}(0, N) \leq -\varrho I$ for all N and R_3 and ∇ are continuous functions of $X_{\theta_2}(0, N)$, we have $\lim_{N\to\infty} R_{3_{\theta_2}}(0, N) \geq D'_{2_{\theta_2}} D_{2_{\theta_2}} > 0$ and $\lim_{N\to\infty} \nabla_{\theta_2}(0, N) \leq -\varrho I$. Thus $\left(\bar{D}'_{\theta_1} J\bar{D}_{\theta_1} + B'_{\theta_1} Y B_{\theta_1}\right)^{-1}$ exists for Y in a neighborhood of $X_{\infty_{\theta_2}}$. Therefore,

$$\lim_{N\to\infty} \left(\bar{D}'_{\theta_1} J\bar{D}_{\theta_1} + \bar{B}'_{\theta_1} X_{\theta_2}(0, N) \bar{B}_{\theta_1}\right)^{-1} = \left(\bar{D}'_{\theta_1} J\bar{D}_{\theta_1} + \bar{B}'_{\theta_1} X_{\infty_{\theta_2}} \bar{B}_{\theta_1}\right)^{-1}. \tag{56}$$

Likewise

$$\lim_{N\to\infty} \left(\bar{D}'_{\theta_1} J\bar{C}_{\theta_1} + \bar{B}'_{\theta_1} X_{\theta_2}(0, N) A_{\theta_1}\right) = \left(\bar{D}'_{\theta_1} J\bar{C}_{\theta_1} + \bar{B}'_{\theta_1} X_{\infty_{\theta_2}} A_{\theta_1}\right). \tag{57}$$

Thus

$$\begin{aligned}X_{\infty\theta_1} &= \lim_{N\to\infty} X_{\theta_1}(0,N+1) &(58)\\
&= \lim_{N\to\infty} \left(A'_{\theta_1}X_{\theta_2}(0,N)A_{\theta_1} + C'_{\theta_1}C_{\theta_1}\right.\\
&- \left(\bar{D}'_{\theta_1}J\bar{C}_{\theta_1} + \bar{B}'_{\theta_1}X_{\theta_2}(0,N)A_{\theta_1}\right)'\left(\bar{D}'_{\theta_1}J\bar{D}_{\theta_1} + \bar{B}'_{\theta_1}X_{\theta_2}(0,N)\bar{B}_{\theta_1}\right)^{-1}\\
&\times \left.\left(\bar{D}'_{\theta_1}J\bar{C}_{\theta_1} + \bar{B}'_{\theta_1}X_{\theta_2}(0,N)A_{\theta_1}\right)\right)\\
&= A'_{\theta_1}X_{\infty\theta_2}A_{\theta_1} + C'_{\theta_1}C_{\theta_1}\\
&- \left(\bar{D}'_{\theta_1}J\bar{C}_{\theta_1} + \bar{B}'_{\theta_1}X_{\infty\theta_2}A_{\theta_1}\right)'\left(\bar{D}'_{\theta_1}J\bar{D}_{\theta_1} + \bar{B}'_{\theta_1}X_{\infty\theta_2}\bar{B}_{\theta_1}\right)\\
&\times \left(\bar{D}'_{\theta_1}J\bar{C}_{\theta_1} + \bar{B}'_{\theta_1}X_{\infty\theta_2}A_{\theta_1}\right).\end{aligned}$$

Since $f(\theta_1) = \theta_2$, equation (51) follows.

Now, we can drop the dependence on the initial condition θ_o in R, L and ∇, that is, $R_{\theta(k)} := R_{\theta_o}(k)$, $L_{\theta(k)} := L_{\theta_o}(k)$, $\nabla_{\theta(k)} := \nabla_{\theta_o}(k)$. As a simple consequence of (40),

$$X_{\infty\theta} \geq 0 \tag{59}$$

and by (41),

$$\nabla_\theta \leq -\varrho I. \tag{60}$$

Thus the best control u_∞ and worst disturbance w_∞ feedback matrices depend only on the current state $\theta(k)$. That is, equations (42) and (43) can be rewritten as

$$u_\infty(k) = F_{u_{\theta(k)}}\begin{bmatrix} x(k) \\ w(k) \end{bmatrix} := -R^{-1}_{3_{\theta(k)}}\begin{bmatrix} L_{2_{\theta(k)}} & R_{2_{\theta(k)}} \end{bmatrix}\begin{bmatrix} x(k) \\ w(k) \end{bmatrix} \tag{61}$$

and

$$w_\infty(k) = F_{w_{\theta(k)}}x(k) := -\nabla^{-1}_{\theta(k)}L_{\nabla_{\theta(k)}}x(k). \tag{62}$$

The function X_∞ given by (50) is uniformly bounded. That is, there exists a $\bar{X}_\infty < \infty$ such that $\|X_{\infty\theta}\| < \bar{X}_\infty$.

Let $X_{2_\theta} = X'_{2_\theta} \geq 0$ be the solution to (17) and define

$$A_{LQ_\theta} := A_\theta - B_{2_\theta}\left(D'_{2_\theta}D_{2_\theta} + B'_{2_\theta}X_{2_{f(\theta)}}B_{2_\theta}\right)^{-1}B'_{2_\theta}X_{2_{f(\theta)}}A_\theta$$

to be the closed-loop state transition matrix with $w = 0$ and $u = u_{LQ}$ given by equation (18). Define

$$\begin{aligned}&v(k)\\
&:= \sum_{i=k}^{\infty}\left(\prod_{j=k}^{i} A_{LQ_{f^j(\theta_o)}}\right)'\left(X_{2_{f^{i+1}(\theta_o)}}B_{1_{f^i(\theta_o)}}w(i) + C'_{f^{i+1}(\theta_o)}D_{1_{f^{i+1}(\theta_o)}}w(i+1)\right)\end{aligned}$$

and

$$\begin{aligned}G_{\theta_o}(w,x_o;k) := &\left(D'_{2_{f^k(\theta_o)}}D_{2_{f^k(\theta_o)}} + B'_{2_{f^k(\theta_o)}}X_{2_{f^{k+1}(\theta_o)}}B_{2_{f^k(\theta_o)}}\right)^{-1}B'_{2_{f^k(\theta_o)}}\\
&\times\left(X_{2_{f^{k+1}(\theta_o)}}A_{f^k(\theta_o)}x(k) - v(k)\right).\end{aligned} \tag{63}$$

