Image Enhancement in the Frequency Domain

- Two dimensional Fourier transform
  - Continuous space
  - Discrete space
- Sampling
- Revisit algorithms for improving the visual appearance of images
  - Image smoothing
  - Image sharpening
  - Homomorphic filtering

Fourier Transform

- Fourier series
  - Periodic signals can be expressed as a sum of complex exponentials
  - Frequencies are integer multiples of a fundamental
- Fourier transform
  - Generalization to nonperiodic signals
  - Original problem: heat transfer
    - Not widely accepted in 1807

Two-Dimensional Fourier Transforms

- Input and Output of 2-D systems are functions of 2 independent variables
  - Coordinates (x,y)
  - f(x,y) is brightness
- Linear systems obey the superposition property
  - Characterized by the impulse response
- Impulse response:
  - $\delta(x,y)$
  - $\iint \delta(x,y)dx\,dy = 1$
  - $\iint \delta(x,y)\delta(x,y) = \delta(0,0)$
  - Separable: $\delta(x,y) = \delta(x)\delta(y)$
  - Even: $\delta(-x,-y) = \delta(x,y)$
Impulse Response

- Impulse response $h(x,y)$ characterizes system
- For input $f(x,y)$ the system output is
  
  \[ g(x,y) = f(x,y) * h(x,y) \]
  
  \[ = \int_{-\infty}^{\infty} f(x-\alpha, y-\beta)h(\alpha, \beta) d\alpha d\beta \]

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Fourier Analysis

- Recall in one dimension
  \[ F(u) = \int f(x) e^{-j2\pi ux} dx \]
  \[ f(x) = \int F(u) e^{j2\pi ux} du \]
- In two dimensions
  \[ F(u,v) = \int \int f(x,y) e^{-j2\pi (ux+vy)} dxdy \]
  \[ f(x,y) = \int \int F(u,v) e^{j2\pi (ux+vy)} dudv \]
- Concept: superimpose infinite terms of the form
  \[ \cos(2\pi (ax+by)) \]
  - Lines of constant amplitude given by $2\pi (ax+by) = k$
- Interpretation:
  - Low frequencies carry macroscopic image information
  - High frequencies carry detail

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Fourier Transform Properties

- Linearity: $\mathcal{F}\{af(x,y) + bf(x,y)\} = aF(u,v) + bF(u,v)$
- Scaling: $\mathcal{F}\{f(ax,by)\} = \frac{1}{\sqrt{ab}} F\left(\frac{u}{a}, \frac{v}{b}\right)$
- Shift: $\mathcal{F}\{f(x-a,y-b)\} = F(u,v)e^{-j2\pi (au+fv)}$
- Convolution: $f_1(x,y) * f_2(x,y) \Leftrightarrow F_1(u,v)F_2(u,v)$
- Conjugates: $f^*(x,y) \Leftrightarrow F^*(-u,-v)$
- Duality: $\mathcal{F}\{F(x,y)\} = f(-x,-y)$
- Parseval’s: $\|f(x,y)\|^2 dxdy = \|F(u,v)\|^2 dudv$
- Separability: a function is separable if
  
  \[ f(x,y) = g(x)h(y) \Leftrightarrow F(u,v) = G(u)H(v) \]

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Common 2-D Functions

- $\text{rect}[x,y] = \text{rect}[x]\text{rect}[y] = \begin{cases} 1 & |x|,|y| \leq 1/2 \\ 0 & \text{else} \end{cases}$
- $\text{sinc}(x,y) = \text{sinc}(x)\text{sinc}(y) = \frac{\sin(\pi x)}{\pi x} \frac{\sin(\pi y)}{\pi y}$
- $\text{comb}(x,y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x-n,y-m)$
- $\text{rep}_{x,y}[f(x,y)] = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(x-nX,y-mY)$
- $\text{comb}_{x,y}[f(x,y)] = f(x,y) \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x-nX,y-mY)$
**Fourier Transform Example**

\[
f(x, y) = \text{rect}(x, y) = \begin{cases} 1 & \text{if } x < \frac{1}{2}, y < \frac{1}{2} \\ 0 & \text{else} \end{cases}
\]

\[
F(u, v) = \iint \text{rect}(x, y)e^{-j2\pi ux}e^{-j2\pi vy} \, dx \, dy
= \int \text{rect}(y)e^{-j2\pi uy} \int \text{rect}(y)e^{-j2\pi vx} \, dy
= \text{sinc}(u)\text{sinc}(v)
\]

