Relationships Between Linear Dynamically Varying Systems and Jump Linear Systems

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Abstract

The connection between linear dynamically varying (LDV) systems and jump linear systems is explored. LDV systems have been shown to be useful in nonlinear tracking. Some nonlinear systems, for example Axiom A systems, admit Markov partitions and can be described by a Markov chain. In this case, the nonlinear system can be approximated as Markovian jump linear systems. It is shown that (i) jump linear controllers for arbitrarily fine partitions exist if and only if the LDV controller exists; (ii) jump linear controllers stabilize the nonlinear dynamical system; (iii) jump linear controllers are an approximations of the LDV controller.

1 introduction

Recently control theorists have focused on two related classes of linear systems; linear parametrically varying (LPV) systems [3], [4], [5], [14], [25] and jump linear systems [17], [15], [12], [13]. While these classes of systems are similar in that they are linear with varying parameters, in some cases there exists a deeper relationship. This relationship is based on linear dynamically varying (LDV) systems and the relationship between dynamical systems and symbolic dynamics. LDV systems are a specification of LPV systems to the case where the parameter dynamics are exactly known [7], [8], [6], [9], [18].

The relationship between dynamical system and symbolic dynamics was developed by Sinai for Anosov diffeomorphisms [23] and by Bowen for more general hyperbolic systems [11]. This work showed that dynamical systems can induce Markov partitions and, hence, symbolic dynamical systems, that is, dynamical systems can be described by Markov chains. This relationship between dynamical systems and Markov chains has been extended to other situations, for example nonuniform hyperbolic systems [19], expanding homeomorphisms [2], and to systems that satisfy a local product structure [22].

We consider the following nonlinear tracking problem: find a $u \in l_2$ such that $\|\varphi(k) - \theta(k)\| \to 0$ where

$$\begin{aligned} \varphi(k+1) &= f(\varphi(k), u(k)) \\ \theta(k+1) &= f(\theta(k), 0), \end{aligned} \tag{1}$$

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where $f \in C^1$ and $f(\Theta, 0) = \Theta$ with Θ a compact set. There are many seemingly distinct approaches to this problem based on linear approximation. In many cases the linear approximation is

$$
\begin{aligned}
x(k+1) &= A_{\theta(k)} x(k) + B_{\theta(k)} u(k) \\
\theta(k+1) &= f(\theta(k), 0).
\end{aligned} \tag{2}
$$

where $A_\theta = \frac{\partial f}{\partial \theta}(\theta, 0)$ and $B_\theta = \frac{\partial f}{\partial u}(\theta, 0)$. The typical LPV approach is to simply assume that $f(\Theta, 0) = \Theta$, and neglect all other information about f. A slight specification of this LPV approach is to make use to the variation in f via $\|f(\theta, 0) - \theta\| < \Delta$ [25]. If all information about f is assumed known, then (2) is an LDV system as discussed in Section 2. A seemingly different approach is to model the parameter dynamics f probabilistically. In this case system (2) is a jump linear system. This type of approximation is detailed in Section 4. As discussed in Section 4.4, a significant problem with the jump linear approach is that it cannot be directly shown that the jump linear controller will stabilize the nonlinear system (1). However, it will be shown that the jump linear controller may be an approximation of the LDV controller (Theorem 8). Since the LDV controller stabilizes (1), if the approximation of the LDV is good enough, the jump linear system will also stabilize (1). Of course, not all dynamical systems permit a symbolic dynamics representation. In this case the Markovian assumption necessary for jump linear systems cannot be meet. However, if a Markovian assumption is incorrectly made, the jump linear controller remains an approximation of the LDV controller. Hence, the Markovian assumption of the parameter dynamics of a jump linear system do not seem to be very critical. Specifically, under certain situations, there is little difference between the optimal jump linear controller under the incorrect Markovian assumption and the optimal controller under the correct non-Markovian assumption. A further implication of the results presented here is that techniques of computing jump linear controllers can be used to compute LDV controllers.

Before detailing the connection between jump linear and LDV systems in Section 5, LDV systems are introduced (Section 2), jump linear systems reviewed (Section 4) and the semi-conjucacy between dynamical systems and symbolic systems must be briefly discussed (Section 3). Section 6 provides an example of the Hénon map.

2 LDV systems

We consider a slight generalization of (2) and define a LDV system as

$$
\begin{aligned}
x(k+1) &= A^{LDV}_{\theta(k)} x(k) + B^{LDV}_{\theta(k)} u(k) \\
z(k) &= \begin{bmatrix} C^{LDV}_{\theta(k)} x(k) \\ D^{LDV}_{\theta(k)} u(k) \end{bmatrix} \\
\theta(k+1) &= f(\theta(k)),
\end{aligned} \tag{3}
$$

where $A^{LDV} : \Theta \to \mathbb{R}^{n \times n}$, $B^{LDV} : \Theta \to \mathbb{R}^{n \times m}$, $C^{LDV} : \Theta \to \mathbb{R}^{p \times n}$, $D^{LDV} : \Theta \to \mathbb{R}^{p \times m}$ and $f : \Theta \to \Theta$, with $f \in C^0$ and Θ compact. If the maps $A, B, C, D \in C^0$, then system (2) is a continuous LDV. The relationship between the a linear approximation of the nonlinear tracking system (1) can be had by assuming $f \in C^1$ and setting $x := \varphi - \theta$, $A_\theta := \frac{\partial f}{\partial x}(\theta, 0)$, and $B_\theta := \frac{\partial f}{\partial u}(\theta, 0)$.

We say that the pair $\left(A^{LDV}, f\right)$ is *exponentially stable* if system (3) is exponentially stable, that is, for $u = 0$ and $\theta_o \in \Theta$, there exists an $\alpha(\theta_o) < 1$ and a $\beta(\theta_o) < \infty$ such that if $\theta(0) = \theta_o$, then $\|x(k)\| < \beta(\theta_o)\,\alpha(\theta_o)^k \|x(0)\|$. Furthermore, the pair $\left(A^{LDV}, f\right)$ is *uniformly exponentially stable* if α and β can be chosen independent of $\theta(0)$. The triple $\left(A^{LDV}, B^{LDV}, f\right)$ is *stabilizable* if there exists a bounded feedback $F : \Theta \to \mathbb{R}^{m\times n}$ such that $\left(A^{LDV} + B^{LDV}F, f\right)$ is exponentially stable. Note that uniform exponential stability is not required for a system to be stabilizable. The triple $\left(A^{LDV}, C^{LDV}, f\right)$ is *uniformly detectable* if there is a uniformly bounded map $H^{LDV} : \Theta \to \mathbb{R}^{n\times p}$ such that $\left(A^{LDV} + H^{LDV}C^{LDV}, f\right)$ is uniformly exponentially stable. That is, there exists $\alpha_d < 1$ and $\beta_d < \infty$ such that for all $\theta(0) \in \Theta$, $\|x(k)\| < \beta_d \alpha_d^k \|x(0)\|$ where $x(k+1) = \left(A^{LDV}_{f^k(\theta_o)} + H^{LDV}_{f^k(\theta_o)} C^{LDV}_{f^k(\theta_o)}\right) x(k)$.

We say that the LDV system $\left(\frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial u}, f\right)$ is the LDV system induced by f. It was shown in [7] that if the LDV system (3) induced by f is uniformly exponentially stabilized by the control $u(k) = F_{\theta(k)} x(k)$, then the nonlinear system (1), with control $u(k) = F_{\theta(k)} x(k)$, is locally uniformly exponentially stable. By definition *locally uniformly exponentially stable* means that there exist $\alpha < 1$, $\beta < \infty$ and $\gamma > 0$ such that if $\|x(0)\| = \|\varphi(0) - \theta(0)\| < \gamma$, then $\|x(k)\| < \beta\alpha^k \|x(0)\|$ where α, β and γ can be taken independent of the initial condition θ_o, i.e. uniformly in θ_o and locally in x. Therefore, we say that the dynamical system f is LDV stabilizable if the LDV system induced by f is stabilizable.