It is possible to show (see page 157, Claim 3 in [26] or Lemma 9.6 in [30]) that, if $w \in l_2$, then

$$G_{\theta_o}(w,x_o) = \arg\inf\left\{\|z_{\theta_o}(u,w,x_o)\|_{l_2} : u \in l_2\right\}, \tag{64}$$

where the notation $z_{\theta_o}(u,w,x_o)$ was introduced at the end of Section 1. Note that $G_{\theta_o}(w,x_o)$ and therefore $z_{\theta_o}(G_{\theta_o}(w,x_o),w,x_o)$ are linear in (w,x_o).

By assumption, there exists a control satisfying Objective B. Thus $G_{\theta_o}(w,0)$ must also satisfy Objective B, that is,

$$\left\|z_{\theta_o}\left(G_{\theta_o}(w,0),w,0\right)\right\|_{l_2}^2-\gamma^2\|w\|_{l_2}^2\le-\varepsilon\|w\|_{l_2}^2 \tag{65}$$

and

$$\left\|z_{\theta_o}\left(G_{\theta_o}(w,0),w,0\right)\right\|_{l_2}<\gamma\|w\|_{l_2}. \tag{66}$$

If $w\equiv 0$, then $v=0$, and by comparing (18) and (63) we see that $G_{\theta_o}(0,x_o;k)=u_{LQ}(k)$, that is, $G_{\theta_o}(0,x_o;k)$ is the optimal linear-quadratic control given by (18). Thus, if $w=0$, then by (19)

$$\left\|z_{\theta_o}\left(G_{\theta_o}(0,x_o),0,x_o\right)\right\|_{l_2}^2=\inf_{u\in l_2}\left\|z_{\theta_o}(u,0,x_o)\right\|_{l_2}^2=x_o'X_{2_{\theta_o}}x_o, \tag{67}$$

where X_{2_θ} is the solution to the Riccati equation (17). It was shown in [8] that X_2 is uniformly bounded. Denote this bound by $\bar{X}_2$, that is, for all $\theta\in\Theta$, $\|X_{2_\theta}\|\le\bar{X}_2<\infty$. Hence,

$$\left\|z_{\theta_o}\left(G_{\theta_o}(0,x_o),0,x_o\right)\right\|_{l_2}^2=x_o'X_{2_{\theta_o}}x_o\le\bar{X}_2|x_o|<\infty. \tag{68}$$

Combining equations (65), (66), and (68) yields

$$\begin{aligned}
&\left\|z_{\theta_o}\left(G_{\theta_o}(w,x_o),w,x_o\right)\right\|_{l_2}^2-\gamma^2\|w\|_{l_2}^2 \qquad (69)\\
&=\left\|z_{\theta_o}\left(G_{\theta_o}(0,x_o),0,x_o\right)+z_{\theta_o}\left(G_{\theta_o}(w,0),w,0\right)\right\|_{l_2}^2-\gamma^2\|w\|_{l_2}^2\\
&=\left\|z_{\theta_o}\left(G_{\theta_o}(0,x_o),0,x_o\right)\right\|_{l_2}^2+2\left\langle z_{\theta_o}\left(G_{\theta_o}(w,0),w,0\right),z_{\theta_o}\left(G_{\theta_o}(0,x_o),0,x_o\right)\right\rangle\\
&+\left\|z_{\theta_o}\left(G_{\theta_o}(w,0),w,0\right)\right\|_{l_2}^2-\gamma^2\|w\|_{l_2}^2\\
&\le\left\|z_{\theta_o}\left(G_{\theta_o}(0,x_o),0,x_o\right)\right\|_{l_2}^2+2\left\langle z_{\theta_o}\left(G_{\theta_o}(w,0),w,0\right),z_{\theta_o}\left(G_{\theta_o}(0,x_o),0,x_o\right)\right\rangle-\varepsilon\|w\|_{l_2}^2\\
&\le x_o'X_{2_{\theta_o}}x_o+2\gamma\|w\|_{l_2}\sqrt{x_o'x_o\bar{X}_2}-\varepsilon\|w\|_{l_2}^2\\
&=x_o'X_{2_{\theta_o}}x_o+\|w\|_{l_2}\left(2\gamma|\sqrt{x_o'x_o\bar{X}_2}-\varepsilon\|w\|_{l_2}\right)\\
&\le x_o'X_{2_{\theta_o}}x_o+\max_{\|w\|\in\mathbb{R}}\left\{\|w\|_{l_2}\left(2\gamma|x_o|\sqrt{\bar{X}_2}-\varepsilon\|w\|_{l_2}\right)\right\}\\
&\le x_o'X_{2_{\theta_o}}x_o+\frac{\gamma^2\bar{X}_2|x_o|^2}{\varepsilon}\le|x_o|^2\left(\bar{X}_2+\frac{\gamma^2\bar{X}_2}{\varepsilon}\right)<\infty.
\end{aligned}$$

Thus

$$\begin{aligned}
&\sup_{w\in l_2}\inf_{u\in l_2}\left\|z_{\theta_o}(u,w,x_o)\right\|_{l_2}^2-\gamma^2\|w\|_{l_2}^2\\
&=\sup_{w\in l_2}\left\|z_{\theta_o}\left(G_{\theta_o}(w,x_o),w,x_o\right)\right\|_{l_2}^2-\gamma^2\|w\|_{l_2}^2\le|x_o|^2\left(\bar{X}_2+\frac{\gamma^2\bar{X}_2}{\varepsilon}\right)<\infty.
\end{aligned}$$