**Notation & Equivalencies**

\[
\sum_{m} e^{-j2\pi mx} = \sum_{m} \delta(x - m)
\]

\[
\delta(ax + b) = \frac{1}{a} \delta(x + \frac{b}{a})
\]

\[
\text{rep}_{x \rightarrow y} \left[ f(x, y) \right] = \sum_{k} f(x - kX, y)
\]

\[
\text{rep}_{y \rightarrow x} \left[ f(x, y) \right] = \sum_{k} f(x, y - kY)
\]

**Comb Example**

\[
\text{comb}(x, y) = \sum_{m} \sum_{n} \delta(x - m, y - n)
= \sum_{m} \sum_{n} \delta(x - m) \delta(y - n)
\]

\[
2 \left\{ \text{comb}(x, y) \right\} = 2 \left\{ \sum_{m} \sum_{n} \delta(x - m) \sum_{n} \delta(y - n) \right\}
= 2 \left\{ \sum_{m} e^{-j2\pi mx} \sum_{n} e^{-j2\pi ny} \right\}
= \sum_{m} \delta(u - m) \sum_{n} \delta(v - n)
= \text{comb}(u, v)
\]

\[
\text{comb}(x, y) = \text{comb}(u, v)
\]

**Rep Example**

\[
2 \left\{ \text{rep}_{x \rightarrow y} \left[ f(x, y) \right] \right\} = 2 \left\{ \sum_{m} f(x - mX, y - nY) \right\}
= \sum_{m} \sum_{n} f(x - mX, y - nY) \sum_{m} e^{-j2\pi mx} \sum_{n} e^{-j2\pi ny}
= F(u, v) \sum_{m} \delta(u - m) \sum_{n} \delta(v - n)
= F(u, v) \sum_{m} \sum_{n} \delta(u - m) \delta(v - n)
= \frac{1}{X} \sum_{m} \sum_{n} \delta(u - m) \delta(v - n)
= \frac{1}{XY} \text{comb}(u, v)
\]

\[
\text{rep}_{x \rightarrow y} \left[ f(x, y) \right] = \frac{1}{XY} \text{comb}(u, v)
\]
**Fourier Transform Example**

- **Signal:** \( f(x, y) = \text{rect}\left(\frac{2y}{Y}\right) \)
- **Note:** \( \text{rect}\left(\frac{2y}{Y}\right) = 1 \cdot \text{rect}\left(\frac{2y}{Y}\right) \)
  \( \delta(u) \Leftrightarrow \delta(u) \)
  \( \text{rect}\left(\frac{2y}{Y}\right) \Leftrightarrow \frac{Y}{2} \sin\left(\frac{Y}{2} \right) \)
- **Thus:** \( \text{rect}\left(\frac{2y}{Y}\right) \Leftrightarrow \delta(u) \frac{Y}{2} \sin\left(\frac{Y}{2} \right) \)
- **Finally:** \( F(u,v) = \frac{1}{2} \text{comb}\left[\frac{1}{2} \sin\left(\frac{Y}{2} \right) \right] \)