The one of the main results from [7] is:

Theorem 1 *Suppose (3) is a continuous, uniformly detectable LDV system with $D^{LDV\prime}_\theta D^{LDV}_\theta > 0$ for all $\theta \in \Theta$. Then system (3) is LDV stabilizable if and only if there exists a bounded function $X : \Theta \to \mathbb{R}^{n\times n}$ with $X'_\theta = X_\theta \geq 0$ that satisfies the functional discrete time algebraic Riccati equation*

$$\begin{aligned} X_\theta &= A^{LDV\prime}_\theta X_{f(\theta)} A^{LDV}_\theta + C^{LDV\prime}_\theta C^{LDV}_\theta \\ &\quad - A^{LDV\prime}_\theta X_{f(\theta)} B^{LDV}_\theta \left(D^{LDV\prime}_\theta D^{LDV}_\theta + B^{LDV\prime}_\theta X_{f(\theta)} B^{LDV}_\theta\right)^{-1} B^{LDV\prime}_\theta X_{f(\theta)} A^{LDV}_\theta. \end{aligned} \tag{4}$$

In this case the control

$$\begin{aligned} u^{LDV}(k) &= F^{LDV}_{\theta(k)} x(k) \\ &= -\left(D^{LDV\prime}_{\theta(k)} D^{LDV}_{\theta(k)} + B^{LDV\prime}_{\theta(k)} X_{f(\theta(k))} B^{LDV}_{\theta(k)}\right)^{-1} \times B^{LDV\prime}_{\theta(k)} X_{f(\theta(k))} A^{LDV}_{\theta(k)}\, x(k) \end{aligned} \tag{5}$$

is optimal in the sense that it minimizes the quadratic cost

$$V(\theta_o, u, x_o) = \sum_{k=0}^{\infty} \left\| C^{LDV}_{f^k(\theta_o)} x(k) \right\|^2 + \left\| D^{LDV}_{f^k(\theta_o)} u(k) \right\|^2.$$

Furthermore, this control uniformly exponentially stabilizes the system and $x'_o X_{\theta_o} x_o = \min_u V(\theta_o, u, x_o)$ and X is a continuous function.

Remark 1 *If f is invertible, the uniform detectability can be weakened to detectability, which is the dual of stabilizability (see [7] for details).*

Remark 2 *With some mild assumptions on the dynamical system f, it is known that f has many structural properties. These properties can be used to determine approximate solutions to (4). In [7] a technique based on recurrence is developed. Another technique based on the probabilistic structure of f is developed here.*

3 Markov partitions for dynamical systems

A well understood class of diffeomorphisms was first introduced by Smale [24]:

Axiom A

1. (Uniformly-hyperbolic) Over the nonwandering set of f, the tangent bundle $T\Theta$ splits smoothly as the sum $E^{+} \oplus E^{-}$, with $df_{\theta}\left(E_{\theta}^{\pm}\right) = E_{f(\theta)}^{\pm}$ and $df|_{E^{+}}$ and $df|_{E^{-}}$ are uniformly expanding and contracting, respectively.
2. The set of periodic points is dense in the nonwandering set.

Systems that satisfy Axiom A have rich properties. For example, such system have a dense set of periodic points, recurrence and transitivity. For our purposes, an important property is that Axiom A systems induce Markov chains.

Definition 1 *The* local stable manifold *of x is* $W_{\varepsilon}^{s}(x) = \left\{ y : \left\| f^{k}(x) - f^{k}(y) \right\| \overset{k\to\infty}{\to} 0 \text{ and } \|x-y\| < \varepsilon \right\}$. *Likewise, the* local unstable manifold *of x is* $W_{\varepsilon}^{s}(x) = \left\{ y : \left\| f^{-k}(x) - f^{-k}(y) \right\| \overset{k\to\infty}{\to} 0 \text{ and } \|x-y\| < \varepsilon \right\}$.

Definition 2 *A subset $R \subset \Theta$ is a* rectangle *if $diam(R) < \delta$ and $W_{\varepsilon}^{s}(x) \cap W_{\varepsilon}^{u}(y) \subset R$ for every $x, y \in R$, where δ and ε are small enough and depend on the system (see [20] for details). A rectangle is* proper *if $R = cl(int(R))$.*

Definition 3 *A family of proper rectangles $\mathcal{R} = \{R_1, R_2, \cdots R_M\}$ is a* Markov partition *if*

1. $\bigcup_{i=1}^{N} R_i = \Theta$;
2. $R_i \cap R_j = \partial R_i \cap \partial R_j$ *for* $i \neq j$;
3. *For every $1 \le i, j \le M$ such that $f(R_i) \cap int(R_j) \neq \emptyset$ and every $x \in R_i \cap f^{-1}(int(R_j))$ we have*
$$f\left(W_{\varepsilon}^{s}(x) \cap R_i\right) \subset W_{\varepsilon}^{s}(f(x)) \cap R_j.$$
4. *For every $1 \le i, j \le M$ such that $f^{-1}(R_i) \cap int(R_j) \neq \emptyset$ and every $x \in R_i \cap f(int(R_j))$ we have*
$$f^{-1}\left(W_{\varepsilon}^{u}(x) \cap R_i\right) \subset W_{\varepsilon}^{u}\left(f^{-1}(x)\right) \cap R_j.$$

Once a Markov partition has been chosen, then there exists a matrix $T = [t_{i,j}]$ with $t_{i,j} \in \{0, 1\}$, a subset of allowable sequences of M symbols

$$\Sigma_T = \left\{ s : \mathbb{Z} \to \{1, 2, \cdots M\} : t_{s(k),s(k+1)} = 1 \ \forall k \right\} \tag{6}$$

and a continuous map

$$h : \Sigma_T \to \Theta$$
$$h(s) = \bigcap_{k=-\infty}^{\infty} f^{-k}\left(R_{s(k)}\right)$$

thus,

$$h(\{s : s(0) = i\}) = R_i.$$

Furthermore,

$$\begin{array}{ccc} \Sigma_T & \xrightarrow{\sigma} & \Sigma_T \\ \downarrow h & & \downarrow h \\ \Theta & \xrightarrow{f} & \Theta \end{array}$$

commutes, where σ is the shift operator defined by $\sigma(s)(k) = s(k+1)$. Hence, f is semi-conjugate to the topological Markov chain (σ, Σ_T). Furthermore, there exists an σ invariant measure on Σ_T such that $s(k)$ is a Markov chain with

$$P\left(s(k+1) = j | \, s(k) = i, s(k-1) = l_1, s(k-2) = l_2, \cdots\right) = p_{i,j} \tag{7}$$

and h is a measure preserving map, i.e. $P\left(s \in h^{-1}(E)\right) = \mu(\theta \in E)$ where μ is an invariant measure for f. Thus the dynamics of f is described by a Markov chain.

Remark 3 *See [19], [20], and [21] for discussions on Markov partitions.*

4 jump linear systems

4.1 basic definitions and results

A jump linear system is defined as follows:

$$\begin{aligned} x(k+1) &= A^{JL}_{s(k)} x(k) + B^{JL}_{s(k)} u(k) \\ z(k) &= \begin{bmatrix} C^{JL}_{s(k)} x(k) \\ D^{JL}_{s(k)} u(k) \end{bmatrix} \end{aligned} \tag{8}$$

where $s(k)$ a Markov chain that takes values in a finite set $\{1, 2, \cdots, M\}$ with transition probabilities

$$P\left(s(k+1) = j | \, s(k) = i, s(k-1) = l_1, s(k-2) = l_2, \cdots\right) = p_{i,j},$$

(in [12], $s(k)$ is allowed to take values in the countable set $\{1, 2, \cdots\}$). Thus the parameters $A^{JL}, B^{JL}, C^{JL}, D^{JL}$ are matrix valued Markov chains. At time k it is assumed that only $s(k)$ and $x(k)$ are known.