Lemma 5.1 implies that

$$x_o'X_{\infty_{\theta_o}}x_o=\sup_{w\in l_2}\inf_{u\in l_2}\left\|z_{\theta_o}(u,w,x_o)\right\|_{l_2}^2-\gamma^2\|w\|_{l_2}^2\le|x_o|^2\left(\bar{X}_2+\frac{\gamma^2\bar{X}_2}{\varepsilon}\right)<\infty. \tag{70}$$

Note that the worst disturbance has the property,

$$\|w_\infty\|_{l_2}^2\le P|x_o|^2, \tag{71}$$

where

$$P:=\frac{4\gamma^2\bar{X}_2}{\varepsilon^2}|x_o|^2. \tag{72}$$

To see this, observe that, if $\|w_\infty\|_{l_2}^2 > P|x_o|^2$, then equation (69) implies that

$$\begin{aligned}
&\|z_{\theta_o}(G_{\theta_o}(w_\infty, x_o), w_\infty, x_o)\|_{l_2}^2 - \gamma^2 \|w_\infty\|_{l_2} \\
&\leq x_o' X_{2_{\theta_o}} x_o + \|w\|_{l_2}\left(2\gamma|x_o|\sqrt{\bar{X}_2} - \varepsilon\|w\|_{l_2}\right) \\
&< x_o' X_{2_{\theta_o}} x_o = \|z_{\theta_o}(G_{\theta_o}(0, x_o), 0, x_o)\|_{l_2}^2 .
\end{aligned}$$

That is, the cost resulting from $w \equiv 0$ is larger than the cost resulting from $w = w_\infty$, which contradicts the maximizing property of w_∞.

For $w = 0$, $u(k) = u_\infty(k)$ uniformly exponentially stabilizes the system.

Since X_∞ is uniformly bounded, $\bar{X}_\infty(\theta)$ defined in Lemma 5.1 is uniformly bounded. The proof of Lemma 5.1 can be applied with no changes to conclude that A_u is uniformly exponentially stable.

The closed-loop system with $u(k) = u_\infty(k)$ and $w(k) = w_\infty(k)$ is uniformly exponentially stable. In other words, the system $x(k+1) = \left(A_{\theta(k)} - \bar{B}_{\theta(k)} R_{\theta(k)}^{-1} L_{\theta(k)}\right) x(k)$ is uniformly exponentially stable.

Let $|x_o| \leq 1$. Define w_∞ as

$$w_\infty(k) = F_{w_{\theta(k)}} x_{\theta_o}(F_u, w_\infty, x_o; k).$$

Then w_∞ is a linear function of x_o. By Lemmas 5.1 and 5.1,

$$\sup_{w \in l_2} \inf_{u \in l_2} \|z_{\theta_o}(u, w, x_o)\|_{l_2}^2 - \gamma^2\|w\|_{l_2}^2 = \|z_{\theta_o}(F_u, w_\infty, x_o)\|_{l_2}^2 - \gamma^2\|w_\infty\|_{l_2}^2 = x_o' X_{\infty_{\theta_o}} x_o \leq \bar{X}_\infty |x_o|^2,$$

and, by equation (71), $\|w_\infty\|_{l_2}^2 \leq P|x_o|^2$. Thus

$$\|z_{\theta_o}(F_u, w_\infty, x_o)\|_{l_2}^2 = \gamma^2\|w_\infty\|_{l_2}^2 + x_o' X_{\infty_{\theta_o}} x_o \leq (\gamma^2 P + \bar{X}_\infty)|x_o|^2 . \tag{73}$$

Note that, if $w(k) = w_\infty(k)$ and $u(k) = F_{u_{\theta(k)}} \begin{bmatrix} x(k) \\ w_\infty(k) \end{bmatrix}$, then $x(k+1) = \left(A_{\theta(k)} - \bar{B}_{\theta(k)} R_{\theta(k)}^{-1} L_{\theta(k)}\right) x(k)$.

Define the system

$$\begin{aligned}
x(k+1) &= \left(A_{\theta(k)} - \bar{B}_{\theta(k)} R_{\theta(k)}^{-1} L_{\theta(k)}\right) x(k) + r(k), \\
v(k) &= \begin{bmatrix} -\nabla_{\theta(k)}^{-1} L_{\nabla_{\theta(k)}} \\ \tilde{C}_{\theta(k)} \end{bmatrix} x(k),
\end{aligned} \tag{74}$$

where

$$\tilde{C} = C - D_1\nabla^{-1}L_\nabla + D_2\left(R_3^{-1}R_2\nabla^{-1}L_\nabla - R_3^{-1}L_2\right).$$

Then $v = \begin{bmatrix} w_\infty \\ z_{\theta_o}(F_u, w_\infty, x_o) \end{bmatrix}$. Fix $j \geq 0$ and set $r(k) = r_o\delta(k-j)$, that is, $r(k) = 0$ for $k \neq j$ and $r(j) = r_o$. Then equation (73) implies that $\left\|z_{f^j(\theta_o)}(F_u, w_\infty, r_o)\right\|_{l_2}^2 \leq (\gamma^2 P + \bar{X}_\infty)|r_o|^2$. Likewise $\|w_\infty\|_{l_2}^2 \leq P|r_o|^2$. Therefore

$$\|v\|_{l_2}^2 = \left\|z_{f^j(\theta_o)}(F_u, w_\infty, r_o)\right\|_{l_2}^2 + \|w\|_{l_2}^2 \leq \left((\gamma^2 P + \bar{X}_\infty) + P\right)|r_o|^2 . \tag{75}$$

Note that there exists a matrix $\begin{bmatrix} H_1 & H_2 \end{bmatrix}$ such that

$$\left(A - \bar{B}R^{-1}L\right) + \begin{bmatrix} H_1 & H_2 \end{bmatrix} \begin{bmatrix} -\nabla^{-1}L_\nabla \\ \left(C - D_1\nabla^{-1}L_\nabla + D_2\left(R_3^{-1}R_2\nabla^{-1}L_\nabla - R_3^{-1}L_2\right)\right) \end{bmatrix} \tag{76}$$

is uniformly exponentially stable. For example, set $H_1 = -B_1 - H_2D_1$ and $H_2 = H_dC\left(C'C\right)^+ C' - B_2\left(D_2'D_2\right)^{-1} D_2'$, where H_d is the feedback such that $A - H_dC$ is uniformly exponentially stable, the existence of which is guaranteed by the detectability assumption, and $(C'C)^+$ is the pseudo-inverse of $C'C$. Since H_d is bounded, $D_2'D_2 > 0$, D_2, B_1, C, D_1 are uniformly continuous, Θ is compact, and $C\left(C'C\right)^+ C'$ is bounded[1], there is a $\bar{H} < \infty$ such that $\begin{bmatrix} H_1 & H_2 \end{bmatrix}' \begin{bmatrix} H_1 & H_2 \end{bmatrix} \leq$

[1] Although $C\left(C'C\right)^+ C$ is not continuous, $\left\|C\left(C'C\right)^+ C\right\| \leq 1$.