**Fourier Transform Example**

- \( f(x, y) = \cos(2\pi(x + y)) \)
- **Note:**
  \( \delta(x, y) \Leftrightarrow 1 \)
  \( \delta(x+1, y+1) \Leftrightarrow e^{j2\pi(x+y)} \) (shift)
  \( e^{j(2\pi x+y)} \Leftrightarrow \delta(-u-1,-v-1) \) (duality)
  \( = \delta(u-1,v+1) \) (even function)
- **Similarly:**
  \( e^{-j2\pi(x+y)} \Leftrightarrow \delta(u+1,v-1) \)
- **Therefore:**
  \( \cos(2\pi(x+y)) \Leftrightarrow \frac{1}{2} \left[ \delta(u-1,v-1) + \delta(u+1,v+1) \right] \)

**Block Example**

- **Approach:** use known transforms and results
  \( \text{rect}\left(\frac{x}{A}\right) \Leftrightarrow c \cdot d \cdot \text{sinc}[cu,dv] \)

**Block Example II**

- **From previous example:**
  \( \text{rep}_{x,y} \left[ \text{rect}\left(\frac{y}{c},\frac{y}{d}\right) \right] \Leftrightarrow \frac{cd}{ab} \text{comb}_{\frac{c}{a}} \left[ \text{sinc}[cu,dv] \right] \)
- **Infinite pattern**
- **Signal of interest:**
  \( f(x, y) = \text{rep}_{x,y} \left[ \text{rect}\left(\frac{y}{c},\frac{y}{d}\right) \right] \text{rect}\left(\frac{y}{A},\frac{y}{B}\right) \)
- **Thus obtain** \( F(u,v) \) **by convolution**
  \( F(u,v) = \frac{cd}{ab} \text{comb}_{\frac{c}{a}} \left[ \text{sinc}[cu,dv] \right] \ast \text{ABsinc}(Au,Bv) \)
**Block Example III**

- Expand:
  \[ F(u, v) = \frac{A \delta cd}{ab} \sum_{m} \sum_{n} \sin \left( \frac{ma}{a} \right) \sin \left( \frac{na}{b} \right) \sin(Au, Bv) \]

- Convolve functions of \((u, v)\):
  \[ F(u, v) = \frac{A \delta cd}{ab} \sum_{m} \sum_{n} \sin \left( \frac{ma}{a} \right) \sin(Au, B) \delta \left( u - \frac{ma}{a}, v - \frac{na}{b} \right) \]

  - Periodicity controlled by \((1/a, 1/b)\)
  - Sinc envelope controls sinc repetitions

**Block Example VI**

**Two Dimensional Sampling (I)**

\[ f(x, y) \rightarrow f_s(x, y) \]

- Sampling interval: \((X, Y)\)
- Assume image is band limited

\[ F(u, v) = 0 \text{ for } u > X, v > Y \]

- Then

\[ \Im \{ f(x, y) \} = \Im \left\{ \sum \delta(x - mX, y - nY) \right\} \]

\[ = F(u, v) \ast \Im \left\{ \sum \delta(x - mX, y - nY) \right\} \]

**Two Dimensional Sampling (II)**

\[ F_s(u, v) = F(u, v) \ast \frac{1}{XY} \sum \delta(u - \frac{mX}{X}, v - \frac{nY}{Y}) \]

- Spectrums repeat at interval: \((1/X, 1/Y)\)
- For no aliasing:

\[ \frac{1}{2X} > B_x \text{ or } F_s > 2B_x \]

\[ \frac{1}{2Y} > B_y \text{ or } F_s > 2B_y \]
Two-Dimensional DFT

- Forward and inverse transforms:
  \[ F(u,v) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-j2\pi(ux/M + vy/N)} \]
  \[ f(x,y) = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v) e^{j2\pi(ux/M + vy/N)} \]

- Convenient to move the spectral origin to the image center:
  \[ 2 \left[ f(x,y)e^{-j\pi x} \right] = F(u-M/2, v-N/2) \]
- Verify frequency shift

Directional Response Example

- Scanning Electron Microscope (SEM) image of integrated circuit board
- Note DC (origin) shift to image center
- Edges correspond to high frequencies
  - Note directionality of edges

Filtering in the Frequency Domain

- Output in the frequency domain:
  \[ G(u,v) = H(u,v) F(u,v) \]
- We focus on real valued transforms
  - Zero phase shift