We say that system (8) is *stochastically stabilizable* if there exists a function $F^{JL} : \{1, 2 \cdots M\} \to \mathbb{R}^{n \times p}$ such that the closed loop jump linear system

$$x(k+1) = \left(A^{JL}_{s(k)} + B^{JL}_{s(k)} F^{JL}_{s(k)}\right) x(k)$$

is stochastically stable, where stochastically stable means that there exists an $\alpha < 1$ and $\beta < \infty$ such that for $1 \le i \le M$,

$$E\left(\|x(k)\| | \, s(0) = i\right) < \beta \alpha^k \|x(0)\|.$$

Similarly, we say that system (8) is *stochastically detectable* if there exists a function $H^{JL} : \{1, 2 \cdots M\} \to \mathbb{R}^{n \times p}$ such that the closed loop jump linear system

$$x(k+1) = \left(A^{JL}_{s(k)} + H^{JL}_{s(k)} C^{JL}_{s(k)}\right) x(k)$$

is stochastically stable, i.e. there exist an $\alpha_d < 1$ and $\beta_d < \infty$ such that $E\left(\|x(k)\| \middle| s(0) = i\right) < \beta_d \alpha_d^k \|x(0)\|$. Note that stochastic detectability is stronger than existence of a time-varying asymptotic observer.

As shown in [17], [15], [12], assuming that $\left(D^{JL\prime}_{s(k)} D^{JL}_{s(k)}\right)$ is invertible and stochastic detectability, the optimal linear quadratic controller for these stochastic systems is characterized by the existence of a function $Y : \{1, 2, \cdots M\} \to \mathbb{R}^{n \times n}$ with $Y_i' = Y_i \geq 0$ such that

$$\begin{aligned} Y_{s(k)} &= A^{JL\prime}_{s(k)} \hat{Y}_{s(k+1)|s(k)} A^{JL}_{s(k)} + C^{JL\prime}_{s(k)} C^{JL}_{s(k)} \\ &- A^{JL\prime}_{s(k)} \hat{Y}_{s(k+1)|s(k)} B^{JL}_{s(k)} \left(D^{JL\prime}_{s(k)} D^{JL}_{s(k)} + B^{JL\prime}_{s(k)} \hat{Y}_{s(k+1)|s(k)} B^{JL}_{s(k)} \right)^{-1} B^{JL\prime}_{s(k)} \hat{Y}_{s(k+1)|s(k)} A^{JL}_{s(k)}, \end{aligned} \tag{9}$$

where

$$\hat{Y}_{s(k+1)|s(k)} = E\left(Y_{s(k+1)} \middle| s(k) \right) = \sum_{j=1}^{M} p_{s(k),j} Y_j.$$

Equation (9) defines a system of coupled Riccati equations. If a solution to (9) exists, then a control is

$$u(k) := F^{JL}_{s(k)} x(k) := -\left(D^{JL\prime}_{s(k)} D^{JL}_{s(k)} + B^{JL\prime}_{s(k)} \hat{Y}_{s(k)} B^{JL}_{s(k)} \right)^{-1} B^{JL\prime}_{s(k)} \hat{Y}_{s(k+1)|s(k)} A^{JL}_{s(k)} x(k).$$

This control is optimal in the sense that

$$u = \arg \min_{u \in U_{JL}} E\left(\sum_{k=1}^{\infty} \left\| C^{JL}_{s(k)} x(k) \right\|^2 + \left\| D^{JL}_{s(k)} u(k) \right\|^2 \right), \tag{10}$$

where U_{JL} is the set of u such that $u(k)$ depends only on $x(l)$ and $s(l)$ for $l \leq k$. That is, $u \in U_{JL}$ implies that $u(k) \in \mathcal{F}_k$ where $\mathcal{F}_k$ denotes the sigma algebra generated by $s(l)$ and $x(k)$, $l \leq k$ and $u(k) \in \mathcal{F}_k$ denotes that $u(k)$ is $\mathcal{F}_k$ measurable.

If a solution to (9) exists and system (8) is stochastically detectable, then the control (10) is stochastically stabilizing. Furthermore, a stochastically stabilizing controller exists if and only if a positive semi-definite solution to (9) exists.

There are many techniques to solve equation (9). The simplest is to set $Y(N, i) = 0$ for all i and for $K = (N-1), \cdots, 0$ iterate

$$\begin{aligned} Y(K, i) &= A_i^{JL\prime} \hat{Y}(K, i) A_i^{JL} + C_i^{JL\prime} C_i^{JL} \\ &- A_i^{JL\prime} \hat{Y}(K, i) B_i^{JL} \left(D_i^{JL\prime} D_i^{JL} + B_i^{JL\prime} \hat{Y}(K, i) B_i^{JL} \right)^{-1} B_i^{JL\prime} \hat{Y}(K, i) A_i^{JL} \end{aligned}$$

where

$$\hat{Y}(K, i) = \sum_{j=1}^{M} p_{i,j} Y(K+1, j)$$

Other techniques are developed in [1] and [10].

The analog of this next lemma is a standard fact for time varying systems. Similar results can be found in the literature, for example in [12]. However, the following proof is simpler.

Lemma 2 *Assume the jump linear system (8) is stochastically stabilizable and detectable and $D_i^{JL\prime} D_i^{JL} > 0$ for all i. Assume that $Y \geq 0$ is the solution to the couple Riccati equations (9). Furthermore, assume there is a $\bar{Y} < \infty$ such that $\|Y_i\| \leq \bar{Y}$ for $1 \leq i \leq M$. In this case, $E\left(\|x(k)\|^2 \middle| \mathcal{F}_l \right) \leq \beta \alpha^{k-l} \|x(l)\|^2$, for $k \geq l$ where α and β can be taken to only depend on α_d, and β_d in the definition of stochastic detectability, and on the bound $\bar{Y}$.*

Proof. With an exogenous input added, the closed loop system, with the optimal jump linear control, is

$$x(k+1) = \left(A^{JL}_{s(k)} + B^{JL}_{s(k)} F^{JL}_{s(k)}\right) x(k) + r(k) \qquad x(0) = x_o,\ s(0) = i$$
$$z(k) = \begin{bmatrix} C^{JL}_{s(k)} x(k) \\ D^{JL}_{s(k)} F^{JL}_{s(k)} x(k) \end{bmatrix}.$$

Thus, if $r \equiv 0$, then, by the Principle of Optimality, $E\left(\|z\|^2_{[k,\infty)} \middle| \mathcal{F}_k \right) = x(k)' Y_{s(k)} x(k)$. And, if $x_o = 0$, then by linearity,

$$E\left(\|z\|^2_{[0,\infty)} \middle| \mathcal{F}_0 \right) \leq \bar{Y} \|r\|^2_{[0,\infty)}. \tag{11}$$

Define

$$y(k+1) = \left(A^{JL}_{s(k)} + B^{JL}_{s(k)} F^{JL}_{s(k)}\right) y(k) + \tilde{H}_{s(k)} \left(\begin{bmatrix} C^{JL}_{s(k)} y(k) \\ D^{JL}_{s(k)} F^{JL}_{s(k)} y(k) \end{bmatrix} - z(k) \right) + r(k),$$

where

$$\tilde{H}_{s(k)} = \begin{bmatrix} H^{JL}_{s(k)} & -B^{JL}_{s(k)} \left(D^{JL\prime}_{s(k)} D^{JL}_{s(k)} \right)^{-1} D^{JL\prime}_{s(k)} \end{bmatrix}$$

with H^{JL}, whose existence is guaranteed by the detectability assumption, is such that $A^{JL}_{s(k)} + H^{JL}_{s(k)} C^{JL}_{s(k)}$ is a stochastically stable system. With $\tilde{H}$ in place, we have

$$y(k+1) = \left(A^{JL}_{s(k)} + H^{JL}_{s(k)} C^{JL}_{s(k)}\right) y(k) - \tilde{H}_{s(k)} z(k) + r(k). \tag{12}$$