$\bar{H}$.

Now, let

$$y(k+1) = \left(\left(A_{\theta(k)} - \bar{B}_{\theta(k)} R_{\theta(k)}^{-1} L_{\theta(k)}\right) + \left[\begin{array}{cc} H_{1_{\theta(k)}} & H_{2_{\theta(k)}} \end{array}\right] \left[\begin{array}{c} \tilde{C}_{\theta(k)} \\ \nabla_{\theta(k)}^{-1} L_{\nabla_{\theta(k)}} \end{array}\right]\right) y(k) \qquad (77)$$
$$- \left[\begin{array}{cc} H_{1_{\theta(k)}} & H_{2_{\theta(k)}} \end{array}\right] v(k) + r_o \delta(k-j),$$

that is, y is an estimate of x. Since system (77) is uniformly exponentially stable, there exists an $R < \infty$, such that

$$\begin{aligned}
\|y\|_{l_2} &\leq R \left\|r - \left[\begin{array}{cc} H_1 & H_2 \end{array}\right] v\right\|_{l_2} \leq R\|r\|_{l_2} + \bar{H}R\|v\|_{l_2} \\
&\leq R|r_o| + \bar{H}R\sqrt{((\gamma^2 P + \bar{X}_\infty) + P)}\,|r_o| \\
&\leq \left(R + R\bar{H}\sqrt{((\gamma^2 P + \bar{X}_\infty) + P)}\right)|r_o|.
\end{aligned}$$

On the other hand, if the system is initially at rest, that is, $y(0) = x(0) = 0$, then $y(k) = x(k)$. Thus $\|y\|_{l_2} = \|x\|_{l_2}$, and therefore $\|x\|_{l_2} \leq \left(R + R\bar{H}\sqrt{((\gamma^2 P + \bar{X}_\infty) + P)}\right)|r_o|$. Let $\Phi_{\theta_o}(k,j)$ be the state transition matrix of system (74) with initial conditions $\theta(0) = \theta_o$, $x(0) = 0$ and let $r(i) = r_o\delta(i-j)$. Then we have

$$\|x\|_{[j,\infty)}^2 = \sum_{i=j}^{\infty} x'(i)\,x(i) = \sum_{i=j}^{\infty} \|\Phi_{\theta_o}(i,j)\,r(j)\|^2 \leq \left(R + R\bar{H}\sqrt{((\gamma^2 P + \bar{X}_\infty) + P)}\right)^2 |r_o|^2.$$

Furthermore, for $i \geq j$, $\|\Phi_{\theta_o}(i,j)\|^2 \leq \left(R + R\bar{H}\sqrt{((\gamma^2 P + \bar{X}_\infty) + P)}\right)^2$. Applying standard techniques we find that

$$\begin{aligned}
K\|\Phi_{\theta_o}(j+K,j)\|^2 &= \sum_{i=j}^{K} \|\Phi_{\theta_o}(j+K,j)\|^2 \\
&\leq \sum_{i=j}^{K} \|\Phi_{\theta_o}(j+K,i)\|^2 \|\Phi_{\theta_o}(i,j)\|^2 \\
&\leq \left(R + R\bar{H}\sqrt{((\gamma^2 P + \bar{X}_\infty) + P)}\right)^2 \sum_{i=j}^{K} \|\Phi_{\theta_o}(i,j)\|^2 \\
&\leq \left(R + R\bar{H}\sqrt{((\gamma^2 P + \bar{X}_\infty) + P)}\right)^4.
\end{aligned}$$

Choosing $K \in \mathbb{Z}$ such that $K \geq \sqrt{2}\left(R + R\bar{H}\sqrt{((\gamma^2 P + \bar{X}_\infty) + P)}\right)^4$ yields $\|\Phi(j+K,j,\theta_o)\|^2 \leq \frac{1}{\sqrt{2}}$. Since this is true for all j and all θ_o, setting $M \in \mathbb{Z}$ with $M \geq 0$ and $k - (j + MK) < K$, we conclude that

$$\begin{aligned}
\|\Phi_{\theta_o}(k,j)\| &\leq \|\Phi_{\theta_o}(k, j+MK)\| \prod_{m=1}^{M} \|\Phi_{\theta_o}(j+mK, j+(m-1)K)\| \\
&\leq \left(R + R\bar{H}\sqrt{((\gamma^2 P + \bar{X}_\infty) + P)}\right)\left(\frac{1}{2}\right)^M \\
&\leq \left(R + R\bar{H}\sqrt{((\gamma^2 P + \bar{X}_\infty) + P)}\right) 2 \left(\frac{1}{2}\right)^{\frac{k-j}{K}}.
\end{aligned}$$

That is, the system (74) is uniformly exponentially stable.

As $N \to \infty$, $X_\theta(0, N+1) \to X_{\infty_\theta}$ uniformly in θ and X_∞ is a continuous function.