Notch Filter

- DC removal
  \[ H(u,v) = \begin{cases} 0 & \text{if } (u,v)=(M/2, N/2) \\ 1 & \text{otherwise} \end{cases} \]
- Edge information (high frequencies) remain
- Most practical filters and have smoother stop-pass band transitions
Lowpass and Highpass Filtering Example

Gaussian Filter Example

- Frequency response:
  - $H(u,v) = Ae^{-\frac{u^2 + v^2}{2\sigma^2}}$
- Impulse response
  - $h(x,y) = \sqrt{2\pi}\sigma e^{-(x^2 + y^2)/2\sigma^2}$
- Highpass filter
  - $H(u,v) = Ae^{-\frac{u^2 + v^2}{\sigma^2}} - Be^{-\frac{u^2 + v^2}{2\sigma^2}}$
  - $h(x,y) = \sqrt{2\pi}\sigma e^{-(x^2 + y^2)/\sigma^2} - \sqrt{2\pi}\sigma e^{-(x^2 + y^2)/2\sigma^2}$
- Increasing $\sigma$
  - Broadens frequency response
  - Narrows spatial response

Ideal Lowpass Filter (ILPF)

- The ILPF bandwidth is controlled by the constant $D_o$
  - $H(u,v) = \begin{cases} 1 & \text{if } D(u,v) < D_o \\ 0 & \text{if } D(u,v) \geq D_o \end{cases}$
- If spectral centering is used:
  - $D(u,v) = \left[\left(u-M/2\right)^2 + \left(v-N/2\right)^2\right]^{1/2}$
- Cutoff frequency can be determined as a function of image power:
  - $P_c = \sum_{u,v} P(u,v)$
- A cut off based on power percentage can be used:
  - $\alpha = 10\log\left(\sum_{u,v} P(u,v)/P_c\right)$

Convolution

- Discrete time two-dimensional convolution:
  - $f(x,y) * h(x,y) = \frac{1}{MN} \sum_{m,n=-MN/2}^{MN/2} f(m,n) h(x - m, y - n)$

- The time-frequency input-output relations are:
  - $f(x,y) * h(x,y) \leftrightarrow F(u,v)H(u,v)$
  - $f(x,y) h(x,y) \leftrightarrow F(u,v) + H(u,v)$

- Sifting property definition of the delta function:
  - $\sum_{x,y} f(x,y) h(x,y) = s(0,0)$
**Ideal Lowpass Filter Representations**

- **Binary output**
  - Simple function of distance to origin

**Sample Image**

- **Spatial and frequency domains**
  - Percentage power circles
  - Low frequencies dominate

**ILPF Results**

- **Power removed:**
  - 8, 5, 4, 3.6, 2, and 0.5%
- **Sinc impulse response explains blurring and bringing**
  - As the bandwidth increases, the impulse response tends to a delta
  - Inverse relation of pulse and sinc width

**Pulse Sinc Transform Pair**

- **Graphs** showing the relationship between pulse and sinc transform.
ILPF Example II

- Frequency and spatial domain example
- Isolated spatial domain points represent fine details
- Convolution simply replicates sincs
  - Width of sinc controls blurring
  - Positive and negative values of sinc cause ringing
  - One-dimensional signals are scan lines

Butterworth Lowpass Filters

- Butterworth lowpass filter (BLPF) frequency response:
  \[ H(u, v) = \frac{1}{1 + (D(u, v)/D_0)^n} \]
  - Order: \( n \)
  - Cutoff frequency: \( D_0 \)
  - Smooth transfer function
    - Minimizes ringing
    - Order controls transition band width

Butterworth Lowpass Filter Transfer Function

BLPF Results

- Same cutoff frequencies as ILPF example
- Filter order: 2
- Significantly reduced ringing
  - Result of smooth transfer function and impulse response
**BLPF Spatial Domain Representation**