Since $\left(D^{JL\prime}_{s(k)} D^{JL}_{s(k)} \right) > \rho$ for some ρ, we have $\left\| \tilde{H}_i \right\| \leq \bar{H} < \infty$ for some $\bar{H}$. By the stochastic detectability assumption, system (12) is stochastically stable. Therefore, there exists a $T < \infty$ that depends only on α_d and β_d such that,

$$E\left(\|y\|^2_{[0,\infty)} \middle| \mathcal{F}_0 \right) \leq E\left(T \left\| -\tilde{H}_{s(k)} z(k) + r(k) \right\|^2_{[0,\infty)} \middle| \mathcal{F}_0 \right) \leq T \left(\bar{H} \sqrt{\bar{Y}} + 1 \right)^2 \|r\|^2_{[0,\infty)},$$

where the right most inequality is due to (11). Similarly, if $r(k) = z(k) = 0$ for $k < K$, $E\left(\|y\|^2_{[K,\infty)} \middle| \mathcal{F}_K \right) \leq E\left(T \left\| -\tilde{H}_{s(k)} z(k) + r(k) \right\|^2_{[K,\infty)} \middle| \mathcal{F}_K \right) \leq T \left(\bar{H} \sqrt{\bar{Y}} + 1 \right)^2 \|r\|^2_{[K,\infty)}$. However, since y is a state estimator of x, if the system is initially at rest, i.e. $y(0) = x(0) = 0$, then $y(k) = x(k)$, and we conclude that

$$E\left(\|x\|^2_{[0,\infty)} \middle| \mathcal{F}_0 \right) \leq T \left(\bar{H} \sqrt{\bar{Z}} + 1 \right)^2 \|r\|^2_{[0,\infty)} = Q \|r\|^2_{[0,\infty)},$$

where $Q := T \left(\bar{H} \sqrt{\bar{Z}} + 1 \right)^2$. Similarly, if $y(0) = x(0) = 0$ and $r(k) = 0$ for $k \leq K$, then

$$E\left(\|x\|^2_{[K,\infty)} \middle| \mathcal{F}_K \right) \leq Q \|r\|^2_{[K,\infty)}.$$

Thus for all j,

$$E\left(\sum_{k=j}^{\infty} \|\Phi(k, j, s(0))\|^2 \middle| \mathcal{F}_j \right) \leq Q,$$

where $\Phi(k, j, s(0))$ is the state transition matrix of the jump linear system for time index j to time index k, starting, at time 0, from $s(0)$, i.e.

$$\Phi(k, j, \theta(0)) = \prod_{i=j}^{k} \left(A_{s(i)}^{JL} + B_{s(i)}^{JL} F_{s(i)}^{JL} \right).$$

Thus $E\left(\left\| \Phi(k+j, j, s(0)) \right\|^2 \middle| \mathcal{F}_j \right) \leq Q$. And for any integer K,

$$\begin{aligned}
& E\left(K \left\| \Phi(K+j, j, s(0)) \right\|^2 \middle| \mathcal{F}_j \right) = E\left(\sum_{i=j}^{K} \left\| \Phi(K+j, j, s(0)) \right\|^2 \middle| \mathcal{F}_j \right) \\
& \leq E\left(\sum_{i=j}^{K} \left\| \Phi(K+j, i, s(0)) \right\|^2 \left\| \Phi(i, j, s(0)) \right\|^2 \middle| \mathcal{F}_j \right) \\
& = \sum_{i=j}^{K} E\left(\left\| \Phi(i, j, s(0)) \right\|^2 E\left(\left\| \Phi(K+j, i, s(0)) \right\|^2 \middle| \mathcal{F}_i \right) \middle| \mathcal{F}_j \right) \\
& \leq \sum_{i=j}^{K} E\left(\left\| \Phi(i, j, s(0)) \right\|^2 Q \middle| \mathcal{F}_j \right) \leq QE\left(\sum_{i=j}^{K} \left\| \Phi(i, j, s(0)) \right\|^2 \middle| \mathcal{F}_j \right) \leq Q^2
\end{aligned}$$

Thus, if $K \geq 2Q^2$, then

$$E\left(\left\| \Phi(K+j, j, \theta(0)) \right\|^2 \middle| \mathcal{F}_l \right) \leq \frac{Q^2}{K} \leq \frac{1}{2}.$$

From which it is straight forward to show that $E\left(\left\| \Phi(k+j, j, \theta(0)) \right\|^2 \middle| \mathcal{F}_l \right) < \beta\alpha^k$ with $\beta = 2Q^K$ and $\alpha = \left(\frac{1}{2}\right)^{\frac{1}{K}}$. ■

4.2 jump linear system inducing an LDV system

A Markovian jump linear system is a particular type of LDV system. Given a Markov chain there is a matrix $T = [t_{i,j}]$ with $t_{i,j} \in \{0, 1\}$, a space of admissible sequences

$$\Sigma_T = \left\{ s : \mathbb{Z} \to \{1, 2, \cdots M\} : t_{s(k), s(k+1)} = 1 \ \forall k \right\}$$

and a stochastic shift $\sigma : \Sigma_T \to \Sigma_T$. Furthermore, for $s \in \Sigma_T$, we have $s(k+1) = \sigma(s)(k)$ and the Markov probability measure is σ-invariant. Define the piecewise constant parameters for $s \in \Sigma_T$,

$$\begin{aligned}
A_s^{LDV} &= A_{s(0)}^{JL} \\
B_s^{LDV} &= B_{s(0)}^{JL}
\end{aligned}$$

Thus $A^{LDV} : \Sigma_T \to \mathbb{R}^{n \times n}$ is constant on $\{s : s(0) = i\}$. We can define a continuous LDV system

$$\begin{aligned}
x^{LDV}(k+1) &= A_{\theta(k)}^{LDV} x^{LDV}(k) + B_{\theta(k)}^{LDV} u(k) \\
\theta(k+1) &= \sigma(\theta(k)).
\end{aligned}$$

Note that $\theta(k) \in \Sigma_T$, $\theta(k)(0) \in \{1, 2, \cdots M\}$ and at time k all that is known is $\theta(k)(0)$ and the probability $P(\theta(k+1)(0) = j | \theta(k)(0) = i) = p_{i,j}$.

4.3 LDV system inducing a jump linear system

We now show how a LDV system may give rise to a jump linear system. As described above in Section 3, depending on the dynamical system f, there may exist a Markov partition and the dynamics of f can be described by a Markov chain $s(k)$ on the finite set of symbols $\{1, 2, \cdots M\}$ with transition probabilities $p_{i,j}$. This leads to a jump linear system as follows: For each cell R_i of the Markov partition $\mathcal{R} = \{R_1, R_2, \cdots R_M\}$, define a point $\phi_i \in int(R_i)$ for $1 \leq i \leq M$. Set

$$A^{JL}_{s(k)} = A^{LDV}_{\phi_{s(k)}} = \frac{\partial f}{\partial x}\left(\phi_{s(k)}, 0\right)$$
$$B^{JL}_{s(k)} = B^{LDV}_{\phi_{s(k)}} = \frac{\partial f}{\partial u}\left(\phi_{s(k)}, 0\right).$$

Then $A^{JL}_{s(k)}$ is a Markov chain which takes values in $\left\{A^{LDV}_{\phi_1}, A^{LDV}_{\phi_2}, \cdots A^{LDV}_{\phi_M}\right\}$ and $B^{JL}_{s(k)}$ is a Markov chain which takes values in $\left\{B^{LDV}_{\phi_1}, B^{LDV}_{\phi_2}, \cdots B^{LDV}_{\phi_M}\right\}$. Thus we have the jump linear system

$$x^{JL}(k+1) = A^{JL}_{s(k)} x(k) + B^{JL}_{s(k)} u(k) \tag{13}$$

with transition probabilities given by (7). In this case we say the jump linear system (13) is induced by f and the partition $\mathcal{R}$. Note that if $\max_i (diam(R_i))$ is small and $h(s) = \theta(0)$, then $A^{JL}_{s(k)} \approx A^{LDV}_{\theta(k)}$ and $B^{JL}_{s(k)} \approx B^{LDV}_{\theta(k)}$, and therefore, $x^{JL}(k) \approx x^{LDV}(k)$. Hence, the jump linear system approximates the LDV system. The smaller the size of the cells R_i the better the approximation and as $\max_i (diam(R_i)) \to 0$, and fixed k, we have $x^{JL}(k) \to x^{LDV}(k)$.