Let $x(k+1) = \left(A_{\theta(k)} - \bar{B}_{\theta(k)} R^{-1}_{\theta(k)} L_{\theta(k)}\right) x(k)$. Then by Lemma 5.1, $x(k) \to 0$ uniformly exponentially fast. Set $w_\infty(k) = F_{w_{\theta(k)}} x(k)$ as in equation (62) and define

$$\tilde{w}_N(k) := \begin{cases} w_\infty(k) & \text{for } k \le N, \\ 0 & \text{otherwise.} \end{cases}$$

Since X_∞ is uniformly bounded and $\nabla_\theta \le -\varrho I$, it follows that F_w is uniformly bounded. Since $x(k) \to 0$ uniformly exponentially fast and F_w is bounded, $w_\infty \to 0$ uniformly exponentially fast. Therefore, $\lim_{N\to\infty} \|w_\infty - \tilde{w}_N\|_{l_2} = \lim_{N\to\infty} \|w_\infty\|_{[N+1,\infty)} = 0$ where the convergence is uniformly exponentially fast.

Recall the following: Equation (31) states that

$$\begin{aligned} & x_o' X_{\theta_o}(0, N+1)\, x_o \\ &= \sup_{w\in l_2[0,N]} \inf_{u\in l_2[0,N]} \left\{ \|z\|^2_{[0,N]} - \gamma^2 \|w\|^2_{[0,N]} + x'(N+1)\, X_{\theta_o}(N+1, N+1)\, x(N+1) \right\}. \end{aligned} \tag{78}$$

From Equation (19) of Theorem 3, it follows that

$$x(N+1)' X_{2_{f^{N+1}(\theta_o)}} x(N+1) = \inf_{u\in l_2} \|z\|^2_{[N+1,\infty)}. \tag{79}$$

From (64),

$$\inf_{u\in l_2} \|z_{\theta_o}(u, w, x_o)\|^2_{[0,\infty)} - \gamma^2 \|w\|^2_{[0,\infty)} = \|z_{\theta_o}(G_{\theta_o}(w, x_o), w, x_o)\|^2_{[0,\infty)} - \gamma^2 \|w\|^2_{[0,\infty)}. \tag{80}$$

Combining (64) and Lemma 5.1 yields

$$\|z_{\theta_o}(G_{\theta_o}(w_\infty, x_o), w_\infty, x_o)\|^2_{[0,\infty)} - \gamma^2 \|w_\infty\|_{l_2} = x_o' X_{\infty_{\theta_o}} x_o. \tag{81}$$

From (66), we have

$$\|z_{\theta_o}(G_{\theta_o}(w, 0), w, 0)\|^2_{[0,\infty)} < \gamma^2 \|w\|^2_{[0,\infty)}, \tag{82}$$

and, from Inequality (71), we have

$$\|w_\infty\|^2_{l_2} \le P\, |x_o|^2. \tag{83}$$

Combining the preceding relations yields the following string:

$$
\begin{aligned}
& x_o' X_{\theta_o}(0, N+1)) x_o \\
& = \sup_{w \in l_2[0,N]} \inf_{u \in l_2[0,N]} \left\{ \left\| z_{\theta_o}(u, w, x_o) \right\|_{[0,N]}^2 - \gamma^2 \|w\|_{[0,N]}^2 + x'(N+1) X_{2_{f^{N+1}(\theta_o)}} x(N+1) \right\} \\
& = \sup_{\{w \in l_2 : w(k)=0,\ k \geq N\}} \inf_{u \in l_2} \left\{ \left\| z_{\theta_o}(u, w, x_o) \right\|_{[0,\infty)}^2 - \gamma^2 \|w\|_{[0,\infty)}^2 \right\} \\
& = \sup_{\{w \in l_2 : w(k)=0,\ k \geq N\}} \left\{ \left\| z_{\theta_o}(G_{\theta_o}(w, x_o), w, x_o) \right\|_{[0,\infty)}^2 - \gamma^2 \|w\|_{[0,\infty)}^2 \right\} \\
& \geq \left\| z_{\theta_o}(G_{\theta_o}(\tilde{w}_N, x_o), \tilde{w}_N, x_o) \right\|_{[0,\infty)}^2 - \gamma^2 \|\tilde{w}_N\|_{[0,\infty)}^2 \\
& = \left\| z_{\theta_o}(G_{\theta_o}(w_\infty, x_o), w_\infty, x_o) + z_{\theta_o}(G_{\theta_o}(\tilde{w}_N - w_\infty, 0), \tilde{w}_N - w_\infty, 0) \right\|_{[0,\infty)}^2 - \gamma^2 \|\tilde{w}_N\|_{[0,\infty)}^2 \\
& = \left\| z_{\theta_o}(G_{\theta_o}(w_\infty, x_o), w_\infty, x_o) \right\|_{[0,\infty)}^2 + \left\| z_{\theta_o}(G_{\theta_o}(\tilde{w}_N - w_\infty, 0), \tilde{w}_N - w_\infty, 0) \right\|_{[0,\infty)}^2 - \gamma \|\tilde{w}_N\|_{[0,\infty)}^2 \\
& + 2 \left\langle z_{\theta_o}(G_{\theta_o}(w_\infty, x_o), w_\infty, x_o), z_{\theta_o}(G_{\theta_o}(\tilde{w}_N - w_\infty, 0), \tilde{w}_N - w_\infty, 0) \right\rangle \\
& \geq \left\| z_{\theta_o}(G_{\theta_o}(w_\infty, x_o), w_\infty, x_o) \right\|_{[0,\infty)}^2 - \gamma^2 \|\tilde{w}_N\|_{[0,\infty)}^2 \\
& + 2 \left\langle z_{\theta_o}(G_{\theta_o}(w_\infty, x_o), w_\infty, x_o), z_{\theta_o}(G_{\theta_o}(\tilde{w}_N - w_\infty, 0), \tilde{w}_N - w_\infty, 0) \right\rangle \\
& \geq \left\| z_{\theta_o}(G_{\theta_o}(w_\infty, x_o), w_\infty, x_o) \right\|_{[0,\infty)}^2 - \gamma^2 \|w_\infty\|_{[0,\infty)}^2 \\
& - 2 \left\| z_{\theta_o}(G_{\theta_o}(w_\infty, x_o), w_\infty, x_o) \right\|_{[0,\infty)} \left\| z_{\theta_o}(G_{\theta_o}(\tilde{w}_N - w_\infty, 0), \tilde{w}_N - w_\infty, 0) \right\|_{[0,\infty)} \\
& \geq x_o' X_{\infty_{\theta_o}} x_o - 2 \left(\sqrt{\gamma^2 \|w_\infty\|^2 + x_o' X_{\infty_{\theta_o}} x_o} \right) \gamma \|w_\infty - \tilde{w}_N\|_{[0,\infty)} \\
& \geq x_o' X_{\infty_{\theta_o}} x_o - 2 |x_o| \left(\sqrt{\gamma^2 P + \bar{X}_\infty} \right) \gamma \|w_\infty - \tilde{w}_N\|_{[0,\infty)} .
\end{aligned}
$$