- Cutoff frequency: 5
- Increasing filter order
  - Impulse response spreads, oscillations introduced
  - Smoothing and ringing introduced

**Gaussian Lowpass Filters (GLPFs)**

- Frequency domain:
  - $H(u,v) = e^{-Duv}$
- Spatial domain also a Gaussian function
  - Nonnegative values
  - No ringing
  - Less cutoff/transition control

**GLPF Results**

- Same cutoff frequencies as previous examples
- Not as much smoothing
  - More gradual transition band
- No ringing

**Application Examples**

- Poor resolution sampled text
  - Scanned material, faxes
  - Broken text, OCR difficulties
  - Gaussian filter
  - Smoothes text, connects broken letters/lines
Cosmetic Smoothing of Images

Spectral Representations of Sharpening Filters

- Simple highpass representation:
  - \( H_s(u,v) = 1 - H_p(u,v) \)
- Spectrally centered examples
  - Ideal
  - Butterworth
  - Gaussian

Spectral Equations

- Ideal highpass filter (IHPF)
  - \( H(u,v) = \begin{cases} 0 & \text{if } D(u,v) < D_0 \\ 1 & \text{if } D(u,v) \geq D_0 \end{cases} \)

- Butterworth highpass filter (BHPF)
  - \( H(u,v) = \frac{1}{\sqrt{1 + \left(\frac{D(u,v)}{D_0}\right)^2n^2}} \)

- Gaussian highpass filter (GHPF)
  - \( H(u,v) = 1 - e^{-D^2(u,v)/2D^2} \)

Spatial Representations

- Spatial representations of typical (a) ideal, (b) Butterworth, and (c) Gaussian frequency sharpening filters, and corresponding point spread functions.
Image Processing
Enhancement in the Frequency Domain
Prof. Barner, ECE Department, University of Delaware

IHPF Result

- Highpass filtering for various cutoff frequencies
- Low cutoff frequency produces ringing
- Image power concentrated in low frequencies

GHPF Results

- Same cutoff frequency is as previous example
- Result:
  - The least ringing of the three filters
  - Alternative approach: difference of Gaussian filters
  - More parameters and greater control of spectral response

BHPF Results

- Filter order: 2
- Same cutoff frequency is as previous example
- Result: reduced ringing

Laplacian Operator

- Note that (prove):
  \[
  \nabla^2 f(x) = (j\omega)^2 F(u,v)
  \]

- Using this to determine the Laplacian:
  \[
  \nabla^2 f(x) = \frac{\partial^2 f(x)}{\partial x^2} + \frac{\partial^2 f(x)}{\partial y^2} = (j\omega)^2 F(u,v)
  \]

- Thus:
  \[
  \nabla^2 f(x) = -(u^2 + v^2) F(u,v)
  \]
  - The Laplacian operator is a filter with frequency response
  \[
  H(u,v) = -(u^2 + v^2)
  \]
Laplacian Operator II

- If spectral centering is used:
  \[ \nabla^2 f(x, y) \cong -\left[ (u-M/2)^2 + (v-N/2)^2 \right] F(u,v) \]
- Recall a sharpened image is given by:
  \[ g(x, y) = f(x, y) - \nabla^2 f(x, y) \]
- Performing the operation in the frequency domain:
  \[ g(x, y) = 3^{-1} \left[ 1 - \left( (u-M/2)^2 + (v-N/2)^2 \right) \right] F(u,v) \]

Laplacian in the Frequency and Spatial Domains

- Highpass filter
  - Consider magnitude
- Spatial response
  - Restricted to axes
- Rotation invariant response
  - Diagonals can be added

Unsharp Masking

- High-boost filtering:
  \[ f_u = Af(x, y) - f_u(x, y) \]
  - Unsharp masking: \( A=1 \)
- Rearranging:
  \[ f_u(x, y) = (A-1)f(x, y) + f_u(x, y) \]
- Composite frequency response:
  \[ H_u(u,v) = (A-1) + H_u(u,v) \]
- High-frequency emphasis:
  \[ H_{\text{hfe}}(u,v) = u + bH_u(u,v) \]
  - Highpass filter that further emphasizes high frequencies