4.4 jump linear control of nonlinear systems

A typical application of control theory is to determine a linear approximation of a nonlinear system around the equilibrium point and apply a controller designed for the linear system to the nonlinear system. This approach is justified by the fact if the closed-loop linear system is exponentially stable, then the closed-loop nonlinear system is locally exponentially stable. In the case where the equilibrium of the nonlinear system varies (e.g. nonlinear tracking), the approach based on linear approximation is slightly complicated. One approach is to approximate the nonlinear system as a jump linear system. We investigate the case where the nonlinear system admits a Markov partition and show that the jump linear approach is more difficult than it might first appear.

Suppose $f \in C^1$ induces Markov partition with arbitrarily small rectangles. Then, for any of these partitions one can develop a jump linear controller. Applying this controller to system (1) yields:

$$x(k+1) = \left(A^{JL}_{s(k)} + B^{JL}_{s(k)} F^{JL}_{s(k)}\right) x(k) + \eta\left(\theta(k), x(k), F^{JL}_{s(k)} x(k)\right), \tag{14}$$

where $x = \varphi - \theta$, and η accounts for error due to linearization and quantization.

Since $f \in C^1$ it is not hard to show that if $x(k)$ is small, then $\eta\left(\theta(k), x(k), F^{JL}_{s(k)} x(k)\right)$ is small. Furthermore, it is true that if $\eta\left(\theta(k), x(k), F^{JL}_{s(k)} x(k)\right)$ is small, the system (14) is stochastically stable. In this case, if $\|x(0)\|$ is small, $E(\|x(k)\|)$ is small for all k. However, there may be a non-zero probability that $\|x(k)\|$ is not small. Thus there may be a non-zero probability that $\eta\left(\theta(k), x(k), F^{JL}_{s(k)}x(k)\right)$ is not small. For any $\varepsilon, \delta > 0$ it is not hard to find examples where if $\eta(x) < \varepsilon$, the system is stochastically stable, but there exists a η with $\eta(x) < \varepsilon$ for $\|x\| < \delta$ such that there is a non-zero probability that the system with this nonlinearity is unstable. For example, consider $x(k+1) = a_i x(k) + \eta(x)$ with $a_0 = 0.1$ and $a_1 = 2$, with $\eta(x) = \begin{cases} 0 \text{ if } x < 1 \\ 10x \text{ if x} \geq 1 \end{cases}$ and transition probabilities $[p_{i,j}] = \begin{bmatrix} 0.9 & 0.1 \\ 0.9 & 0.1 \end{bmatrix}$.Then for all $x(0)$, there is a positive probability that $x > 1$ and in which case the system is unstable with probability one. Of course, using Chebyshev's inequality, one can show that by limiting $\|x(0)\|$ the closed loop system, with nonlinear system with perturbation is stable with a probability close to one.

Furthermore, if the desired trajectory $\{\theta(k)\}$ is a fixed point, and the probability of staying in to cell with the fixed point is not one, then the probability of staying in the cell for all time is zero. Stochastically stable does not directly imply that the jump linear system is stable at the fixed point. The difficulty is that stochastic stability implies stability over the average orbit $\{\theta(k)\}$. But when a particular orbit is chosen, stochastic stability cannot say anything about the stability along this orbit.

These difficulties can be avoided by using techniques described in [16]. However, the method in [16] appears to be overly conservative since simulations show that the standard jump linear controller works for a fine enough partition. The next section shows that the simulations are correct and that for fine enough partition, the jump linear controller stabilizes the nonlinear system.

5 main results

Next it will be shown that if the nonlinear system is LDV stabilizable, then for a fine enough partition, the jump linear system stabilizes the nonlinear system (Proposition 4). Conversely, if as the partition is refined, the solution to the jump linear coupled Riccati equations (9) remain bounded, then the system is LDV stabilizable (Proposition 6). In this case, as the partition is refined, the jump linear controller approaches the LDV controller (Theorem 8). Moreover, this process is robust to errors in the Markov partition. Specifically, if the partition is incorrectly assumed to be Markov, then the resulting jump linear controller still approximates the LDV controller and, if the partition is fine enough, stabilizes the nonlinear system.

Once we set a partition $\mathcal{R} = \{R_1, R_2, \cdots R_M\}$, we define p to be the transition probabilities assumed, perhaps incorrectly, to be Markov. We define $p_{i,j} := P(s(k+1) = j | s(k) = i)$ and it is possible that

$$p_{i,j} \neq P(s(k+1) = j | s(k) = i, s(k-1) = l_1, s(k-2) = l_2, \cdots).$$

Define $E_{\mathcal{R},p}(\cdot)$ to be the expectation operator with the probabilistic structure induced by $p_{i,j}$. Note that since the partition does not need to be Markov, it suffices to define $p_{i,j} = \frac{\sum_k 1_{\{f^k(\theta)\in R_i, f^{k+1}(\theta)\in R_j\}}}{\sum_k 1_{\{f^k(\theta)\in R_i\}}}$ where θ is a transitive point and $1_{\{f^k(\theta)\in R_i\}}$ is the indicator function.

With $\mathcal{R}$ chosen, define $h : \Sigma_T \to \Theta$ as in Section 3. Recall that $h(\{s \in \Sigma_T : s(0) = i\}) =$

R_i. Define

$$g : \Theta \to \{1, 2, \cdots M\}$$
$$g(\theta) = i \text{ if } \theta \in R_i \text{ and } \theta \notin R_j \text{ for } j \neq i.$$

If $\theta \in R_i \cap R_j$, then we have a choice as to the value of $g(\theta) = i$ or j. By invoking the axiom of choice we can define g such that $g(\theta) = i$ implies that $\theta \in R_i$.

As done in Section 4.3, for each cell R_i define $\phi_i \in$int(R_i). Thus $\phi_{s(k)}$ is a Markov chain that takes values in $\{\phi_1, \phi_2, \cdots, \phi_M\}$. Since $R_i \cap R_j = \partial R_i \cap \partial R_j$, we have $g(\phi_i) = i$. This suggests the notation

$$\tilde{g}^{-1} : \{1, 2, \cdots M\} \to \Theta$$
$$\tilde{g}^{-1}(i) = \phi_i.$$

and define

$$A_i^{JL} := A_{\tilde{g}^{-1}(i)}^{LDV} \qquad B_i^{JL} := B_{\tilde{g}^{-1}(i)}^{LDV}$$
$$C_i^{JL} := C_{\tilde{g}^{-1}(i)}^{LDV} \qquad D_i^{JL} := D_{\tilde{g}^{-1}(i)}^{LDV}.$$

Clearly, if the mesh of the partition is small, i.e. $mesh(\mathcal{R}) = \max_i (diam(R_i))$ is small, then $\tilde{g}^{-1}(g(\theta)) \approx \theta$ and

$$A_\theta^{LDV} \approx A_{g(\theta)}^{JL} \qquad B_\theta^{LDV} \approx B_{g(\theta)}^{JL}$$
$$C_\theta^{LDV} \approx C_{g(\theta)}^{JL} \qquad D_\theta^{LDV} \approx D_{g(\theta)}^{JL}$$

Likewise, if k is finite and $mesh(\mathcal{R})$ is small, then $f^k(\theta) \approx f^k\left(\tilde{g}^{-1}(g(\theta))\right)$. Note that the functions h, g and $\tilde{g}^{-1}$ depend on the partition $\mathcal{R}$. Thus these functions should be written $h_{\mathcal{R}}$, $g_{\mathcal{R}}$ and $\tilde{g}_{\mathcal{R}}^{-1}$. However, to reduce cluter the dependence on the partition is dropped.

Define
$$supp\left(p_{g(\theta),g(\cdot)}\right) = \overline{\left\{\varphi \in \Theta : p_{g(\theta),g(\varphi)} \neq 0\right\}}$$

and define $supp\left(p_{g(\theta),g(\cdot)}^k\right)$ similarly, where $p_{i,j}^k$ is the i, j element of the k^{th} power of the matrix $[p_{i,j}]$.

Lemma 3 *For fixed $k < \infty$, as $mesh(\mathcal{R}) \to 0$,*

$$diam\left(supp\left(p_{g(\theta),g(\cdot)}^k\right)\right) \to 0$$

uniformly in θ.