Lemma 5.1 implies that $X_{\infty_{\theta_o}} - X_{\theta_o}(0, N+1) \geq 0$. Thus

$$
0 \leq x_o' \left(X_{\infty_{\theta_o}} - X_{\theta_o}(0, N+1) \right) x_o \leq 2\gamma |x_o| \left(\sqrt{\gamma^2 P + \bar{X}_\infty} \right) \|w_\infty - \tilde{w}_N\|_{[0,\infty)} .
$$

Since $\|w_\infty - \tilde{w}_N\|_{[0,\infty)} \to 0$ uniformly in θ and exponentially in N and $2\gamma \left(\sqrt{\gamma^2 P + \bar{X}_\infty} \right)$ does not depend on θ_o, we have $X_{\theta_o}(0, N+1) \to X_{\infty_{\theta_o}}$ uniformly in θ and exponentially in N.

Since X_2 is continuous, $X_\theta(0, N+1)$ is continuous in θ for $N < \infty$. Since Θ is compact, and $X(0, N+1) \to X_\infty$ in the uniform metric, Theorem 7.1.4 in [24] implies that X_∞ is continuous.

The time-invariant version of the first claim of this lemma can be found in [31].

5.2 sufficiency

Suppose that the assumptions of the theorem hold and that (32), (33), and (34) hold. It will be shown that the control given by equation (36) is internally stabilizing and, if $u = u_\infty$ as defined by (42), then Objective B is satisfied. This proof is similar to the proof given in [17].

Under the above conditions, Equation (32) can be written as

$$
X_{\infty_\theta} = C_\theta' C_\theta + A_\theta' X_{\infty_{f(\theta)}} A_\theta - L_{2_\theta}' R_{3_\theta}^{-1} L_{2_\theta} - L_{\nabla_\theta}' \nabla_\theta^{-1} L_{\nabla_\theta} .
$$

It follows that

$$
\begin{aligned}
& \begin{bmatrix} A_\theta' & C_\theta' \\ B_{2_\theta}' & D_{2_\theta}' \end{bmatrix} \begin{bmatrix} X_{\infty_{f(\theta)}} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_\theta & B_{2_\theta} \\ C_\theta & D_{2_\theta} \end{bmatrix} \\
& = \begin{bmatrix} X_{\infty_\theta} + L_{\nabla_\theta}' \nabla_\theta^{-1} L_{\nabla_\theta} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} L_{2_\theta}' \\ R_{3_\theta} \end{bmatrix} R_{3_\theta}^{-1} \begin{bmatrix} L_{2_\theta} & R_{3_\theta} \end{bmatrix} .
\end{aligned}
$$

Multiplying both sides of this equality by $\begin{bmatrix} I & 0 \\ -R_{3_\theta}^{-1}L_{2_\theta} & I \end{bmatrix}$ on the right and by the transpose on the left and taking the (1,1) block yields

$$\begin{aligned} X_{\infty_\theta} &= \left(A_\theta - B_{2_\theta}R_{3_\theta}^{-1}L_{2_\theta}\right)' X_{\infty_{f(\theta)}} \left(A_\theta - B_{2_\theta}R_{3_\theta}^{-1}L_{2_\theta}\right) \\ &- L'_{\nabla_\theta}\nabla_\theta^{-1}\nabla_\theta\nabla_\theta^{-1}L_{\nabla_\theta} + \\ &\left(C_\theta - D_{2_\theta}R_{3_\theta}^{-1}L_{2_\theta}\right)' \left(C_\theta - D_{2_\theta}R_{3_\theta}^{-1}L_{2_\theta}\right). \end{aligned} \tag{84}$$

Since $A_\theta - \bar{B}_\theta R_\theta^{-1} L_\theta = A_\theta - B_{2_\theta}R_{3_\theta}^{-1}L_{2_\theta} - \left(B_{1_\theta} - B_{2_\theta}R_{3_\theta}^{-1}L_{2_\theta}\right)\nabla_\theta^{-1}L_{\nabla_\theta}$ is assumed to be uniformly exponentially stable, we conclude that the triple

$$\left(\left(A_\theta - B_{2_\theta}R_{3_\theta}^{-1}L_{2_\theta}\right), \left(\nabla_\theta^{-1}L_{\nabla_\theta}\right), f\right)$$

is uniformly detectable. Since $\nabla_\theta \le -\varrho I$, ∇_θ^{-1} is uniformly bounded. Since X_∞ is uniformly bounded, $\nabla_\theta^{-1}L_{\nabla_\theta}$ is uniformly bounded. Thus (84) is a Lyapunov equation and Corollary 3 implies that

$$\xi(k+1) = \left(A_{\theta(k)} - B_{2_{\theta(k)}}R_{3_{\theta(k)}}^{-1}L_{2_{\theta(k)}}\right)\xi(k) \tag{85}$$

is a uniformly exponentially stable system. Therefore, the control $u = u_\infty$ is uniformly exponentially stabilizing.