Laplacian Example

- Shown
  - Original
  - Laplacian
  - Scaled Laplacian
  - Enhanced result
- Result previously obtained in the spatial domain

Unsharp Masking

- High-boost filtering:
  \[ f_u = Af(x, y) - f_u(x, y) \]
  - Unsharp masking: \( A=1 \)
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  \[ H_{\text{hfe}}(u,v) = u + bH_u(u,v) \]
  - Highpass filter that further emphasizes high frequencies
High-Boost Filtering Example

Original image characteristics:
- Blurry – x-ray images can’t be “focused”
- Narrow histogram
- High-frequency emphasis
- Improves edge characteristics
- Histogram equalization
- Improves contrast and gray level distribution

Homomorphic Filtering

- Recall illumination and reflectance image model:
  - \( f(x, y) = i(x, y)r(x, y) \)
  - Not directly separable:
    - \( \Im \{ f(x, y) \} \neq \Im \{ i(x, y) \} \Im \{ r(x, y) \} \)

- Solution: separate by utilizing log
  - \( z(x, y) = \ln f(x, y) = \ln i(x, y) + \ln r(x, y) \)
  - \( \Im \{ z(x, y) \} = \Im \{ \ln f(x, y) \} = \Im \{ \ln i(x, y) \} + \Im \{ \ln r(x, y) \} \)
  - \( Z(u, v) = F_i(u, v) + F_r(u, v) \)

Homomorphic Filtering II

- Filtering the results
  - \( S(u, v) = H(u, v)Z(u, v) \)
  - \( = H(u, v)F_i(u, v) + H(u, v)F_r(u, v) \)

- In the spatial domain
  - \( s(x, y) = i'(x, y) + r'(x, y) \)
    - \( i'(x, y) = \Im^{-1} \{ H(u, v)F_i(u, v) \} \)
    - \( r'(x, y) = \Im^{-1} \{ H(u, v)F_r(u, v) \} \)
Homomorphic Filtering III

- Inverting the log operation
  \[
  g(x, y) = e^{u(x, y)} = e^{\log_i(x, y)} = e^{i(x, y)}
  \]
  \[
  h(x, y) = e^{v(x, y)}
  \]
  \[
  e(x, y) = e^{u(x, y)}
  \]

- Homomorphic Filter Characteristics
  - Illumination component
    - Slow spatial variations (low frequencies)
  - Reflectance component
    - Abruptly, especially at object borders (high frequencies)
  - Homomorphic filter characteristics
    - Attenuate illumination component (low frequencies)
    - Amplify reflectance component (high frequencies)

Homomorphic Example

- Filter realization:
  - Result:
    - Reduced dynamic range of brightness
    - Increased contrast
    - Better balance of gray values
  - Other applications: multiplicative processes

Two-Dimensional Fourier Transform Properties

- Translation:
  - \[ f(x, y)e^{2\pi j(Mx/N + Nz/N)} \iff F(u - u_0, v - v_0) \]
  - \[ f(x - x_0, y - y_0) \iff F(u, v)e^{-2\pi j(Mx_0/N + Nz_0/N)} \]
  - Special case: \( u_0 = M/2 \) and \( v_0 = N/2 \)
    - \[ e^{2\pi j(Mx_0/M + Nv_0/N)} \iff e^{2\pi j(x_0y_0)} = (-1)^{x_0y_0} \]
  - Centering result:
    - \[ f(x, y)(-1)^{-x_0} \iff F(u - M/2, v - N/2) \]
    - \[ f(x - M/2, y - N/2) \iff F(u, v)(-1)^{-y_0} \]
Two-Dimensional Fourier Transform Properties II

- **Distributivity**
  \[ \mathcal{F}[f(x, y) + f_2(x, y)] = \mathcal{F}[f(x, y)] + \mathcal{F}[f_2(x, y)] \]