Proof. Set $\varepsilon > 0$. Since f is uniformly continuous, there exists a $\delta > 0$ such that if $mesh(\mathcal{R}) < \delta$, then for each $\theta \in \Theta$, $diam(f(\{\varphi \in \Theta : g(\theta) = g(\varphi)\})) \leq diam\left(f\left(R_{g(\theta)}\right)\right) < \varepsilon$. Thus $diam\left(supp\left(p_{g(\theta),g(\cdot)}\right)\right) < \varepsilon + 2\delta$, where the 2δ term is due to points x such that $x \notin f\left(R_{g(\theta)}\right)$, but $g(x) = g(\varphi)$ and $\varphi \in f\left(R_{g(\theta)}\right)$, so $\|x - \varphi\| \leq mesh(R) < \delta$. (Note the 2δ is not needed if the partition is Markov.) Hence, the lemma holds for $k = 1$. Since k is finite the same reasoning can be applied k times. ■

Proposition 4 *Assume that the map f is LDV stabilizable. In this case there exists a $\delta > 0$ such that if mesh $(\mathcal{R}) < \delta$, then the jump linear system induced by f and $\mathcal{R}$ is stochastically stabilizable. Furthermore, assuming that $C^{LDV} : \Theta \to \mathbb{R}^{n\times m}$ and $D^{LDV} : \Theta \to \mathbb{R}^{n\times m}$ are continuous and $D^{LDV\prime} D^{LDV} > 0$, if mesh $(\mathcal{R}) < \delta$, then there is a bound $\bar{Y}$ on Y the solution to the coupled Riccati equations that is independent of the partition.*

Proof. Let F^{LDV} be the optimal LDV feedback for $C^{LDV} = I$ and $D^{LDV} = I$. Define $F(k, s(0)) := F^{LDV}_{f^k(\tilde{g}^{-1}(s(0)))}$, that is F is the optimal LDV feedback assuming $\theta(0) = \tilde{g}^{-1}(s(0))$. Note that $F(k, s(0)) \in \mathcal{F}_0 \subset \mathcal{F}_k$. Thus $u(k) = F(k, s(0))\, x(k) \in U_{JL}$. Since the LDV feedback uniformly exponentially stabilizes the LDV system, there exists an $N < \infty$ such that for all $\theta \in \Theta$,

$$\left\| \prod_{k=0}^{N-1} \left(A^{LDV}_{f^k(\theta)} + B^{LDV}_{f^k(\theta)} F^{LDV}_{f^k(\theta)} \right) \right\| \leq \frac{1}{4}. \tag{15}$$

Since equation (15) is continuous in A^{LDV} and B^{LDV}, there exists a $\gamma > 0$ such that if

$$\left\| \tilde{A}_k - A^{LDV}_{f^k(\theta)} \right\|, \left\| \tilde{B}_k - B^{LDV}_{f^k(\theta)} \right\| < \gamma,$$

then $\left\| \prod_{k=0}^{N-1} \left(\tilde{A}_k + \tilde{B}_k F^{LDV}_{f^k(\theta)} \right) \right\| \leq \frac{1}{2}$. Since A^{LDV} and B^{LDV} are uniformly continuous, there exists a $\lambda > 0$ such that if $\|\varphi - \theta\| < \lambda$, then $\left\|A^{LDV}_\varphi - A^{LDV}_\theta\right\|, \left\|B^{LDV}_\varphi - B^{LDV}_\theta\right\| < \gamma$. By Lemma 3, there exists $\delta_1 > 0$ such that if $mesh\,(\mathcal{R}) < \delta_1$ then $diam\left(supp\left(p^k_{g(\theta),g(\cdot)}\right)\right) < \lambda$ for all $k \leq N$. Let $\mathcal{R}$ be a partition such that $mesh\,(\mathcal{R}) < \delta_1$, and let $\{s(k) : 0 \leq k \leq N-1\}$ be an admissible path[1], i.e. $p_{s(i),s(i+1)} > 0$, starting from $s(0) = g(\theta)$. Then $\left\|\tilde{g}^{-1}(s(k)) - f^k(\theta)\right\| < \lambda$ for $k \leq N$, thus $\left\|A^{JL}_{s(k)} - A^{LDV}_{f^k(\theta)}\right\|, \left\|B^{JL}_{s(k)} - B^{LDV}_{f^k(\theta)}\right\| < \gamma$ and therefore $\left\|\prod_{k=0}^{N-1}\left(A^{JL}_{s(k)} + B^{JL}_{s(k)} F^{LDV}_{f^k(\theta)}\right)\right\| < \frac{1}{2}$ and

$$E_{\mathcal{R},p}\left(\left\| \prod_{k=0}^{N-1} \left(A^{JL}_{s(k)} + B^{JL}_{s(k)} F^{LDV}_{f^k(\theta)} \right) \right\| \middle| s(0) = g(\theta) \right) < \frac{1}{2}. \tag{16}$$

Since δ_1 was chosen independent of θ, we conclude that the Markov system is stochastically stabilizable for $mesh\,(\mathcal{R}) < \delta_1$.

With C^{LDV} and D^{LDV} given, bounds on C^{JL} and D^{JL} can be found that do not depend on $mesh\,(\mathcal{R})$. Inequality (16) can be used to show that there is a $\bar{Y} < \infty$ such that if Y solves the coupled Riccati equations (9), then for $mesh\,(\mathcal{R}) < \delta_1$, $\|Y_i\| < \bar{Y}$ for $1 \leq i \leq M$, where $\bar{Y}$ is independent of partition. ∎

The following can be proved in the same fashion as the above proposition.

Proposition 5 *Assume that $C^{LDV} : \Theta \to \mathbb{R}^{n\times m}$ is continuous and $\left(A^{LDV}, C^{LDV}, f\right)$ is LDV detectable. There exists a $\delta > 0$ such that if mesh $(\mathcal{R}) < \delta$, then the jump linear system induced by A^{LDV}, C^{LDV}, f and $\mathcal{R}$ is stochastically detectable. Furthermore, for mesh $(\mathcal{R}) < \delta$, the α_d and β_d in the definition of stochastic detectability can be taken independent of the partition.*

[1] By admissible path, we mean any path with nonzero transition probabilties. Note that since the probabilties may have been incorrectly assumed to be Markov, some admissible paths may not be possible in terms of the actual system. However, we still included these paths as admissible.

Let $\{\mathcal{R}^t\}$ be a sequence of partitions of Θ. Then, if they exists, each partition induces a jump linear controller $F^{JL,t}$ and the solution Y^t to the coupled Riccati equations (9).

Proposition 6 *Assume that $C^{LDV} : \Theta \to \mathbb{R}^{p\times n}$, $D^{LDV} : \Theta \to \mathbb{R}^{q\times m}$ are continuous, (A^{LDV}, C^{LDV}, f) is uniformly detectable and there exists a sequence of partitions $\{\mathcal{R}^t\}$ with mesh $(\mathcal{R}^t) \to 0$ such that $\|Y_i^t\| < \bar{Y}$ for all i. In this case, the LDV induced by f is stabilizable.*

Proof. Since $\|Y^t\| < \bar{Y}$ for all t, by Lemma 2 there exists an $N < \infty$ such that for all t, if the optimal jump linear feedback $F^{JL,t}_{s(k)}$ is applied, we have $E\left(\|x(N)\| \,|\, s(0)\right) < \frac{1}{4}\|x(0)\|$ for all $s(0)$ where $x(k+1) = \left(A^{JL}_{s(k)} + B^{JL}_{s(k)}F^{JL}_{s(k)}\right)x(k)$. Thus for each $s(0) = g(\theta_o)$ there exists an admissible path $\{s(k) : 0 \le k \le N\}$ with $s(0) = g(\theta_o)$ such that

$$\left\|\prod_{k=0}^{N}\left(A^{JL,t}_{s(k)} + B^{JL,t}_{s(k)}F^{JL,t}_{s(k)}\right)\right\| < \frac{1}{4}. \tag{17}$$

Since equation (17) is uniformly continuous in $A^{JL,t}$ and $B^{JL,t}$, it is possible to show that there exists a $\delta > 0$ such that if $mesh\left(\mathcal{R}^t\right) < \delta$, then $\left\|\prod_{k=0}^{N}\left(A^{LDV}_{f^k(\theta_o)} + B^{LDV}_{f^k(\theta_o)}F^{JL,t}_{s(k)}\right)\right\| < \frac{1}{2}$. Since this applies for all θ_o, we concluded that $F^{JL,t}$ stabilizes the LDV. ■

Combining propositions 4 and 6 yields:

Theorem 7 *Let $C^{LDV} : \Theta \to \mathbb{R}^{n\times m}$ and $D^{LDV} : \Theta \to \mathbb{R}^{n\times m}$ be continuous with $D^{LDV\prime}_\theta D^{LDV}_\theta > 0$, and let (A^{LDV}, C^{LDV}, f) be uniformly detectable. The LDV induced by f is stabilizable if and only if for any sequence of partitions $\mathcal{R}^t$ such that mesh $(\mathcal{R}^t) \to 0$ the Markov jump linear systems induced by f and $\mathcal{R}^t$ are stabilizable with bounded optimal quadratic cost, where the bound does not depend on t.*

Thus the existence of a stabilizing LDV controller is linked to the existence of a series of stabilizing jump linear controllers. Now we show that actually these controllers are nearly identical.