Since system (85) is uniformly exponentially stable, if $u = u_\infty$ and $w \in l_2$, then $x \in l_2$ and $\lim_{k\to\infty} x(k) = 0$. Thus, if $u = u_\infty$, then equation (28) implies that, for all N,

$$\begin{aligned} &\|z\|_{[0,N]}^2 - \gamma^2\|w\|_{[0,N]}^2 + x'(N+1)X_{\infty_{f^{N+1}(\theta_o)}}x(N+1) \\ &= x'(0)X_{\infty_{\theta_o}}x(0) + \\ &+ \sum_{k=0}^{N}(w(k) - w_\infty(k))'\nabla_{f^k(\theta_o)}(w(k) - w_\infty(k)), \end{aligned} \tag{86}$$

where $w_\infty(k) := -\nabla_{f^k(\theta_o)}^{-1}L_{\nabla_{f^k(\theta_o)}}x(k)$. Since $x, w \in l_2$, it follows that $u, z \in l_2$. Furthermore, ∇ is bounded. Thus, we can let $N \to \infty$ in equation (86) and for $x_o = 0$,

$$\|z\|_{l_2}^2 - \gamma\|w\|_{l_2}^2 = \sum_{k=0}^{\infty}(w(k) - w_\infty(k))'\nabla_{f^k(\theta_o)}(w(k) - w_\infty(k)). \tag{87}$$

Since system (85) is stable and causal, the closed-loop system with $u = u_\infty$, viz.,

$$\begin{aligned} &\begin{bmatrix} x(k+1) \\ w(k) - w_\infty(k) \end{bmatrix} \\ &= \begin{bmatrix} \left(A_{\theta(k)} - B_{2_{\theta(k)}}R_{3_{\theta(k)}}^{-1}L_{2_{\theta(k)}}\right) & \left(B_{1_{\theta(k)}} - B_{2_{\theta(k)}}R_{3_{\theta(k)}}^{-1}R_{2_{\theta(k)}}\right) \\ \left(\nabla_{\theta(k)}^{-1}L_{\nabla_{\theta(k)}}\right) & I \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix}, \end{aligned} \tag{88}$$

is l_2-stable and causal. The inverse of this system [36] is

$$\begin{bmatrix} \xi(k+1) \\ w(k) \end{bmatrix} = \begin{bmatrix} \tilde{A}_{\theta(k)} & -\tilde{B}_{\theta(k)} \\ \tilde{C}_{\theta(k)} & \tilde{D}_{\theta(k)} \end{bmatrix} \begin{bmatrix} \xi(k) \\ w(k) - w_\infty(k) \end{bmatrix}$$

with

$$\begin{aligned} \tilde{A}_\theta &= A_\theta - B_{2_\theta}R_{3_\theta}^{-1}L_{2_\theta} - \left(B_{1_\theta} - B_{2_\theta}R_{3_\theta}^{-1}R_{2_\theta}\right)\nabla_\theta^{-1}L_{\nabla_\theta} \\ &= A_\theta - \bar{B}_\theta R_\theta^{-1}L_\theta, \\ \tilde{B}_\theta &= -\left(B_{1_\theta} - B_{2_\theta}R_{3_\theta}^{-1}R_{2_\theta}\right), \\ \tilde{C}_\theta &= -\nabla_\theta^{-1}L_{\nabla_\theta}, \\ \tilde{D}_\theta &= I. \end{aligned}$$

Since $\xi(k+1) = \left(A_{\theta(k)} - \bar{B}_{\theta(k)} R_{\theta(k)}^{-1} L_{\theta(k)}\right) \xi(k)$ is uniformly exponentially stable, the inverse of system (88) is uniformly exponentially stable and hence l_2 stable. Thus there exists a $\delta > 0$ such that, for all $\theta_o \in \Theta$,

$$\|w\|_{l_2}^2 \leq \frac{1}{\delta} \|w - w_\infty\|_{l_2}^2 .$$

Since $\nabla \leq -\varrho I$, equation (87) implies that

$$\|z\|_{l_2}^2 - \gamma^2 \|w\|_{l_2}^2 \leq -\varrho \|w - w_\infty\|_{l_2}^2 \leq -\delta\varrho \|w\|_{l_2}^2 = -\varepsilon \|w\|_{l_2}^2 \ .$$

6 controlling nonlinear systems with linear dynamically varying H^∞ controllers

In the preceding section, a technique for stabilizing an LDV system subject to an H^∞ disturbance rejection requirement was developed. Here, it will first be shown that the LDV controller for the *linearized* tracking error (LDV) system can also be used to stabilize the *nonlinear* tracking error dynamics, in a scheme that works along every trajectory, provided that the initial tracking error be small enough (Section 6.1). Next, some issues quite specific to the H^∞ implementation of the tracking scheme (to be published elsewhere) will be briefly surveyed.

6.1 stability of closed loop nonlinear system

To make H^∞ design relevant to nonlinear tracking performance improvement, the guiding idea is to write the nonlinearity η in the tracking error dynamics (6) as a feedback from an output $z(k)$ to a disturbance $w(k)$. To this end, introduce the factorization

$$\eta(x,u,\theta) = \begin{bmatrix} \eta_x(x,u,\theta) & \eta_u(x,u,\theta) \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = B_{1_\theta} \tilde{\eta}(x,u,\theta) \begin{bmatrix} C_\theta & D_{2_\theta} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}. \tag{89}$$

Here, we simply take

$$B_{1_\theta} := I_{n\times n}, \quad C_\theta := \begin{bmatrix} I_{n\times n} \\ 0_{m\times n} \end{bmatrix}, \quad D_{2_\theta} := \begin{bmatrix} 0_{n\times m} \\ I_{m\times m} \end{bmatrix}, \tag{90}$$
$$\tilde{\eta}(x,u,\theta) := \begin{bmatrix} \eta_x(x,u,\theta) & \eta_u(x,u,\theta) \end{bmatrix},$$

where η_x and η_u are defined by (8) and (9). The error due to linearization can be modeled as a feedback from z to w:

$$\begin{aligned} \eta(x(k),u(k),\theta(k)) &= B_{1_{\theta(k)}} w(k), \\ w(k) &= \tilde{\eta}(x(k),u(k),\theta(k))\, z(k), \\ z(k) &= C_{\theta(k)} x(k) + D_{2_{\theta(k)}} u(k). \end{aligned}$$

Here $D_1 = 0$. Clearly $B_1 : \Theta \to \mathbb{R}^{n\times n}$, $C : \Theta \to \mathbb{R}^{(n+m)\times n}$, and $D_2 : \Theta \to \mathbb{R}^{(n+m)\times m}$ are continuous functions and the triple (A, C, f) is detectable. Hence, assuming that (A, B_2, f) is stabilizable, Theorem 4.2 or Corollary 4.2 can be applied to generate a controller.