- **Scaling**
  \[ a f(x, y) \Leftrightarrow a \mathcal{F}(u, v) \]
  \[ f(ax, by) \Leftrightarrow \frac{1}{ab} \mathcal{F}(u/a, v/b) \]

- **Rotation:** change to polar coordinates
  \[ x = r \cos(\theta); y = r \sin(\theta); u = \omega \cos(\phi); v = \omega \sin(\phi) \]
  \[ f(r, \theta) \leftrightarrow F(\omega, \phi) \]
  \[ f(x, y) \leftrightarrow F(u, v) \]

Rotation property
  \[ f(r \cos(\theta) + \phi) \leftrightarrow F(u \cos(\phi) + \theta) \]

Two-Dimensional Fourier Transform Properties III

- **The DFT and inverse DFT are periodic**
  \[ F(u, v) = F(u + M, v) = F(u, v + N) = F(u + M, v + N) \]
  \[ f(x, y) = f(x + M, y) = f(x, y + N) = f(x + M, y + N) \]

Fourier Series and DFT

- For non-periodic signals, generate periodic version by replication

- For periodic signal \( f(x, y) \)
  \[ f(x, y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) e^{j \frac{2\pi vx}{M}} e^{j \frac{2\pi uy}{N}} \]

- **FS coefficients:** \( F(u, v) \)
  \[ F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j \frac{2\pi ux}{M}} e^{-j \frac{2\pi vy}{N}} \]

FS Example

- One period of signal: \( f(x, y) = \delta(x, y) \)
  \[ F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} \delta(u, v) e^{-j \frac{2\pi ux}{M}} e^{-j \frac{2\pi vy}{N}} \]
  \[ = 1 \text{ for all } (u, v) \]

- Therefore
  \[ f(x, y) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} \delta(x, y) e^{-j \frac{2\pi ux}{M}} e^{-j \frac{2\pi vy}{N}} \]
**Periodic Extensions and the DFT**

- A finite extent signal \( f(x,y) \) is make periodic by
  \[
  f_p(x,y) = \sum_{k_x} \sum_{k_y} f(x-k_x M, y-k_y N)
  \]
- \( f_p(x,y) \) is a periodic extension of \( f(x,y) \) if
  \[
  f(x,y) = \begin{cases} 
  f_p(x,y) & \text{if } 0 \leq x < M, 0 \leq y < N \\
  \text{else} & 
  \end{cases}
  \]
- Similarly
  \[
  F(u,v) = \begin{cases} 
  F(f(u,v)) & \text{if } 0 \leq u < M, 0 \leq v < N \\
  \text{else} & 
  \end{cases}
  \]
- DFT coefficients = FS coefficients

**Circular Convolution**

- Let
  \[
  f_p(x,y) \Leftrightarrow F_p(u,v)
  \]
  \[
  h_p(x,y) \Leftrightarrow H_p(u,v)
  \]
  \[
  y_p(u,v) = H_p(u,v)F_p(u,v)
  \]
- Determine the time domain relation
  \[
  \text{Let } W_k = e^{-j\pi k}
  \]
  \[
  \text{Then } y_p(x,y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} H(u,v)F(u,v)W_u W_v
  \]
- Recall
  \[
  F_p(u,v) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n)W^m W^n
  \]

**Circular Convolution II**

- Combining the above
  \[
  y_p(x,y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n) W^m W^n
  \]
- Result convolution, but
  \[
  y(x,y) = \begin{cases} 
  y_p(x,y) & \text{if } 0 \leq x < M, 0 \leq y < N \\
  \text{else} & 
  \end{cases}
  \]
- Then
  \[
  y(x,y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n) W^m W^n
  \]
- Circular convolution