Theorem 8 *Let $C^{LDV} : \Theta \to \mathbb{R}^{n\times m}$ and $D^{LDV} : \Theta \to \mathbb{R}^{n\times m}$ be continuous with $D^{LDV\prime}_\theta D^{LDV}_\theta > 0$, and let (A^{LDV}, C^{LDV}, f) be uniformly detectable. Assume that f is LDV stabilizable or, equivalently, assume that there exists a bounded sequence of solutions Y^t to the coupled Riccati equations (9) associated with a sequence of partitions with mesh $(\mathcal{R}^t) \to 0$; then*

$$\sup_{\theta\in\Theta}\left\|X_\theta - Y^t_{g(\theta)}\right\| \to 0 \text{ as } t \to \infty$$

where X solves the functional Riccati equation and Y^t solve the couple Riccati equations.

Proof. Set $\varepsilon > 0$. First we show that there exists a $\delta_1 > 0$ such that if $mesh\left(\mathcal{R}^t\right) < \delta_1$, then $Y^t_{g(\theta)} \le X_\theta + \varepsilon I$. As in the proof of proposition 4 the feedback $F^{LDV}_{f^k(\tilde{g}^{-1}(s(0)))} \in \mathcal{F}_0 \subset \mathcal{F}_k$, that is F is the optimal LDV feedback assuming $\theta(0) = \tilde{g}^{-1}(s(0))$. Thus $u(k) = F^{LDV}_{f^k(\tilde{g}^{-1}(s(0)))}x(k) \in U_{JL}$.

Set $\|x_o\| = 1$ and set $\theta = \tilde{g}^{-1}(s(0))$. Define $x^{LDV}(F, k+1) = \left(A^{LDV}_{f^k(\theta)} + B^{LDV}_{f^k(\theta)} F_k\right) x^{LDV}(F, k)$ with $x^{LDV}(F, 0) = x_o$. And define $x^{JL}(F, k+1) = \left(A^{JL}_{s(k)} + B^{JL}_{s(k)} F_k\right) x^{JL}(F, k)$ with $x^{JL}(F, 0) = x_o$.

Since the LDV feedback is exponentially stabilizing, there exists an $N < \infty$ such that for all $\theta \in \Theta$

$$\left\|x^{LDV}\left(F^{LDV}, N\right)\right\| < \frac{\varepsilon}{4\bar{Y}}.$$

Since $x^{LDV}\left(F^{LDV}, N\right)$ is uniformly continuous in A^{LDV} and B^{LDV} and by Lemma 3, there exists a $\delta_1 > 0$, such that if $mesh\,(\mathcal{R}) < \delta_1$, then for each admissible path $\left\{s : s(0) = \theta,\ p_{s(k),s(k+1)} > 0\right\}$

$$\left\|x^{JL}\left(\tilde{F}, N\right)\right\|^2 < \frac{\varepsilon}{2\bar{Y}}, \tag{18}$$

where $\tilde{F}_k := F^{LDV}_{f^k(\theta)} = F^{LDV}_{f^k(\tilde{g}^{-1}(s(0)))}$. Hence,

$$E_{\mathcal{R},p}\left(\left\|x^{JL}\left(\tilde{F}, N\right)\right\|^2 \middle| s(0) = g(\theta)\right) \leq \frac{\varepsilon}{2\bar{Y}}. \tag{19}$$

Similarly, since

$$\begin{aligned}&\sum_{k=0}^{N-1}\left\|C^{LDV}_{f^k(\theta)} x^{LDV}\left(F^{LDV}, k\right)\right\|^2 + \left\|D^{LDV}_{f^k(\theta)} F^{LDV}_{f^k(\theta)} x^{LDV}\left(F^{LDV}, k\right)\right\|^2 \\ &= x(0)' X_\theta x(0) - x^{LDV}\left(F^{LDV}, N\right)' X_{f^N(\theta)} x^{LDV}\left(F^{LDV}, N\right).\end{aligned}$$

And since the left-hand side is continuous in A^{LDV}, B^{LDV}, C^{LDV} and D^{LDV}, by an augment similar to that which lead to (18), there exists a $\delta_2 > 0$ such that if $mesh\,(\mathcal{R}) < \delta_2$, then for any admissible path $\left\{s : s(0) = \theta,\ p_{s(k),s(k+1)} > 0\right\}$,

$$\begin{aligned}&\left(\sum_{k=0}^{N-1}\left\|C^{JL}_{s(k)} x^{JL}\left(\tilde{F}, k\right)\right\|^2 + \left\|D^{JL}_{s(k)} F^{LDV}_{f^k(\theta)} x^{JL}\left(\tilde{F}, k\right)\right\|^2\right) \\ &\leq x_o' X_\theta x_o - x^{LDV}\left(F^{LDV}, N\right)' X_{f^N(\theta)} x^{LDV}\left(F^{LDV}, N\right) + \frac{\varepsilon}{2}.\end{aligned} \tag{20}$$

Assume Y is the positive definite solution the coupled Riccati equations (9), that is,

$$x(0)' Y_{s(0)} x(0) = E_{\mathcal{R},p}\left(\sum_{k=0}^{N-1}\left\|C^{JL}_{s(k)} x^{JL}\left(F^{JL}, k\right)\right\|^2 + \left\|D^{JL}_{s(k)} F^{JL}_{s(k)} x^{JL}\left(F^{JL}, k\right)\right\|^2 + x^{JL}(N)\, Y_{s(N)} x^{JL}(N) \middle| s(\right.$$

Therefore, if the control $u(k) = F^{LDV}_{f^k(\tilde{g}^{-1}(s(0)))} x(k)$ is applied for $k < N$, the control $u(k) = F^{JL}_{s(k)} x(k)$ is applied for $k \geq N$, and the $mesh\,(\mathcal{R}) < \min(\delta_1, \delta_2)$, then applying (20) and

(19) yields

$$
\begin{aligned}
&x(0)' Y_{s(0)} x(0) \\
&= E_{\mathcal{R},p}\left(\sum_{k=0}^{N-1}\left\|C_{s(k)}^{JL} x^{JL}\left(F^{JL},k\right)\right\|^{2}+\left\|D_{s(k)}^{JL} F_{s(k)}^{JL} x^{JL}\left(F^{JL},k\right)\right\|^{2}+x^{JL}\left(F^{JL},N\right)' Y_{s(N)} x^{JL}\left(F^{JL},N\right)\middle| s(0)\right) \\
&\leq E_{\mathcal{R},p}\left(\sum_{k=0}^{N-1}\left\|C_{s(k)}^{JL} x^{JL}\left(\tilde{F},k\right)\right\|^{2}+\left\|D_{s(k)}^{JL} F_{f^{k}(\theta)}^{LDV} x^{JL}\left(\tilde{F},k\right)\right\|^{2}+x^{JL}\left(\tilde{F},k\right)' Y_{s(N)} x^{JL}\left(\tilde{F},k\right)\right) \\
&\leq x_{o}' X_{\theta} x_{o}-x^{LDV}\left(F^{LDV},N\right)' X_{f^{N}(\theta)} x^{LDV}\left(F^{LDV},N\right)+\frac{\varepsilon}{2}+\frac{\varepsilon}{2\bar{Y}}\bar{Y} \\
&\leq x(0) X_{\theta} x(0)+\varepsilon .
\end{aligned}
$$