Let (4) hold. Define A, B_1, B_2, C, D_1, and D_2 as above and assume that the triple (A, B_2, f) is stabilizable. Let F be the H^∞ controller such that $\sup_{w\in l_2} \frac{\|z\|_{l_2}}{\|w\|_{l_2}} < \gamma$ for some $\gamma < \infty$. Then there exists a $R_{Capture} > 0$ such that, if $u(k) = F_{\theta(k)}(\varphi(k) - \theta(k))$ and $|\varphi(0) - \theta(0)| < R_{Capture}$, then $|\varphi(k) - \theta(k)| \to 0$ as $k \to \infty$.

Define $\bar{\eta}(\bar{x},\bar{u}) := \sup\{\|\tilde{\eta}(x,u,\theta)\| : |x| \leq \bar{x}, |u| \leq \bar{u}, \theta \in \Theta\}$. By (10) and (11),

$$\bar{\eta}(\bar{x},\bar{u}) \to 0 \text{ as } \bar{x},\bar{u} \to 0. \tag{91}$$

It follows that there exist $x^*, u^* > 0$ such that $\bar{\eta}(x^*, u^*) \leq \frac{1}{\gamma}$. Now define

$$h(x, u, \theta) := \begin{cases} \tilde{\eta}(x, u, \theta), & \text{for } |x| < x^* \text{ and } |u| < u^*, \\ 0_{n\times(n+m)}, & \text{otherwise.} \end{cases} \tag{92}$$

Thus, for all x and u, $\sup_{\theta\in\Theta} \|h(x, u, \theta)\| \leq \frac{1}{\gamma}$. Consider the following closed-loop LDV system:

$$\begin{aligned} \xi(k+1) &= A_{\theta(k)}\xi(k) + B_{1_{\theta(k)}}\omega(k) + B_{2_{\theta(k)}}\upsilon(k), \\ \zeta(k) &= C_{\theta(k)}\xi(k) + D_{2_{\theta(k)}}\upsilon(k), \\ \omega(k) &= h(\xi(k), \upsilon(k), \theta(k))\,\zeta(k), \\ \upsilon(k) &= F_{\theta(k)}\xi(k), \\ \theta(k+1) &= f(\theta(k)). \end{aligned} \tag{93}$$

Since $\sup_{w\in l_2} \frac{\|z\|_{l_2}}{\|w\|_{l_2}} < \gamma$, the Small Gain Theorem [32] implies that system (93) is externally l_2 stable. Since $A_\theta + B_{2_\theta}F_\theta$ is uniformly exponentially stable, system (93) is also internally l_2 stable. Therefore, there exist a $G_x \geq 1$ and a $G_u > 0$ such that $\|\xi\|_{l_\infty} \leq \|\xi\|_{l_2} < G_x|\xi(0)|$ and $\|\upsilon\|_{l_\infty} \leq \|\upsilon\|_{l_2} < G_u|\xi(0)|$. Now set

$$R_{Capture} := \min\left(\frac{x^*}{G_x}, \frac{u^*}{G_u}\right).$$

If $|\xi(0)| \leq R_{Capture}$, then

$$\|\xi\|_{l_\infty} < G_x|\xi(0)| \leq x^*$$

and

$$\|\upsilon\|_{l_\infty} < G_u|\xi(0)| \leq u^*.$$

By the above inequalities and (92), we conclude that, for all k, we have $h(\xi(k), \upsilon(k), \theta(k)) = \tilde{\eta}(\xi(k), \upsilon(k), \theta(k))$. Thus, if

$$|x(0)| < \min\left(\frac{x^*}{G_x}, \frac{u^*}{G_u}\right),$$

then, by the uniqueness of solutions to difference equations, the closed-loop LDV system

$$\begin{aligned} x(k+1) &= A_{\theta(k)}x(k) + B_{1_{\theta(k)}}w(k) + B_{2_{\theta(k)}}u(k), \\ z(k) &= C_{\theta(k)}x(k) + D_{2_{\theta(k)}}u(k), \\ w(k) &= \tilde{\eta}(x(k), u(k), \theta)\,z(k), \\ u(k) &= F_{\theta(k)}x(k), \\ \theta(k+1) &= f(\theta(k)) \end{aligned} \tag{94}$$

is l_2 stable, and furthermore $\|x\|_{l_\infty} < x^*$ and $\|u\|_{l_\infty} < u^*$. Since (94) is the tracking error of the closed-loop nonlinear system, we conclude that $|x(k)| = |\varphi(k) - \theta(k)| \to 0$ as $k \to \infty$.

6.2 further considerations

Writing the nonlinearity $\eta(x, u, \theta)$ as a bounded feedback $\tilde{\eta}(x, u, \theta)$ from an output z to the input w (see (10), (11), (89)) yields an H^∞ design that attenuates the effect of the nonlinearity and hence amplifies the initial allowable tracking error. It is further possible to optimize this procedure by factoring $\tilde{\eta}$ in such a way that $\|\tilde{\eta}\| \leq 1$ (see [7] for details). The suboptimal H^∞ controller is continuous and, therefore, an approximation of the LDV H^∞ controller can be constructed the same way as an approximation of the LDV quadratic controller was constructed in [8]. The fact that the H^∞ controller is guaranteed to be continuous under the condition that it be suboptimal does not prove that the suboptimality condition for continuity is necessary. In fact, an example based on the Hénon map shows that the suboptimal controller becomes discontinuous as γ approaches γ_o, the optimal H^∞ tolerance.

7 conclusion

Suboptimal H^{∞} controllers for LDV systems have been developed. Like the LDV quadratic controllers, these H^{∞} controllers are continuous functions. The H^{∞} method has the distinct advantages over the linear-quadratic method in that H^{∞} can be tuned to minimize the effect of linearization and it is possible to find a lower bound on the maximum allowable initial tracking error.

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