**Circular Convolution Example**

- \( 2 \times 2 \) DFT:
  \[
  F(u,v) = \sum_{x=0}^{1} \sum_{y=0}^{1} f(x,y)e^{-j\frac{2\pi uv}{2}}e^{-j\frac{2\pi uv}{2}} (-1)^u (-1)^v
  \]
- \( F(0,0) = 2+1+1+0 = 4 \)
- \( F(1,0) = f(0,0) - f(0,1) + f(1,0) - f(1,1) = 2-1+1-0 = 2 \)
- \( F(1,0) = 1 \)
- \( h(x,y) = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \)
Circular Convolution Example II

- \( F(0,1) = f(0,0) + f(1,0) - f(0,1) - f(1,1) \)
  \[= 2 + 1 - 1 - 0 = 2 \]
- \( F(1,1) = f(0,0) - f(1,0) - f(0,1) + f(1,1) \)
  \[= 2 - 1 - 1 + 0 = 0 \]

Circular Convolution Example III

- Let \( G(u,v) = H(u,v)F(u,v) \)
- Taking the image 2x2 DFT

Linear and Circular Convolution

- Goal: have circular and linear convolution the equivalent over the desired range: \([0,M-1],[0,N-1]\)
- Solution: zero padding
  - \( f_e(x) = \begin{cases} f(x) & 0 \leq x \leq A-1 \\ 0 & A \leq x \leq P \end{cases} \)
  - \( g_e(x) = \begin{cases} g(x) & 0 \leq x \leq B-1 \\ 0 & B \leq x \leq P \end{cases} \)
- No convolution overlap for \( P \geq A+B-1 \)
Convolution Example

- For appropriate zero padding, circular and linear convolution are equivalent.
- Over fixed range of interest.
- Result can be obtained by multiplying DFTs.

Two-Dimensional Extensions

- Image 1: \( f(x,y) \) Size: \( AxB \)
- Image 2: \( h(x,y) \) Size: \( CxD \)
- Zero padded images:
  \[
  f_p(x,y) = \begin{cases} 
  f(x,y) & 0 \leq x \leq A-1 \text{ and } 0 \leq y \leq B-1 \\ 
  0 & \text{otherwise}
  \end{cases}
  \]
  \[
  h_p(x,y) = \begin{cases} 
  h(x,y) & 0 \leq x \leq C-1 \text{ and } 0 \leq y \leq D-1 \\ 
  0 & \text{otherwise}
  \end{cases}
  \]
- Padding constraints:
  \[ P \geq A+C-1 \]
  \[ Q \geq B+D-1 \]

Two-Dimensional Example of Zero Padding

Filtering Through Padding Example

- Padded lowpass filter and resulting output image.
- Cropping is used to obtain the final area of interest.
- Note directional ringing (in padded region).
### Convolution and Correlation

- **Recall convolution:**
  - $f(x,y) * h(x,y) = \sum_{m,n} f(m,n) * h(-m,-n)$
  - $f(x,y) * h(x,y) = F(u,v)H(u,v)$
  - $f(x,y)h(x,y) \Leftrightarrow F(u,v) + \overline{H(u,v)}$

- **Correlation definition:**
  - $f(x,y) \leftrightarrow F(u,v)$
  - $f(x,y) * h(x,y) = \sum_{m,n} f(m,n) * h(m+n)$
  - Spatial and frequency domain relations:
    - $f(x,y) * h(x,y) = F(u,v)H(u,v)$
    - $F^*(u,v)H^*(u,v)$
  - Autocorrelation special case
    - $f(x,y) \leftrightarrow F(u,v)$
    - $f(x,y) \leftrightarrow \overline{F(u,v)}$

### Correlation Example

- Correlation measures statistical similarity
- **Common application:**
  - Template matching
  - Zero pad image and template
  - Multiply DFTs (conjugate image DFT)
  - Invert resulting DFT
  - Find peak location
    - Identifies best template match

### FFT Calculation

- **Implementation:** FFT
  - $N \log N$ operations vs $N^2$
  - Uses symmetry of the problem
  - Power of two length sequences most common
    - $N = 2^n$