Since x_o and θ are arbitrary,

$$
Y_{s(g(\theta))} \leq X_{\theta}+\varepsilon I. \tag{21}
$$

Now since Y^t is bounded, for $mesh\left(\mathcal{R}^t\right)$ small enough, by Corollary 5 and Lemma 2, there exists a $N<\infty$ such that for all $\theta \in \Theta$, and with the optimal jump linear control applied

$$
E_{\mathcal{R}^{t},p}\left(\left\|x^{JL}\left(F^{JL},N\right)\right\|^{2}\middle| s(0)=g(\theta)\right) \leq \frac{\varepsilon}{2\bar{X}}, \tag{22}
$$

where $\|X_{\varphi}\| \leq \bar{X}<\infty$ for all $\varphi \in \Theta$. Since F^{LDV} is the optimal control, and X_θ is the minimum quadratic cost, for any sequence $\left\{n(l) \in \mathcal{R}^{t}: 0 \leq l<N\right\}$

$$
X_{\theta} \leq \sum_{k=0}^{N-1}\left\|C_{f^{k}(\theta)}^{LDV} x^{LDV}\left(\hat{F},k\right)\right\|^{2}+\left\|D_{f^{k}(\theta)}^{LDV} \hat{F}_{k} x^{LDV}\left(\hat{F},k\right)\right\|^{2}+x^{LDV}\left(\hat{F},k\right)' X_{f^{N}(\theta)} x^{LDV}\left(\hat{F},k\right),
$$

where $\hat{F}_k = F_{n(k)}^{JL}$. Since the right hand side of this expression is continuous in $A^{LDV}, B^{LDV}, C^{LDV}$ and D^{LDV}, there exists a $\delta_4 > 0$ such that if $mesh(\mathcal{R}) < \delta_4$ and if $\left\{s : s(0)=\theta,\ p_{s(k),s(k+1)}>0\right\}$ is an admissible path, then

$$
\begin{aligned}
&\sum_{k=0}^{N-1}\left\|C_{f^{k}(\theta)}^{LDV} x^{LDV}\left(\hat{F},k\right)\right\|^{2}+\left\|D_{f^{k}(\theta)}^{LDV} F_{n(l)}^{JL} x^{LDV}\left(\hat{F},k\right)\right\|^{2}+x^{LDV}\left(\hat{F},N\right)' X_{f^{N}(\theta)} x^{LDV}\left(\hat{F},N\right) \\
&-\sum_{k=0}^{N-1}\left\|C_{s(l)}^{JL} x^{JL}\left(F^{JL},k\right)\right\|^{2}+\left\|D_{s(l)}^{JL} F_{n(l)}^{JL} x^{JL}\left(F^{JL},k\right)\right\|^{2}+x^{JL}\left(F^{JL},k\right)' X_{f^{N}(\theta)} x^{JL}\left(F^{JL},k\right) \\
&<\frac{\varepsilon}{2}
\end{aligned}
$$

Thus combining these last two expressions yields

$$
X_{\theta} \leq \sum_{k=0}^{N-1}\left\|C_{s(l)}^{JL} x^{JL}\left(F^{JL},k\right)\right\|^{2}+\left\|D_{s(l)}^{JL} F_{n(l)}^{JL} x^{JL}\left(F^{JL},k\right)\right\|^{2}+x^{JL}\left(F^{JL},N\right)' X_{f^{N}(\theta)} x^{JL}\left(F^{JL},N\right)+\frac{\varepsilon}{2} I
$$

This equation remains true for any weighted sum of admissible paths, in particular

$$
\begin{aligned}
X_\theta &\leq E_{\mathcal{R},p}\left(\sum_{k=0}^{N-1}\left\|C_{s(k)}^{JL}x^{JL}\left(F^{JL},k\right)\right\|^2+\left\|D_{s(k)}^{JL}F_{s(k)}^{JL}x^{JL}\left(F^{JL},k\right)\right\|^2\right.\\
&\left.+x^{JL}\left(F^{JL},N\right)'X_{f^N(\theta)}x^{JL}\left(F^{JL},N\right)\right)+\frac{\varepsilon}{2}I\\
&\leq Y_{g(\theta)}+E_{\mathcal{R},p}\left(x^{JL}\left(F^{JL},N\right)'X_{f^N(\theta)}x^{JL}\left(F^{JL},N\right)\right)+\frac{\varepsilon}{2}I\\
&\leq Y_{g(\theta)}+\varepsilon I,
\end{aligned}
$$

where the last inequality follows from (22). Thus

$$X_\theta \leq Y_{g(\theta)}+\varepsilon I$$

Combining this with (21) yields if $mesh\left(\mathcal{R}\right)<\min\left(\delta_1,\delta_2,\delta_3\right)$, then for all $\theta\in\Theta$

$$\left\|X_\theta - Y_{g(\theta)}\right\| < \varepsilon.$$

■

6 example

The Hénon system is defined as

$$\begin{bmatrix}\theta_1(k+1)\\ \theta_2(k+1)\end{bmatrix}=\begin{bmatrix}f_1(\theta(k),u(k))\\ f_2(\theta(k),u(k))\end{bmatrix}=\begin{bmatrix}1-(a+u(k))\,\theta_1(k)^2+\theta_2(k)\\ b\theta_1(k)\end{bmatrix}$$

where u is the control input. In this example, $a=1.4$ and $b=0.3$. For these parameter values and $u\equiv 0$, it is known that the Hénon map has an attractor Θ, that is, these exists an open set $V\supsetneq\Theta$ such that $\lim_{k\to\infty}d\left(f^k\left(\theta_o\right),\Theta\right)=0$ for all $\theta_0\in V$. This attractor is the crescent shaped object shown in Figure 1. A Markov partition for the Hénon system is not known. However, an arbitrary partition can be made, transition probabilties can found and a jump linear control designed. Furthermore, Theorem 8 guarantees that as the partition is refined, the controller convergences to the optimal LDV controller. Figure 1 shows the partition and the 1-1 component of the quadratic cost.

7 conclusion

When addressing nonlinear tracking problems two seemingly distinct approaches are LDV model and the jump linear model. The relationship between these approaches has been developed. In the case when the nonlinear system admits a Markov partition, a jump linear approach is justified and it has been shown that the LDV system is stabilizable if and only if the jump linear system is stabilizable for a fine enough enough partition. Furthermore, the jump linear controller is an approximation of the LDV controller in the sense that as the partition is refined, the jump linear controller converges to the LDV controller. These results have further application to the case where either the nonlinear system does not admit a Markov partition or the Markov partition is unknown. Another application of these results is that methods to compute jump linear controllers can be used to computer LDV controllers.

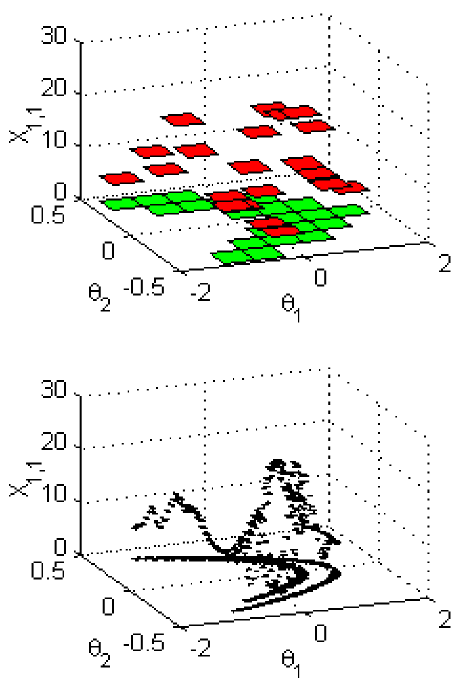

Figure 1: The above shows the 1-1 element of the quadratic cost of the optimal jump linear controller as the partition is refined. The cost converges to a continuous function which is the cost of the optimal LDV controller. For reference, the partition is shown in the $z = 0$ plane.

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